Improved frequency response measurements using local parametric models

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***Abstract*—The concept of local parametric modelling has sparked renewed attention in frequency response function (FRF) measurements. Essentially, these approaches assume a particular parametric structure and approximate the FRF and the leakage errors in a small window around the frequency of interest. Following the successful application of the idea in the local polynomial method (LPM), the local rational method (LRM) was developed. The smoothness of the transfer function in lowly damped systems can be insufficient for polynomial approximation to be effective, provoking us to consider rational functions instead. The power of the LRM has previously been demonstrated in both simulations and experiments. At the cost of increased computation, the LRM reduces the leakage errors with several orders of magnitude w.r.t. its alternatives while the sensitivity to disturbing noise remains comparable to that of classical methods.**

***Keywords*—Frequency response function, local parametric modelling, polynomial functions, rational functions, approximation**

I. Introduction

Measuring the FRF to characterize the dynamic behaviour of a linear time invariant (LTI) system is a well-studied problem. Spectral analysis methods based on correlation techniques have long been used as standard tools in engineering practice, see e.g. [1], [2]. Any non-parametric method suffers from leakage errors and noise errors. Leakage errors put a major restriction on the classical methods, even in the absence of measurement or process noise. For that reason the problem again attracted considerable attention in more recent years. With the so-called local polynomial method (LPM), the concept of local parametric modelling was introduced in [3], [4], [5]. Basically, LPM locally uses polynomial approximations of the transfer function and the transient behaviour of the system caused by finite data length and initial conditions effects. The idea was subsequently extended [6] replacing polynomial by rational approximants, resulting in what is referred to as the local rational method (LRM). The potential advantage of rational approximations is that they may behave better in the vicinity of poles, and, in particular, may be used to extrapolate or interpolate a function beyond poles that would block the convergence of a polynomial, see e.g. [7], [8]. In this paper we investigate rational approximations of a transfer function in an effort establish an error bound that relates tuning parameters to system properties. In this paper we focus on the LRM. We regard it as a particular instance of a more general local parametric approach. Inspired by the promising results in simulations [6] and in experiments [9] we discuss and compare the different local parametric methods.

II. Local modelling

The local parametric methods focus directly on the relation between input and output DFT spectra. Their essential advantage lies in their ability to dramatically reduce leakage errors. The central idea is to estimate the frequency response by considering a narrow interval around each frequency. Inside that interval the system and transient model can be represented by low order approximations. As such a reduced system identification problem is solved at each frequency of interest.

A. Problem formulation

The data consists of the N-point discrete Fourier transform (DFT) [10] spectra of input and output signals, possibly disturbed by additive noise

$$U(k)=U_0(k) \tag{1}$$
$$Y(k)=Y_0(k)+N_Y(k). \tag{2}$$

Inside the window $B=[k-n,k+n]$ the spectra are related according to

$$\begin{aligned} Y(k) &= G(k)U(k)+T_G(k)+ \\ &\quad H(k)E(k)+T_H(k), \end{aligned} \tag{3}$$

with

$$\begin{aligned} G(k) &= B_G(k)/A_G(k), \\ T(k) &= T_G(k)+T_H(k)=I(k)/A_G(k). \end{aligned} \tag{4}$$

$T_G(\Omega_k)$ and $T_H(\Omega_k)$ are, respectively, the system and noise transient terms. These are rational functions of the generalized frequency variable, Ω_k, that share the same set of poles with respectively $G(\Omega_k)$ and $H(\Omega_k)$. Note that k and Ω_k as the arguments to a function $f(\cdot)$ are used interchangeably throughout the text. The transient terms cause leakage errors in the frequency response function measurements [4].
The unobserved (band-limited) white noise source is assumed to be independent and identically distributed (iid) at the sampling points, resulting in a DFT spectrum $E(k)$ with favourable properties (see [4]). The noise variance is given by

$$\sigma_V^2(k)=\text{Var}(V(k)) \text{ with } V(k)=H(\Omega)E(k) \tag{5}$$

The goal is now to obtain a non-parametric estimate of the frequency response matrix $G(k)$ and the noise variance $\sigma_V^2(k)$ from the measured input and output spectra in the frequency band of interest. The major difficulty in obtaining accurate estimates is the suppression of the leakage terms.

