QUANTIFICATION AND EPSILON-INVARIANCE IN SOME EPSILON CALCULI

ZOLTÁN GÁBOR MOLNÁR

Doctoral Thesis Booklet

Budapest University of Technology and Economics

Department of Algebra

Budapest, 2020

1. Introduction

The *epsilon symbol* was first used by the members of the Göttingen Logic School, primarily in attempting to prove the consistency of Peano Arithmetic (a.k.a. Hilbert's Second Problem) which was one of the main goals of the Hilbert Program. Thus, quite early, in the second volume of the *Grundlagen der Mathematik,* Hilbert and Bernays examined the relationship between the epsilon symbol and the classical quantification [Hilbert–Bernays, 1939].

As it is well-known, if φ is a first-order formula and x is an individual variable, then

$$(\varepsilon x)\varphi(x)$$

denotes an individual in the universe of discurse, "that has the property $\varphi(x)$, if there are individuals with property $\varphi(x)$ at all". Here, ε is Hilbert's epsilon symbol and $(\varepsilon x)\varphi(x)$ is an epsilon term.

The former wording is problematic not only because it is not specified which object this term refers to, but also because, although, it sounds understandable, we do not use such descriptions in the natural language often. To better understand, consider the following arithmetical case. Let $A(n)$ denotes an arbitrary arithmetical one-variable formula. Then, what does $A(1)$ mean? It is completely understandable: "1 satisfies property $A(n)$". But what does

$$B((\varepsilon n)A(n))$$

mean for an arbitrary arithmetical formula $B(n)$? What kind of thing is it talking about at all, and how could this statement be verified?

Even Hilbert himself did not use such a natural linguistic wording, but tied the meaning of epsilon terms to an axiom. The *First Epsilon Axiom* states that, for every individual variable x and first-order formula $\varphi(x)$,

$$(\exists x)\varphi(x) \to \varphi((\varepsilon x)\varphi(x)).$$

The axiom describes, in an exact way, what the meaning of an epsilon term would be, however leaves open the question of what its natural linguistic meaning would be, and if it could be described with the usual logical terms, then what this reformulaton would be. Although the Göttingen School often used this term, we do not know the exact answer. In the *Grundlagen*, the Second Epsilon Theorem states, as we would say it today, that the epsilon calculus is a *conservative expansion* of the classical predicate logic [Zach, 2017]. As Hilbert puts it:

> [...] there is no need to include the ε-symbol in the final deductive structure of logical-mathematical formalism. Rather, operating on the ε-symbol can be considered as a mere helper calculus, which is of considerable benefit to many meta-mathematical considerations [Hilbert–Bernays, 1939, p. 13].

Obviously, this is a partial answer to the question of the meaning of epsilon terms. The zero approach to the problem of the meaning of epsilon terms, then, is to clarify the logical relationship between the existential quantifier and the epsilon symbol. The above early theorem is a result of classical logic. It is not necessarily the case in the intuitionistic logic, where the Law of Excluded Middle is generally not valid. After Bell's article, i.e. the expansion of intuitionistic arithmetic (Heyting Arithmeric, HA) with epsilon symbol is not conservative over HA (it does not remain a mere helper calculus), it becomes clear that the syntactic realization of intuitionistic epsilon logic drastically determines what properties the system will have [Bell, 1993a]. And indeed, the results in [Baaz–Zach, 2019], on the one hand, and in [Mints, 1977], on the other hand, show a strange discrepancy (which is pointed out by Baaz and Zach, themselves). Before Mints, in 1971, Shirai also defined an epsilon expansion that was, in a strange way, conservative over the intuitionistic logic [Shirai, 1971].

The problem now, is to point out the reasons that result in non-conservativity. With some modification of the typing rules also found in [Abadi et al., 2004], which can be achieved with a special inference rule found in [Sorensen–Urzyczyn, 1998], I proved via Curry–Howard Isomorphism that, in a special typed calculus, which mimics a fragment of first-order logic, over the logic of intuitionistic implication and existential quantifier, the epsilon-expansion is not conservative (Theorem 4). However, I proved that in the reverse case the conservativity still holds (Theorem 3). Neither [Abadi et al., 2004] nor the unpublished [Baaz–Zach, 2019] mentions or elaborates these results, where my methods are based on natural deduction system mostly in the spirit of the well-known textbook [Troelstra–Schwichtenberg, 2000].

Turning to a closer discussion of the problem outlined above it is worth noting that in the evolution of modern logic, any initiative that considered the formal language and its semantics as the basic starting point for mathematical activity overcome the problem of the *descriptive terms.* Gottlob Frege, Bertrand Russell and David Hilbert, all three made their own suggestions on how to understand, in mathematical theories, the sentences of the form "The F is G.", where the term "the F" is the definite description. So I think of sentences like "The smallest positive prime number is even." or "The greatest prime number is odd". Although it turned out that the interpretation of these sentences is essentially a task of quantification, as any language philosophical problem, this question has also not been answered without a doubt. The biggest problem is obviously that, in the absence of a single and existing F, we do not have any intuition about what this thing or these things may be like, whether it is G (they are Gs) or not. Russell's proposal for the meaning of such sentences is:

There is an F, there is at most one F, and every F is a G

[Russell, 1905]. Note that, Russell, in the *Principia*'s formal language used the term

$$(\iota x)\varphi(x)$$

instead of "the F" [Whitehead-Russell, 1956]. The problem of the meaning of epsilon terms was best illustrated by Kneebone:

> An ε-term may be thought of as formalizing an indefinite description, somewhat as an ι-term formalizes a definite description[...] ([Kneebone, 1963, p. 101, ftn. 1])

Since Russell gave quantification reading for sentences containing ι-terms, we can hope that the situation is similar with epsilon terms. And indeed, Caiceado, and later Blass and Gurevich proved that, if a sentence is epsilon-invariant (it is independent of the choice of the reference of the epsilon terms) in predicate logic, then the sentence has a plain firs-order logic meaning [Caicedo, 1995] [Blass–Gurevich, 2000]. However the explicit meaning of sentences containing epsilon terms is not known. Based on the proposal of Moser and Zach, I constructed a *set theoretical semantics* that is suitable for making the equivalent explicit plain first-order reformulation under special conditions (Theorem 6) [Molnár, 2013].

