QUANTIFICATION AND EPSILON-INVARIANCE IN SOME EPSILON CALCULI

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1. Introduction

The epsilon symbol was first used by the members of the Göttingen Logic School, primarily in attempting to prove the consistency of Peano Arithmetic (or Hilbert's Second Problem) which was one of the main goals of the Hilbert Program. Early on, in the second volume of their *Grundlagen der Mathematik,* Hilbert and Bernays examined the relationship between the epsilon symbol and predicate logic. In this tractate, the Second Epsilon Theorem states, as we would put it today, that the epsilon calculus is a conservative expansion of the predicate logic [Hilb, p. 130][Zach]. It is well-known that the epsilon terms are special kind of Skolem functions, and using this representation conservativity results have been proven both in semantic and syntactic ways [Mints][Zach][Monk].

These are all results of classical logical, since they are not necessarily the case in intuitionistic logic. After Bell's result, i.e. the expansion (HA_ε) of Heyting Arithmetic (HA) with epsilon symbol is not conservative over HA, it becomes clear that the syntactic realization of intuitionistic epsilon logic drastically determines what properties the system will have [Bell]. And indeed, the results in [Baaz], on the one hand, and in [Mints], on the other hand, show a strange discrepancy (which is pointed out by Baaz and Zach, themselves). Before Mints, in 1971, Shirai also defined an epsilon expansion that was, in a strange way, conservative over the intuitionistic logic [Shir]. The problem now, then, is to point out the reasons that result in non-conservativity. With some modification of the typing rules also found in [Abad], which can be achieved with a special inference rule found in [SoreUn], I proved via Curry–Howard Isomorphism that, in a special typed calculus, which mimics a fragment of First-Order Logic (FOL), over the logic of intuitionistic implication and existential quantifier, the epsilon-expansion is not conservative (Section 2). However, I proved that in the reverse case the conservativity still holds (Section 2). Neither [Abad] nor the unpublished [Baaz] mentions or elaborates these results, where my methods are based on natural deduction system and types of two kinds.

The problem of the meaning of epsilon terms was best illustrated by Kneebone:

> An ε-term may be thought of as formalizing an indefinite description, somewhat as an ι-term formalizes a definite description[...] ([Knee, p. 101, ftn. 1])

Since Russell gave quantification reading for sentences containing ι-terms, we can hope that the situation is similar with ε-terms [Russ]. And indeed, Caiceado, and later Blass and Gurevich proved that, if a sentence is epsilon-invariant (it is independent of the choice of the reference of the ε-terms) in predicate logic, then the sentence has a plain FOL meaning [Caic][Blas]. But, since, the proof uses Craig's non-constructive interpolation theorem, this meaning is not an explicit one. Based on the proposal of Moser and Zach, I constructed a semantics that is suitable for making the equivalent explicit plain FOL reformulation under special conditions (Section 3) [Moln2013].

Tarski's algebraic program can be considered as another manifestation of Russell's quantification program – at least from a distance. Hilbert's Second Epsilon Axiom, via an algebraic mapping, generates a cylindric algebra in the universe of the epsilon model. The algebraic mapping connects the cylindric set algebra of the model with the algebra generated in the base set of the model. With this completely new approach, in the case of Boolean or monadic algebras, I obtained an algebraic connection between the model and its cylindric set algebra, of course only in the case of extensional models of epsilon calculi. Although, this is not a natural transformation like in the case of the double duals, isomorphism still exists in some cases (Section 4) [Moln2011].

The partial result in Section 3 revealing the explicit quantification reading, which is a FOL-based solution, contains rather cumbersome conditions in the statement of the theorem. The reason is to be found in the fact that in any solution based on the syntactic structure of formulas, substitution slips out of structural induction. Hence, instead of the meta-concept of substitution, it is worth finding a syntactic solution. My suggestion for it is the application in lambda calculus. With the help of this, I found a formal representation in which I was able to state the theorem much more generally and with fewer technical conditions (Section 5) [Moln2017].