Both transfer function and transient term in model (3-4) are estimated by minimizing the residuals $\mathcal{E}(\ell)$ inside the interval B for all $\ell = k + r$ with $r = -n, \ldots, 0, \ldots, n$

$$\mathcal{E}(\ell) \quad = \quad Y(l) - G(\ell, \theta_G)U(\ell) - T_G(\ell, \theta_{T_G}), \qquad (6)$$

using a weighted least-squares cost function:

$$V_c(k) = \sum_B W(\ell)|\mathcal{E}(\ell)|^2, \qquad (7)$$

that is minimized with respect to θ_G and θ_{T_G}, the model parameters. Under mild conditions $\sigma_Y^2(k+r) = \sigma_Y^2(k)+\mathcal{O}(r/N)$ ([4], Appendix B), so that we may reasonably assume that the output noise variance is constant in the interval B. In the following we revise several options for the weighting function $W(\ell)$ and justify our choice in the final algorithm.

B. Weighting the least-squares cost

a) Local polynomial method (LPM): $W(\ell) = 1$, $\hat{G} = B_{\hat{G}}$, $\hat{T} = I$. The nonlinear problem is transformed into a linear least-squares problem by setting the denominator equal to one $A_{\hat{G}}(\ell) = 1$. The minimizer of the simplified cost function

$$V_{LPM}(k) = \sum_B |Y(\ell) - B_{\hat{G}}(\ell)U(\ell) - I(\ell)|^2. \qquad (8)$$

is found by solving an overdetermined set of linear equations.

b) Iterative local rational method (ILRM): $W(\ell) = 1$, $\hat{G} = B_{\hat{G}}/A_{\hat{G}}$, $\hat{T} = I/A_{\hat{G}}$. Due to the presence of the denominator $A_{\hat{G}}(\ell)$, the objective function remains truly nonlinear:

$$V_{ILRM}(k) = \sum_B |Y(\ell) - \frac{B_{\hat{G}}(\ell)}{A_{\hat{G}}(\ell)}U(\ell) - \frac{I(\ell)}{A_{\hat{G}}(\ell)}|^2. \qquad (9)$$

The optimization problem must now be solved iteratively.

c) Local (linearized) rational method (LRM): $W(\ell) = |A_{\hat{G}}(\ell)|^2$, $\hat{G} = B_{\hat{G}}/A_{\hat{G}}$, $\hat{T} = I/A_{\hat{G}}$. The objective function is linearized by this particular choice of the weighting function, resulting in another linear least-squares problem, [6],[11]:

$$V_{LRM}(k) = \sum_B |A_{\hat{G}}(\ell)Y(\ell) - B_{\hat{G}}(\ell)U(\ell) - I(\ell)|^2. \qquad (10)$$

C. Comparison of properties

With a short discussion of the properties of the three outlined methods, we highlight the pertinence of the LRM.

a) Iterative local rational method: The ILRM appears to be the most straightforward, and the increased computational cost does not entail an unreasonable burden. A more critical issue is its high noise sensitivity. Excess poles and zeros can create large spikes in case of closely spaced poles and zeros. These artifacts can only be avoided by dedicated model tuning (at each single frequency). Obviously, this slashes the robustness of the overall process. The effect of the cost function on pole-zero cancellation is the subject of Section V.

b) Local polynomial method: The simplified approach using polynomial approximation proves to be very attractive. The identification problem is linear in the parameters, and it turns out that model selection is not critical. Clearly, a polynomial can only approximate a rational form adequately in a finite frequency window B. It was shown in [12] that in case of lowly damped systems the approximation error E_{LPM} is bounded by $(B_{LPM}/B_{3\text{dB}})^{R+2}$. In this expression B_{LPM} is the local bandwidth of the LPM, R the degree of the local polynomial (assumed to be even), and $B_{3\text{dB}}$ the 3-dB bandwidth around a resonance, a system property. Compared to the spectral windowing methods, a huge gain is made in the reduction of leakage errors. The length of the local window is subject to a trade-off [13]. On the one hand the interval should be as small as possible to reduce model errors. On the other hand it must contain enough frequencies to estimate all complex coefficients in the polynomial approximants. In addition, increasing the length of B reduces the sensitivity of the estimate to noise.

c) Local rational method: For systems with low damping, as often occurs in advanced mechanical applications (and other applications dealing with high resonances), the constraint $B \leq B_{3dB}$ may not be satisfied. The potential advantage of rational approximations is that they may behave better in the vicinity of poles, and, in particular, may be used to extrapolate or interpolate a function beyond poles that would block the convergence of a polynomial, see e.g. [7], [8].