Tarski's algebraic program can be considered as another manifestation of Russell's quantification program – at least from a distance. Hilbert's Second Epsilon

Axiom

$$(\forall x)(\varphi \leftrightarrow \psi) \to (\varepsilon x)\varphi = (\varepsilon x)\psi$$

via an algebraic mapping, generates a cylindric algebra in the universe of the epsilon model. Here, the notion of model corresponds to Monk's choice structures [Monk, 1976]. The algebraic mapping connects the cylindric set algebra of the model with the algebra generated in the base set of the model. With this completely new approach, in the case of Boolean or monadic algebras, I obtained an algebraic connection between the model and its cylindric set algebra, of course only in the case of extensional models of epsilon calculi. Although, this is not cononical connection like in the case of the double duals, an isomorphism still exists in finite cases (Theorem 7, Proposition 8) [Molnár, 2011].

The partial first-order result in Theorem 6 revealing the explicit quantification reading, which is a first-order-based solution, contains rather cumbersome conditions in the statement of the theorem. The reason is to be found in the fact that in any solution based on the syntactic structure of formulas, substitution slips out of structural induction. Hence, instead of the meta-concept of substitution, it is worth finding a syntactic solution. My suggestion for it is the application in lambda calculus. With the help of this, I found a formal representation in which I was able to state the theorem much more generally and with fewer technical conditions (Theorem 8) [Molnár, 2017].

2. Conservativity Connections in Intuitionistic Epsilon Logic

In this section, I intend to approach the proof theoretic meaning of the epsilon symbol using the system of natural deduction in a fragment of the intuitionistic predicate calculus. Here I only consider implication and extending it by the rules of existential quantification and some corresponding epsilon rules. As it is well-known, the intuitionistic logic is a conservative expansion of the implicational one, hence it does not seem to be an oversimplification if we consider only implication. I will take two connections under investigation. The first will be when I consider the logic of implication plus existential quantifier expanded by epsilon rules. The second will be when the implicational plus epsilon logic is expanded by the existential quantifier and its inference rules. I will prove that in the former case the expansion will not be conservative. This is supported by Bell's work, which proved that the Heyting

Arithmetic becomes "classical" in the presence of epsilon rules [Bell, 1993a]. Not surprisingly, the not derivable sentence is the (so called) *existential presupposition* of the classical epsilon terms:

$$(\exists x)((\exists x)\varphi(x) \to \varphi(x))$$

The *language of implicational-existential logic with epsilon* consist of a first-order style symbolism, with set of formulas $\mathrm{Fm}(L_{\exists\varepsilon})$ using $\to$, $\exists$ and ε, as well as the set of terms $\mathrm{Tm}(L_{\exists\varepsilon}))$ using functional signs and ε. Then the set proof terms of the calculus consist of

$$\mathrm{Exp} ::= \mathrm{V} \mid (\mathrm{Exp}\mathrm{Exp}) \mid (\lambda \mathrm{V}.\mathrm{Exp}) \mid (\mathsf{pair}_{(\bullet\mathrm{Var})\mathrm{Fm}(L_{\exists\varepsilon})}(\mathrm{Exp}, \mathrm{Tm}(L_{\exists\varepsilon}))) \mid$$
$$\mid (\mathsf{ind}_{(\bullet\mathrm{Var})\mathrm{Fm}(L_{\exists\varepsilon}),\mathrm{Fm}(L_{\exists\varepsilon})}(\mathrm{Exp}, \mathrm{V}.\mathrm{Var}.\mathrm{Exp}))$$

where the index $\bullet$ has value $\exists$ or ε.
Furthermore, the resticted languages of pure existential and epsilon *proof expressions* are also defined:

$$\mathrm{Exp}_\exists = \mathrm{Exp}\restriction_{\exists,\to}$$
$$\mathrm{Exp}_\varepsilon = \mathrm{Exp}\restriction_{\varepsilon,\to}.$$

A *context* is a finite function

$$\Gamma = \{(u_1,\varphi_1),\ \dots\ ,\ (u_n,\varphi_n),\ (t_1,\iota),\ \dots\ ,\ (t_m,\iota)\}$$

where $\{u_i\}_{i=1\dots n} \subseteq \mathrm{V}$, $\{\varphi_i\}_{i=1\dots n} \subseteq \mathrm{Fm}(L_{\varepsilon,\exists})$, and $\{t_i\}_{i=1\dots m} \subseteq \mathrm{Var} \cup \mathrm{Const}$. The set of all contexts is denoted by $\mathrm{Cntx}(L_{\varepsilon,\exists})$.