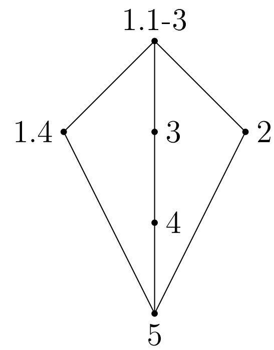
1.1. **The structure of the work.** Epsilon calculus can have at least three different formal representations in mathematical logic.

1.1-3—3—4—5: for those interested in model theory, I recommend this path, which begins with the classical intentional epsilon calculus, continues with cylindrical algebraic aspects, and ends with Montague semantics.

1.1-3—2—5: those interested in lambda calculus can choose the proof-theoretic approach via Curry–Howard Isomorphism to intensional epsilon logic and Montague

semantics.
1.1-4—5: Finally, for those interested in the formal linguistic approach, this path is recommended.

1.1-3. Introduction
1.4. The Context of the Problem
2. Conservativity Connections in Intuitionistic Epsilon Logic
3. The Intensional Semantics
4. The Extensional Semantics and the Algebraic Approach
5. Epsilon Logic in Simply Typed Lambda Calculus



1.2. **What is Hilbert's Epsilon Symbol?**

As it is well-known, if φ is a formula, then

$$(\varepsilon x)\varphi(x)$$

denotes an individual in the universe of discurse, "that has the property $\varphi(x)$, if there are individuals with property $\varphi(x)$ at all". Here, ε is Hilbert's Epsilon Symbol and $(\varepsilon x)\varphi(x)$ is an epsilon term or epsilon expression. The former wording is problematic not only because it is not specified which object this term refers to, but also because, although, it sounds understandable, we do not use such descriptions in the natural language often. To better understand, consider the following arithmetical problem. Let $\psi(x)$ denotes an arbitrary arithmetical one-variable formula. Then, what does $\psi(1)$ mean? – it is completely understandable: "1 satisfies property $\psi(x)$". But what does $\psi((\varepsilon x)\varphi(x))$ mean? What kind of thing is it talking about at all, and how could this statement be verified? Even Hilbert himself did not use such a natural linguistic wording, but tied the meaning of epsilon terms to an axiom. The *First Epsilon Axiom* (or Axiom Intensionality or Transfinite Axiom) states that

$$(\exists x)\varphi(x) \to \varphi((\varepsilon x)\varphi(x)).$$

The axiom describes, in an exact way, what the meaning of an epsilon term would be, but leaves open the question of what its natural linguistic meaning would be, and if it could be described with the usual logical terms, then what it would be.

Although the Göttingen School often used this term, we do not know the exact answer. The situation is similar to the status of finitistic arithmetic. Since Hilbert did not define what "finite" means, we can only make thesis-like statements about the nature of finitistic arithmetic. In the axiomatic framework, however, we can give serious answers to well-defined questions about epsilon terms. As Hilbert puts it:

> [...] there is no need to include the ε-symbol in the final deductive structure of logical-mathematical formalism. Rather, operating on the ε-symbol can be considered as a mere helper calculus, which is of considerable benefit to many meta-mathematical considerations [Hilb, p. 13].

Based on this conjecture, I examine under what circumstances, on the one hand, the use of the epsilon term is unnecessary, and, on the other hand, in what logically equivalent form, a sentence in which it appears can be expressed.

1.3. **The Main Results.**

Hilbert's conjecture above suggests us that epsilon logic is a conservative expansion of the predicate calculus, i.e., if a sentence cannot be derived in the epsilon-free logic, then it cannot be derived in epsilon logic either. Indeed, Monk proved that the epsilon logic considered as a Skolem expansion is conservative over the classical predicate logic [Monk, p. 213, 481]. However, it is known that the intuitionistic arithmetic (the Heyting Arithmetic, HA) becomes classical (Peano Arithmetic, PA) in the presence of epsilon symbol with the two epsilon axioms [Bell].