III. A CLOSER VIEW ON THE LOCAL RATIONAL METHOD

The basic idea of the LRM is very simple. The frequency response G and the transient term T are smooth functions of the frequency. Furthermore, for a linear time invariant system, they have a rational form. As a result, they can be approximated in a narrow band around a particular frequency k by low-order complex rational functions. The complex parameters are estimated in a local band and the function value at the center frequency k is retrieved as the FRM at that point.

A. Basic equations

The crucial property used in local parametric approaches is that $G(\Omega_k)$, $H(\Omega_k)$, $T(\Omega_k)$ are smooth functions of the frequency. Returning to the local frequency variable $r = -n, \ldots, 0, \ldots, n$ the output spectrum at line $k + r$ can be written as:

$$Y(k+r) = G(k+r)U(k+r) + T(k+r) + V(k+r), \qquad (11)$$

where $T(k) = T_G(k) + T_H(k)$ since there is no means of discriminating between system and noise transients at that level. $G(k)$ and $T(k)$ are assumed to be rational functions, consequently having continuous derivatives up to any order. Then $G(k + r)$ and $T(k + r)$ can be expanded at $k + r$ as:

$$G(k+r) \quad = \quad G(k) + \sum_{s=1}^{R} g_s(k)r^s + O((r/N)^{(R+1)}), \qquad (12a)$$

$$T(k+r) \quad = \quad \sum_{s=1}^{R} t_s(k)r^s + N^{-\frac{1}{2}}O((r/N)^{(R+1)}). \qquad (12b)$$

This expression originates from applying Taylor's formula with remainder to (11) and is the basis for the LPM. The

remainders are, respectively, the system interpolation error and the sum of the residual system and noise leakage errors.

In the LRM the above polynomial approximation is substituted for a (linearized) rational approximation $\hat{G}(k) = B(k)/A(k)$. Note that the subscript $\hat{G}$ is dropped to unburden the notation. Neglecting the remainders results in

$$\begin{aligned} Y(k+r) &= \frac{B(k+r)}{A(k+r)}U(k+r) + \\ &\quad \frac{I(k+r)}{A(k+r)} + V(k+r) \end{aligned} \tag{13}$$

where $\mathcal{X}(k+r) = \mathcal{X}(k) + \sum_{s=1}^{R} x_s(k) r^s$ with $\mathcal{X} = I, B, A$ and $x = i, b, a$ denote the polynomials corresponding to the transient term, and the transfer function's denominator and numerator. Note that for the expansion of the system transfer function numerator and denominator, as well as the leakage term, different orders are allowed. However, it turns out that if an R-th order approximation is suitable for $\hat{G}$ (effectively A), it is also appropriate for $\hat{T}$. The reason is that G and T have the same poles, and that the system leakage term T_G is dominant with respect to the noise leakage term T_H.

B. Linearized least squares

Rearranging (13) (without the noise contribution) leads to

$$A(k+r)Y(k+r) = B(k+r)U(k+r) + I(k+r). \tag{14}$$

In order to avoid the trivial solution $\Theta = 0$, $A(k)$ is set equal to one. With this additional assumption the equation becomes:

$$\begin{aligned} Y(k+r) &= B(k+r)U(k+r) + I(k+r) \\ &\quad -\tilde{A}Y(k+r) + \tilde{V}(k+r) \\ &= \Theta K(k+r) + \tilde{V}(k+r), \end{aligned} \tag{15}$$

where we have introduced $\tilde{A}(k+r) = A(k+r) - 1$, and $\tilde{V} = A(k+r)V(k+r)$. Θ is the $1 \times (3R+2)$ vector of the unknown complex parameters

$$\begin{aligned} \Theta = [&b_0(k)\, b_1(k)\, b_2(k)\, \ldots\, n_R(k) \ldots \\ &i_0(k)\, i_1(k)\, i_2(k)\, \ldots\, i_R(k) \ldots \\ &a_1(k)\, a_2(k)\, \ldots\, a_R(k)\,] \end{aligned} \tag{16}$$