The *derivability* relation $\vdash$ on $\mathrm{Cntx}(L_{\varepsilon,\exists}) \times \mathrm{Exp} \times \mathrm{Fm}(L_{\varepsilon,\exists})$ is defined by the following recursive manner.
Variable typing rule. If $u \in \mathrm{Var}$, $\varphi \in \mathrm{Fm}(L_{\varepsilon,\exists})$ and $\Gamma \in \mathrm{Cntx}(L_{\varepsilon,\exists})$, then

$$\frac{}{\Gamma \cup \{(u:\varphi)\} \vdash u:\varphi}\,\text{var type}$$

Variable kinding rule. If $x \in \mathrm{V}$, $\Gamma \in \mathrm{Cntx}(L_{\varepsilon,\exists})$, then

$$\frac{}{\Gamma \cup \{(x:\iota)\} \vdash x:\iota}\,\text{var kind}$$

Constructive term kinding rule. If $t_1, \ldots, t_n \in \mathrm{Tm}(L_{\varepsilon,\exists})$, $\Gamma \in \mathrm{Cntx}(L_{\varepsilon,\exists})$, and f is a function symbol with arity n, then

$$\frac{\Gamma \vdash t_1 : \iota, \ldots, \Gamma \vdash t_n : \iota}{\Gamma \vdash f t_1 \ldots t_n : \iota}\text{term kind}$$

Constructive epsilon-term kinding rule. If $\varphi \in \mathrm{Fm}(L_{\varepsilon,\exists})$, $x \in \mathrm{V}$ and $\Gamma \in \mathrm{Cntx}(L_{\varepsilon,\exists})$, then

$$\frac{}{\Gamma \vdash (\varepsilon x)\varphi : \iota}\text{epsilon kind}$$

Implication introduction and elimination rules. If $u \in \mathrm{Var}$, $\varphi, \psi \in \mathrm{Fm}(L_{\varepsilon,\exists})$, $\Gamma \in \mathrm{Cntx}(L_{\varepsilon,\exists})$ and $P, Q \in \mathrm{Exp}$, then

$$\frac{\Gamma \cup \{u : \varphi\} \vdash P : \psi}{\Gamma \vdash \lambda u.P : \varphi \to \psi} \to \text{intro}$$

$$\frac{\Gamma \vdash P : \varphi \to \psi, \quad \Gamma \vdash Q : \varphi}{\Gamma \vdash PQ : \psi} \to \text{elim}$$

Existential quantifier introduction and elimination rules. If $u \in \mathrm{Var}$, $\varphi, \psi \in \mathrm{Fm}(L_{\varepsilon,\exists})$, $\Gamma \in \mathrm{Cntx}(L_{\varepsilon,\exists})$ and $P, Q \in \mathrm{Exp}$, then

$$\frac{\Gamma \vdash P : \varphi[x/t] \qquad \Gamma \vdash t : \iota}{\Gamma \vdash \mathsf{pair}_{(\exists x)\varphi}(P, t) : (\exists x)\varphi}\exists\text{intro}$$

$$\frac{\Gamma \vdash P : (\exists x)\varphi \qquad \Gamma \cup \{u : \varphi, x : \iota\} \vdash Q : \psi}{\Gamma \vdash \mathsf{ind}_{(\exists x)\varphi,\psi}(P, u.x.Q) : \psi}\exists\text{elim}$$

in the former $x \notin \mathrm{FreeVar}(t)$, and the later $x \notin \mathrm{FreeVar}(\mathrm{ran}(\Gamma) \cup \psi)$.

Epsilon introduction and elimination rules. If $u \in \mathrm{Var}$, $\varphi, \psi \in \mathrm{Fm}(L_{\varepsilon,\exists})$, $\Gamma \in \mathrm{Cntx}(L_{\varepsilon,\exists})$ and $P, Q \in \mathrm{Exp}$, then

$$\frac{\Gamma \vdash P : \varphi[x/t] \qquad \Gamma \vdash t : \iota}{\Gamma \vdash \mathsf{pair}_{(\varepsilon x)\varphi}(P, t) : \varphi[x/(\varepsilon x)\varphi]}\varepsilon\text{intro}$$

$$\frac{\Gamma \vdash P : \varphi[x/(\varepsilon x)\varphi] \qquad \Gamma \cup \{u : \varphi, x : \iota\} \vdash Q : \psi}{\Gamma \vdash \mathsf{ind}_{(\varepsilon x)\varphi,\psi}(P, u.x.Q) : \psi}\varepsilon\text{elim}$$

in the former $x \notin \mathrm{FreeVar}(t)$, and the later $x \notin \mathrm{FreeVar}(\mathrm{ran}(\Gamma) \cup \psi)$.

Notation 1. The rules above define the proof theory of the *intuitionistic implicational existential epsilon-logic* over the language $\mathrm{Fm}(L_{\varepsilon,\exists})$ and proof expressions Exp, and it is denoted by

$$\mathrm{IL}^{\rightarrow}_{\exists\varepsilon}$$

Notation 2. The set of rules

$$\{\text{var type}, \text{var kind}, \text{term}, \rightarrow \text{intro}, \rightarrow \text{elim}, \exists\text{intro}, \exists\text{elim}\}$$

over the language $\mathrm{Fm}(L_{\exists})$ and proof expressions $\mathrm{Exp}_{\exists}$ consists the proof theory of intuitionistic implicational existential logic and it is denoted by

$$\mathrm{IL}^{\rightarrow}_{\exists}$$

Notation 3. The set of rules

$$\{\text{var type}, \text{var kind}, \text{term}, \text{epsilon kind}, \rightarrow \text{intro}, \rightarrow \text{elim}, \varepsilon\text{intro}, \varepsilon\text{elim}\}$$

over the language $\mathrm{Fm}(L_{\varepsilon})$ and proof expressions $\mathrm{Exp}_{\varepsilon}$ consists the proof theory of intuitionistic implicational epsilon logic and it is denoted by

$$\mathrm{IL}^{\rightarrow}_{\varepsilon}$$

The statement of the first epsilon axiom is valid in the system. Indeed the logic above is an intensional epsilon logic.