$K(k+r)$ is a vector of size $(3R+2) \times 1$:

$$K(k+r) = \begin{bmatrix} \kappa(r,\beta)U(k+r) \\ \kappa(r,\iota) \\ -\kappa(r,\alpha)Y(k+r) \end{bmatrix} \tag{17}$$

where $\beta, \iota = [0\, 1 \ldots R]^T$ and $\alpha = [1\, 2 \ldots R]^T$. $\kappa(r,\chi)$ is defined as

$$\kappa(r,\chi) = r^{\chi} \in \mathbf{Z}^{n_\chi \times (2n+1)}, \tag{18}$$

with n_X the number of elements of $\underline{X}$. Collecting (15) for $r = -n, -n+1, \ldots, 0, \ldots, n-1, n$ gives

$$Y_n = \Theta K_n + \tilde{V}_n \tag{19}$$

where Y_n, K_n and V_n are, respectively, $1 \times (2n+1)$, $(3R+2 \times (2n+1)$ and $1 \times (2n+1)$ matrices

$$X_n = (X(k-n)X(k-n+1) \ldots X(k) \ldots X(k+n)) \tag{20}$$

with $X = Y,\ K,\ \tilde{V}$.
When $2n+1 \geq 2(R+1)+R$. Eq. (19) is an overdetermined set of equations that can be solved in the least squares sense as

$$\hat{\Theta} = Y_n K_n^H (K_n K_n^H)^{-1}. \tag{21}$$

x^H denotes the Hermitian transpose of x.

The local rational estimate of the frequency response function and the transient term at frequency k is then obtained via

$$\hat{G}(k) = \hat{\Theta}\,(1\,\mathbf{0})^H = \hat{\Theta}_{[1]} \tag{22}$$

$$\hat{I}(k) = \hat{\Theta}\,(\mathbf{0}\,1\,\mathbf{0})^H = \hat{\Theta}_{[(R+1)+1]} \tag{23}$$

where $X_{[1,j]}$ selects the j-th element of X. Because the DFT lines $k+r$, for $r = -n, n-n+1, \ldots, n$ are used for estimating the FRF at k, $\hat{G}(k)$ is correlated with $\hat{G}(k+r)$ for $r = -n, -n+1, \ldots 0, \ldots, n$.

IV. Approximation error

When the system under test exhibits strongly resonant behaviour, the LRM performs remarkably better than the classical spectral analysis techniques, and even, the LPM. Figure 1 supports that claim. Given that for a certain class of systems the interpolation errors dominate stochastic errors, it is of prime importance to understand the source of approximation errors induced by rational approximation.

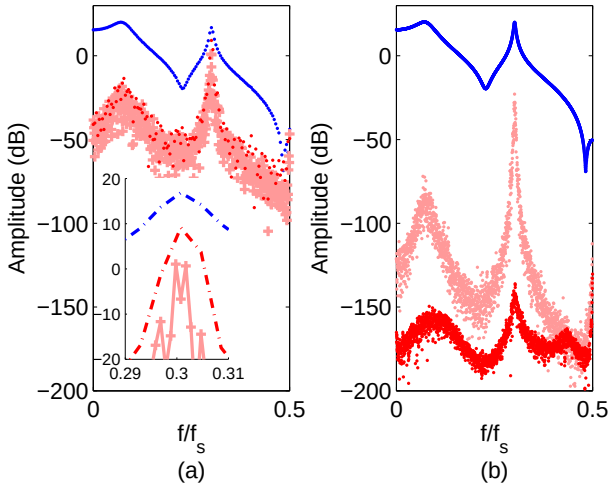

Fig. 1. A comparison of errors in FRF measurements for a simple numerical example using undisturbed data. A system composed of a highly and a lowly damped resonance is excited with filtered white noise (bandwidth of $0.4 f_s$) The full record contains 4096 samples. Figure (a) shows the results for a spectral analysis method with Hann window applied on subrecords of length $N = 256$ (red) and $N = 1024$ (pink) samples, both with an overlap of $R = 2/3 \times N$. In the inset of (a) a zoom around the second resonance is given. The LPM (pink) and LRM (red) in figure (b) is applied to the full length record. Observe that the errors of the LPM/LRM are an order of magnitude smaller than those of the Hanning method. The LRM outperforms the LPM.