Theorem 1. The derivability system $\mathrm{IL}^{\rightarrow}_{\exists\varepsilon}$ is an intensional epsilon logic: if $\varphi \in \mathrm{Fm}(L_{\varepsilon,\exists})$, then

$$\vdash_{\mathrm{IL}^{\rightarrow}_{\exists\varepsilon}} ((\exists x)\varphi) \rightarrow \varphi[x/(\varepsilon x)\varphi].$$

A property of the elimination and introduction rules of intuitionistic logic is suitable for cutting superfluous inferences from derivations. Concerning the proof systems above the three cut eliminating situations are the following. If a $\rightarrow$ introduction rule is followed by a modus ponens, then the reduction itself is the so called beta reduction of typed lambda calculus:

$$\frac{\dfrac{\Gamma \cup \{u : \varphi\} \vdash P : \psi}{\Gamma \vdash \lambda u.P : \boxed{\varphi \rightarrow \psi}} \qquad \Gamma \vdash Q : \varphi}{\Gamma \vdash (\lambda u.P)Q : \psi} \rightarrow_{\beta} \Gamma \vdash P[u/Q] : \psi$$

or just looking at the proof expressions:

$$(\lambda u.P)Q \to_\beta P[u/Q].$$

The reduction rule for the existential quantifier is:

$$\dfrac{\dfrac{\Gamma \vdash P : \varphi[x/t] \qquad \Gamma \vdash t : \iota}{\Gamma \vdash \mathsf{pair}_{(\exists x)\varphi}(P,t) : \boxed{(\exists x)\varphi}} \qquad \Gamma \cup \{x : \iota, u : \varphi\} \vdash Q : \psi}{\Gamma \vdash \mathsf{ind}_{(\exists x)\varphi,\psi}(\mathsf{pair}_{(\exists x)\varphi}(P,t), u.x.Q) : \psi} \to_\exists \Gamma \vdash Q[u/P][x/t] : \psi$$

or encapsulated into the proof expression:

$$\mathsf{ind}_{(\exists x)\varphi,\psi}(\mathsf{pair}_{(\exists x)\varphi}(P,t), u.x.Q) \to_\exists Q[u/P][x/t].$$

Proof expressions of the form

$$(\lambda u.P)Q$$

and

$$\mathsf{ind}_{(\exists x)\varphi,\psi}(\mathsf{pair}_{(\exists x)\varphi}(P,t), u.x.Q)$$

are called *redexes*. A proof expression N is called a *normal,* if it does not contain any subexpression which is a redex.

Theorem 2. The system $\mathrm{IL}_{\exists}^{\to}$ is normalizable: for every $M \in \mathrm{Exp}_\exists$, $\Gamma \in \mathrm{Cntx}(L_\exists)$, $\varphi \in \mathrm{Fm}(L_\exists)$ such that $\Gamma \vdash M : \varphi$, there is a normal expression $N \in \mathrm{Exp}_\exists$ such that $\Gamma \vdash M : \varphi$.

It is easy to prove that the classical existential condition for the epsilon terms does not hold.

Proposition 1.

$$\nvdash_{\mathrm{IL}_{\exists}^{\to}} (\exists x)(((\exists x)\varphi) \to \varphi)$$

Theorem 3. $\mathrm{IL}_{\exists\varepsilon}^{\to}$ is conservative over $\mathrm{IL}_{\varepsilon}^{\to}$:

$$\text{if } \{\varphi\} \cup \Gamma \subseteq \mathrm{Fm}(L_\varepsilon) \text{ and } \Gamma \vdash_{\mathrm{IL}_{\exists\varepsilon}^{\to}} \varphi, \text{ then } \Gamma \vdash_{\mathrm{IL}_{\varepsilon}^{\to}} \varphi.$$

The counterpart of the theorem does not hold.

Theorem 4. $\mathrm{IL}_{\exists\varepsilon}^{\to}$ is not conservative over $\mathrm{IL}_{\exists}^{\to}$.

In this construction, the rules of system $\mathrm{IL}_{\exists}^{\to}$ have been chosen so weak that the cardinal formula $(\exists x)(((\exists x)\varphi) \to \varphi)$ cannot be deduced, and the rules of system $\mathrm{IL}_{\exists\varepsilon}^{\to}$ have been chosen so strong that the formula in question can be deduced.

3. The Intensional Semantics

I proved that, if a monadic predicate is syntactically independent from an epsilon-term and if the sentence obtained by substituting the variable of the predicate with the epsilon-term is epsilon-invariant, then the sentence has an explicit first-order reformulation (Theorem 6).

Definition 1 (Epsilon-invariance over a class). The formula $\varphi \in \mathrm{Form}(\mathcal{L}_\varepsilon)$ is *epsilon-invariant over the class* K *of first-order models*, if for every model $\mathfrak{M} \in \mathsf{K}$, for every $a \in {}^\omega M$ and for all choice functions f and g such that $(\mathfrak{M}, f), (\mathfrak{M}, g) \in \mathsf{Ext}(\mathcal{L}_\varepsilon)$

$$(\mathfrak{M}, f) \models \varphi[a] \quad \text{iff} \quad (\mathfrak{M}, g) \models \varphi[a]$$

holds.

Blass, Gurevich and Caicedo proved the following theorem concerning epsilon-invariance valid for extensional models (Ext), i.e. models, where the semantic value of epsilon determined by a usual choice function.

Theorem 5 (Blass–Gurevich–Caicedo). If a formula $\varphi \in \mathrm{Form}(\mathcal{L}_\varepsilon)$ is epsilon-invariant, then there is a first-order (epsilon-free) formula $\psi \in \mathrm{Form}(\mathcal{L}_\exists)$ such that

$$\varphi^{(\mathfrak{M},f)} = \psi^{\mathfrak{M}}$$

holds for all $(\mathfrak{M}, f) \in \mathsf{Ext}(\mathcal{L}_\varepsilon)$. (Cf. [Blass–Gurevich, 2000, Prop. 3.2])

However, it is worth introduce the notion of intensional models (Int), where the choice functions differ form term-to-term. For this, one has to define the notion of epsion matrix.

Definition 2 (Epsilon-matrix). Let us suppose that the epsilon-term t has an occurrence in the formula φ. If this occurrence of t is also an occurrence of t in another epsilon-term s occurring in φ, then it is said to be *interior* in φ, otherwise it is an *exterior* occurrence of t in φ. The epsilon-term

$$(\varepsilon x)\psi(x, y_1, \ldots, y_n)$$

is *a matrix of the epsilon-term* $(\varepsilon x)\varphi$ if

(1) $y_1, ..., y_n \notin \mathrm{Var}((\varepsilon x)\varphi)$,
(2) $\psi(x, y_1, \ldots, y_n)$ is an $(n + 1)$-variable formula, and

(3) the distinct epsilon-terms $t_1, \dots, t_k$ have exterior occurrence in φ such that

$$(\varepsilon x)\varphi = (\varepsilon x)\psi(x, y_1, \dots, y_n)[t_1/y_1, \dots, t_n/y_n].$$

With this, the following theorem holds.

Proposition 2 (Molnár 2013)**.** Let K be a set of first-order models and let $\varphi \in \mathrm{Sent}(\mathcal{L}_\varepsilon)$. If $\mathsf{K} \underset{\mathrm{Int}}{\models} \varphi$, then φ is epsilon-invariant over the class K.

Now, I give the meaning of the sentences containing epsilon terms.

Definition 3. Let

$$\mathsf{InvSub}(\varphi, \vartheta)$$

denotes the formula

$$(\,(\exists x)\varphi \land (\forall x)(\varphi \to \vartheta)\,) \lor (\,\neg(\exists x)\varphi \land (\forall x)\vartheta\,).$$

Our aim is to show, step by step, a meta-equivalence like

$$\vdash \vartheta[(\varepsilon x)\varphi/x] \quad \text{iff} \quad \vdash \mathsf{InvSub}(\varphi, \vartheta)$$

without semantic conditions. First, note that, in the epsilon-calculus, $\mathsf{InvSub}(\varphi, \vartheta)$ implies $\vartheta[(\varepsilon x)\varphi/x]$ without any assumptions. ECε denotes the classical epsilon calculus, i.e. the classical predicate calculus with the first epsilon axiom.

Proposition 3 (Molnár 2013)**.** If φ and ϑ are monadic formulas of the variable x, then

$$\underset{\mathrm{EC}\varepsilon}{\vdash} \vartheta[(\varepsilon x)\varphi/x] \;\leftarrow\; \mathsf{InvSub}(\varphi, \vartheta)$$

Some necessary definitions.

Definition 4. The wf expression α *omits the set* $\mathrm{mat}((\varepsilon x)\varphi)$ if α has a wf expression construction $(\alpha_1, \alpha_2, \dots, \alpha_n)$ such that

$$\{\alpha_i \mid 1 \le i \le n\} \cap \mathrm{mat}((\varepsilon x)\varphi) = \emptyset.$$

[Molnár, 2013]

Definition 5. Let K be a class of first-order models of type t. The formula ϑ is *epsilon-invariant in* $(\varepsilon x)\varphi$ *over* K, if for all models $\mathfrak{M} \in \mathsf{K}$ and for every choice function f, g such that $(\mathfrak{M}, f), (\mathfrak{M}, g) \in \mathsf{Int}(\mathcal{L}_\varepsilon)$, $f = g$ on the set $(\mathrm{Mat} \setminus \{\mathrm{mat}((\varepsilon x)\varphi)\}) \times \mathcal{P}(M) \times {}^{<\omega}M$ and for every $a \in {}^{\omega}M$

$$(\mathfrak{M}, f) \models \vartheta[a] \text{ iff } (\mathfrak{M}, g) \models \vartheta[a].$$

[Molnár, 2013]

Now we prove the formula $\vartheta[(\varepsilon x)\varphi/x] \leftrightarrow \mathsf{InvSub}(\varphi,\vartheta)$ in a given model. The phrase "ψ is epsilon-invariant in $(\varepsilon x)\varphi$ over the model $\mathfrak{M}$" means ψ is epsilon-invariant in $(\varepsilon x)\varphi$ over the class $\{(\mathfrak{M}, f) \in \mathsf{Int}(\mathcal{L}_\varepsilon) \mid f \in \mathrm{pr}_2\, \mathsf{Int}(\mathcal{L}_\varepsilon)\}$ with the *fixed* model $\mathfrak{M}$.