V. Pole-zero cancellation

In Section II-C we mentioned that closely spaced zeros and poles can create large spikes in the ILRM. We used this argument to reject the ILRM in favour of the LRM. Here we support that observation with a simplified analysis of their

respective cost functions. We consider the simplest transfer function one can imagine: a SISO transfer function with one pole-zero pair that cancels itself. Then we add a disturbance on the zero and study the effect on the respective cost functions.

A. Simplified analysis

We consider the simplest transfer function one can imagine: a SISO transfer function with one pole-zero pair that cancels itself. Then we add a disturbance on the zero and study the effect on the respective cost functions.

$$\frac{B}{A} = \frac{1+bs}{1+as} \quad \text{with} \quad \begin{cases} b &= (d+\epsilon)j \\ a &= (d+0)j \end{cases} \tag{24}$$

In case of the ILRM we work with the cost function

$$E_{ILRM} = Y - \frac{B}{A}U, \tag{25}$$

and find that the perturbed transfer function is equal to

$$\frac{1+djs+\epsilon js}{1+djs} = 1 + \frac{\epsilon js}{1+djs}. \tag{26}$$

Evaluating the above in $s = j(1/d + \Delta)$ gives an error expression

$$E_{ILRM} = \epsilon \frac{1/d+\Delta}{\Delta} U \tag{27}$$

In case of the LRM the error function has the form

$$E_{LRM} = AY - BU. \tag{28}$$

Evaluating again at $s = j(1/d+\Delta)$, this leads to an error term

$$E_{LRM} = \epsilon(1/d+\Delta)U \tag{29}$$

Summing over all neighbouring frequencies $r = -n, \ldots, 0, \ldots, n$ we get

$$\sum_r E^2_{ILRM,r} = \sum_r \epsilon^2 \frac{|1/d+\Delta_r|^2}{\Delta_r^2} |U|^2 \tag{30}$$

$$\sum_r E^2_{LRM,r} = \sum_r \epsilon^2 (1/d+\Delta_r)^2 |U|^2 \tag{31}$$

Figure 2 compares the error terms (30) and (31). From this graphical display it becomes clear that the ILRM is much more sensitivity to residual effects due to closely spaced poles and zeros.

VI. Conclusion

In this contribution we introduce the local parametric modelling approaches to FRF estimation from the point of view of the cost function in the subordinate identification problems. We investigate the errors due to rational approximation and consider the effect of pole-zero cancellation. We find that rational functions can suppress approximation errors, consequently lowering the requirements on frequency resolution, shortening measurement time, while simultaneously being robust to perturbations.

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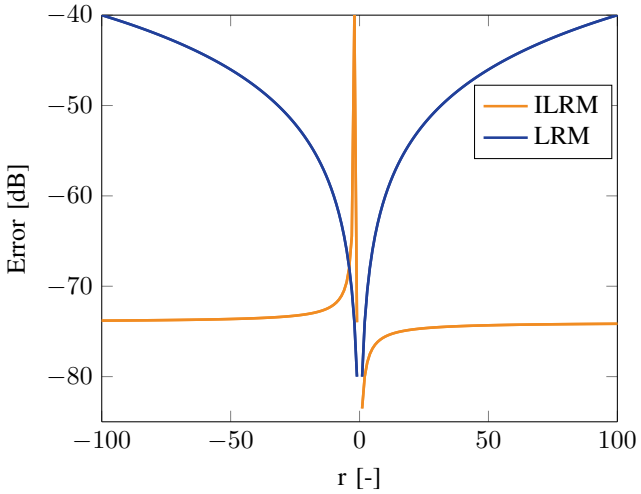

Fig. 2. Pole-zero cancellation and the effect of small perturbations on the error function. In this numerical example $c = 0.01$, $\epsilon = 0.0001$ and $n = 100$.

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