Proposition 4 (Molnár 2013)**.** Let φ and ϑ be monadic predicates of the variable x. If the formulas ϑ and φ omit the set $\mathrm{mat}((\varepsilon x)\varphi)$, and $\vartheta[(\varepsilon x)\varphi/x]$ is epsilon-invariant in $(\varepsilon x)\varphi$ over the model $\mathfrak{M}$, then for every f such that $(\mathfrak{M}, f) \in \mathsf{Int}(\mathcal{L}_\varepsilon)$

$$(\mathfrak{M}, f) \models \vartheta[(\varepsilon x)\varphi/x] \rightarrow \mathsf{InvSub}(\varphi,\vartheta)$$

holds.

Theorem 6 (Molnár 2013)**.** If the monadic formulas φ and ϑ of the variable x omit the set $\mathrm{mat}((\varepsilon x)\varphi)$ and Γ consists of epsilon-invariant sentences, then

$$\Gamma \underset{\mathrm{EC}\varepsilon}{\vdash} \vartheta[(\varepsilon x)\varphi/x] \quad \text{iff} \quad \Gamma \underset{\mathrm{EC}\varepsilon}{\vdash} \mathsf{InvSub}(\varphi,\vartheta).$$

4. The Extensional Semantics and the Algebraic Approach

One of the benefit properties implied by the Second Epsilon Axiom (i.e. in the case of *extensional calculi*) of Hilbert's epsilon calculus is that the calculus becomes complete with respect to the choice structures as semantics. Another implication of the axiom, discussed in the section, is that an algebra is generated over the universe of the canonical model of a complete and consistent theory, which is isomorphic to a quotient algebra of the Lindenbaum–Tarski algebra of the theory. Especially, in the case of Boolean or monadic algebras, the canonical model of the theory of a finite model is isomorphic to the algebra generated by the Second Epsilon Axiom.

Let $(\mathfrak{M}, f)$ be an extensional model (with the choice function f over the non-empty subsets of M), then $\mathbf{LT}\,\mathfrak{M}f$ denotes the Lindenbaum–Tarski algebra of the formulae with the universe

$$\mathrm{LT}\,\mathfrak{M}f = \mathrm{Form}(\mathcal{L}_\varepsilon)\big/_{\mathfrak{M}\models .\leftrightarrow .}$$

Proposition 5 (Molnár, 2011)**.** Let $\mathfrak{M}$ be a first-order model, Γ is a complete and consistent set of sentences in an extensional epsilon calculus.

(a) For all $\varphi, \psi \in \mathrm{Fm}_{v_i}(\mathcal{L}_\varepsilon)$

$$\text{if } \varphi(v_i)^\Gamma = \psi(v_i)^\Gamma \text{ then } ((\varepsilon v_i)\varphi)/_{=_\Gamma} \;=\; ((\varepsilon v_i)\psi)/_{=_\Gamma}$$

(b) There is a choice function f such that for all $\varphi \in \mathrm{Fm}(\mathcal{L}_\varepsilon)$, $t \in \mathrm{Tm}(\mathcal{L}_\varepsilon)$ and valuation $a = (a_1/_{=_\Gamma}, a_2/_{=_\Gamma}, \ldots)$ of $\mathfrak{M}$

$$(\mathfrak{M}, f) \models \varphi[a] \quad \text{iff} \quad \Gamma \vdash \varphi[a_1/v_1, \ldots, a_n/v_n]$$

and

$$\Gamma \vdash t^{\mathfrak{M}f}[a] = t[a_1/v_1, \ldots, a_n/v_n]$$

moreover, on the set

$$\{\{(t/_{=_\Gamma}) \in M \mid \Gamma \vdash \varphi[t/v_i]\} \mid \varphi \in \mathrm{Fm}_{v_i}(\mathcal{L}_\varepsilon)\}$$

f is unique [Molnár, 2011].

The next proposition describes the connection between the neat-1-reduct $\mathbf{Nr}_1\,\mathbf{LT}\,\mathfrak{M}f$ of the cylindric algebra $\mathbf{LT}\,\mathfrak{M}f$ and the universe of the canonical model.

If $(\mathfrak{M}, f) \in \mathrm{Mod}(\mathcal{L}_\varepsilon)$ then the function

$$\Phi : \mathrm{Nr}_1\,\mathbf{LT}\,\mathfrak{M}f \to \mathrm{Can}\,\mathfrak{M}f, \quad \Phi(\varphi/_{\mathfrak{M}\models .\leftrightarrow .}) = ((\varepsilon v_0)\varphi)/_{=_{\mathrm{Th}\,\mathfrak{M}f}}$$

is a *surjection.*

Definition 6. Let $\mathbf{Can}\,(\mathfrak{M}, f)$ denotes the monadic algebra generated by Φ.

Proposition 6 (Molnár, 2011)**.**

$$\mathbf{Nr}_1\,\mathbf{LT}\,\mathfrak{M}f/\mathrm{Ker}\,\Phi \cong \mathbf{Can}\,\mathfrak{M}f$$

Definition 7. We define the *canonical injection* η of the complete and consistent theory Γ. Let $(\mathfrak{N}, g)$ be a model of Γ and let us denote the canonical model of Γ by $\mathfrak{Can}(\Gamma)$ and its universe by $\mathrm{Can}(\Gamma)$ then the canonical injection is

$$\eta : \mathrm{Can}(\Gamma) \to N, \; \eta((\varepsilon v_i)\varphi)^{\mathfrak{Can}\,\Gamma} = ((\varepsilon v_i)\varphi)^{\mathfrak{N}g}$$

Proposition 7. If $(\mathfrak{N}, g)$ is a choice structure then the canonical injection η : $\mathrm{CanTh}(\mathfrak{N}, g) \to N$ is an elementary embedding from $\mathfrak{Can}\mathrm{Th}(\mathfrak{N}, g)$ to $(\mathfrak{N}, g)$.

We know that a Boolean algebra is isomorphic to its second dual, through the Stone-correspondence. The question is what can be said about the relationship with its first dual. More precisely, how does a Boolean algebra as a model relate to the Lindenbaum–Tarski algebra of the given model. When the model $(\mathfrak{M}, f)$ is a finite Boolean algebra, thenthere is a close algebraic relationship between the cylindric set algebra of $(\mathfrak{M}, f)$ and the canonical model $\mathfrak{Can}\,\mathfrak{M}f$. The theorem below describes the main connection between $\mathfrak{Can}\,\mathfrak{M}f$ and $\mathbf{Can}\,\mathfrak{M}f$, and then the proposition states the connection between a finite Boole-algebra its Lindenbaum–Tarski algebra.

Theorem 7 (Molnár, 2011)**.** Let $(\mathfrak{B}, g) \in \mathsf{BA}$ and $(\mathfrak{M}, f) \in \mathsf{CA}_1$ such that $\mathfrak{Can}\,\mathfrak{B}g$ and $\mathfrak{Can}\,\mathfrak{M}f$ are finite algebras. Then

(1) $\mathfrak{Can}\,\mathfrak{B}g \cong \mathbf{Can}\,\mathfrak{B}g \restriction \mathcal{L}_\varepsilon^{\mathrm{BA}}$
(2) $\mathfrak{Can}\,\mathfrak{M}f \cong \mathbf{Can}\,\mathfrak{M}f \quad$ iff $\quad \mathbf{Nr}_0\,\mathfrak{Can}\,\mathfrak{M}f \cong \mathbf{2}$.

Proposition 8 (Molnár, 2011)**.** Let $(\mathfrak{B}, g) \in \mathsf{BA}$ be finite, then

$$\mathbf{Cs}\,(\mathfrak{B}, g) \restriction \mathcal{L}_\varepsilon^{\mathrm{BA}} / \mathrm{Ker}\,\Phi \cong \mathfrak{Can}\,\mathfrak{B}g.$$

5. Epsilon Logic in Simply Typed Lambda Calculus

This section is about the lambda calculus representation of the extensional semantics of the epsilon calculus. For FOL+ε, the quantification reformulation problem has been positively answered in [Molnár, 2013], however with the application of a lot of technical conditions. When one changes FOL to lambda calculus the picture becomes much more clear. The point is that, in FOL the substitution $\psi[x/(\varepsilon x)\varphi]$ is only a meta-language operation, but in the lambda-calculus it is encoded into the object-language via the application MN, where M is an expression of the lambda-language and N is an epsilon-term of the form $(\varepsilon x)P$. Note that, sinse the epsilon invariance is a purely semantic concept, these theorems are valid only in set theoretic models, namely in the Montague semantics.
Suppose that expression T is in the long normal form with a sentential final type i.e.

$$\Gamma \vdash \lambda x_1 \ldots \lambda x_n.P : \alpha_1 \to \cdots \to \alpha_n \to o$$

where $P = \lambda x_1 \ldots \lambda x_n.P$. Furthermore, let $R_1, \ldots, R_n$ be typed by so that

$$\Gamma \vdash R_1 : \alpha_1 \to o, \ldots, \Gamma \vdash R_n : \alpha_n \to o$$

In the following, I will prove that, if all the non-epsilon-term constituents of

$$(\lambda x_1 \ldots \lambda x_n.T)((\varepsilon x_1)R_1) \ldots ((\varepsilon x_n)R_n)$$

are epsilon-invariant, then it has an explicite plain quantificational form.

Theorem 8. Let $P, R \in \mathrm{Exp}(\mathcal{L}_\lambda^{\forall\varepsilon})$, $\mathfrak{M}$ be a Montague model, Γ a context, $\Gamma \vdash \lambda x.P : \alpha \to o$, $\Gamma \vdash \lambda x.R : \alpha \to o$, furthermore, let $(\lambda x.P)(\varepsilon_\alpha x)R$, P and R be epsilon-invariant over the model $\mathfrak{M}$. Then for every assignment a of type Γ and choice function g over $\mathfrak{M}$:

$$[\![(\lambda x.P)(\varepsilon_\alpha x)R]\!]_a^{(\mathfrak{M},g)} = [\![((\forall x)(\neg R)\,\&\,(\forall x)P) \vee (((\exists x)R)\,\&\,(\forall x)(R \to P))]\!]_a^{(\mathfrak{M},g)}.$$

The situation is similar to Russell's Theory of Descriptions. This approach not provides a single formula for how sentences containing epsilon terms can be interpreted, but gives rise the opportunity to find out what such sentences mean under several conditions. Some logical expressions which will be equivalent to a such complex sentence:

$$\mathsf{cases}(\lambda x_1 \ldots \lambda x_n.P, R_1, \ldots, R_n) =$$

$$= \bigvee_{(e_1,\ldots,e_n)\in\{0,1\}^n} (\mathsf{Q}^{e_1} x_1)R_1\,\&\ldots\&\,(\mathsf{Q}^{e_n} x_n)R_n\,\&\,((\forall x_1)\ldots(\forall x_n)(R_1^{e_1}\,\&\ldots\&\,R_n^{e_n}) \to P)$$

where

$$\mathsf{Q}^0 = \neg\exists$$

$$\mathsf{Q}^1 = \exists$$

$$R_i^0 = (\forall z)(z = z)$$

$$R_i^1 = R_i$$

where z is a (fresh) variable (or R_i^0 could be any true sentence).

Proposition 9. Let $\mathfrak{M}$ be a Montague model, Γ a context, $\Gamma \vdash \lambda x_1 \ldots \lambda x_n.P : \alpha_1 \to \cdots \to \alpha_n \to o$, $\Gamma \vdash \lambda x_1.R_1 : \alpha_1 \to o, \ldots, \Gamma \vdash \lambda x_n.R_n : \alpha_n \to o$, and $\mathrm{FV}(R_i) \subseteq \{x_i\}$. Suppose that all of the non-epsilon term constituents of $(\lambda x_1 \ldots \lambda x_n.P)(\varepsilon x_1)R_1 \ldots (\varepsilon x_n)R_n$ is epsilon-invariant over the model $\mathfrak{M}$. Then for every assignment a of type Γ and choice function g over $\mathfrak{M}$:

$$[\![(\lambda x_1 \ldots \lambda x_n.P)(\varepsilon x_1)R_1 \ldots (\varepsilon x_n)R_n]\!]_a^{(\mathfrak{M},g)} = [\![\mathsf{cases}(\lambda x_1 \ldots \lambda x_n.P, R_1, \ldots, R_n)]\!]_a^{(\mathfrak{M},g)}$$

References

[Abadi et al., 2004] Abadi M., Gonthier G., Werner B., *Choice in Dynamic Linking.* In: Walukiewicz I. (eds) Foundations of Software Science and Computation Structures. FoSSaCS 2004. Lecture Notes in Computer Science, vol 2987. Springer, Berlin, Heidelberg.

[Baaz–Zach, 2019] Baaz, M., Zach, R. *The First Epsilon Theorem in Pure Intuitionistic and Intermediate Logics* `https://arxiv.org/abs/1907.04477` unpublished (2019)

[Bell, 1993a] Bell, J. L. *Hilbert's epsilon operator and classical logic*, Journal of Philosophical Logic 22 (1):1–18 (1993)

[Blass–Gurevich, 2000] Blass, A. & Gurevich, Y., *The Logic of Choice*, The Journal of Symbolic Logic, Vol. 65, No. 3 (Sep., 2000), pp. 1264-1310

[Caicedo, 1995] Caicedo, Xavier, *Hilbert's* ε *symbol in the presence of generalized quantifiers* In Quantifiers: Logics, Models, and Computation II Synthese Library, Vol. 249 (1995) p. 63-78.

[Hilbert–Bernays, 1939] Hilbert, D. & Bernays, P., *Grundlagen der Mathematik,* Vol. 2 (1939) Berlin, Springer.

[Kneebone, 1963] Kneebone, G. T., *Mathematical Logic and the Foundation of Mathematics*, Van Nostrand, London, (1963).

[Mints, 1977] Mints, G., *Heyting predicate calculus with epsilon symbol,* Journal of Soviet Mathematics, (1977) vol. 8, no. 3, pp. 317–323.

[Molnár, 2011] Molnár, Z., *Induced Cylindric Algebras of Choice Structures,* Bulletin of the Section of Logic, 2011, 40, 3-4/2011, pp. 119-28.

[Molnár, 2013] Molnár, Z., *Epsilon-Invariant Substitutions and Indefinite Descriptions,* Logic Journal of the IGPL, (2013) 21 (5), pp. 812-829.

[Molnár, 2017] Molnár, Z., *Indefinite descriptions in typed lambda calculus,* In: K + K = 120 Papers dedicated to László Kálmán and András Kornai on the occasion of their 60th birthdays. Editors: Beáta Gyuris, Katalin Mády, and Gábor Recski. Research Institute for Linguistics, Hungarian Academy of Sciences. ISBN 978-963-9074-73-6, 2017.

[Monk, 1976] Monk, J. D., *Mathematical Logic,* Springer-Verlag, New York–Heidelberg–Berlin (1976)

[Russell, 1905] Russell, B., *On Denoting,* in: Mind, New Series, Vol. 14, No. 56 (Oct. 1905), pp. 479-493.

[Shirai, 1971] Shirai K., *Intuitionistic predicate calculus with epsilon-symbol,* Annals of the Japan Association for Philosophy of Science, (1971) vol. 4, no. 1, pp. 49–67.

[Sorensen–Urzyczyn, 1998] Sorensen, M. H. B., Urzyczyn, P., *Curry–Howard Isomorphism* (1998) unpublished `http://disi.unitn.it/~bernardi/RSISE11/Papers/curry-howard.pdf`

[Troelstra–Schwichtenberg, 2000] Troelstra, A. S., Schwichtenberg, H., *Basic Proof Theory,* second edition. Cambridge: Cambridge University Press. (2000)

[Whitehead-Russell, 1956] Whitehead, A. N., Russell, B., *Principia Mathematica,* Vol I. Cambridge: Cambridge University Press, 1956.

[Zach, 2017] Zach, R. *Semantics and Proof Theory of the Epsilon Calculus.* In: Ghosh S., Prasad S. (eds) Logic and Its Applications. ICLA 2017. Lecture Notes in Computer Science, vol 10119. Springer, Berlin, Heidelberg.