The proof-theoretic representation is usually used to establish the epsilon symbol's relation to the existential quantifier. The most obvious framework for this is the intuitionistic proof-theory, where the quantifiers are inherently independent of each other. In Section 2, I prove that the expansion of the intuitionistic logic of implication and existential quantifier with the epsilon symbol (regarding as a natural deduction system) is not conservative over the base logic. Though, if we do the opposite (expanding the intuitionistic logic of implication and epsilon with the existential quantifier), the expansion becomes conservative.

In Section 3, I turn to the model theory of epsilon logic. I prove that the *substitutive semantics* and Moser–Zach's choice structural semantics is equivalent, hence a completeness result arises regarding the intensional epsilon logic. With the tool of epsilon-invariance, I answer the question of logical meaning of epsilon substitutions. According to Proposition 6, if the constituents of

$$\psi((\varepsilon x)\varphi(x))$$

are epsilon-invariants and independent in a technical sense, then it has an equivalent plain quantificational reformulation. The model theoretic approach is also able to distinguish differences that occur even in classical logic. Systems in which the First Epsilon Axiom is true are called *intensional epsilon logics.* The completeness of the classical intensional semantics is not a trivial problem, in this approach the epsilon expressions are invariant under the term substitution (substitutive semantics).

By requiring another axiom, we get a much simpler semantics. Systems that satisfies the *Second Epsilon Axiom* (or the axiom of Extensionality)

$$(\forall x)(\varphi \leftrightarrow \psi) \to (\varepsilon x)\varphi = (\varepsilon x)\psi$$

are called *extensional epsilon logics.* In these, even cylindric algebraic considerations make sense. In Section 4, I examine the cylindric set algebras of the models of the extensional epsilon logic. It will turn out that the Second Epsilon Axiom gives rise of an algebraic connection between the cylindric set algebra of a Boolean structure and the Boolean structure itself.

In Section 5, I return to the problem of epsilon-invariance. I represent the problem in the simply typed lambda calculus, and in Montague semantics. It will turn out that, in the typed lambda calculus, the problem of reformulating the sentence into a quantified form is way easier than the FOL case of the same.

The paranthetical Subsection 1.4 deals with the historical and language philosophical concerns of the problem of epsilon symbol.

If we think of the question on the meaning of epsilon terms as a conjecture and the result as a response to it, then the following story emerges from what we have found. After Hilbert's statement, we can formulate the main naïve semantic conjecture:

> **Naïve Semantic Conjecture** – If a sentence containing epsilon terms is independent of the choice of the reference of the epsilon terms, then the sentence is equivalent to a plain quantified one.

Indeed, in the epsilon calculus the epsilon terms are merely a tool for quantification in a lot of sentences, such as

$$\varphi((\varepsilon x)\varphi) \leftrightarrow (\exists x)\varphi, \quad \varphi((\varepsilon x)\neg\varphi) \leftrightarrow (\forall x)\varphi,$$

$$(\varepsilon x)(x \neq x) = (\varepsilon x)(x \neq x) \leftrightarrow (\forall x)(x = x)$$

Blass, Gurevich and Caicedo gave a positive answer for the FOL representation of the problem, but the answer was not constructive. It did not give the explicit form of the quantified sentence. In case of substituting a single epsilon term, I gave a constructive answer ([Moln2013]), but the statement of the main theorem (Proposition 6 in the present work) contains cumbersome technical conditions. However, the inconvenience of the solution disappears immediately after we move on to the lambda calculus representation. The semantics becomes relatively simpler, the technical conditions disappear and the case containing several epsilon expressions can be solved too. It is known that, in the typed lambda calculus, every expression has a unique long normal form. Therefore, here, under reasonable conditions, the question can be answered as broadly as possible for the expressions of the form:

$$P((\varepsilon x_1)R_1)\ldots((\varepsilon x_n)R_n)$$

where the concatenation is the application, and the epsilon operator is expressed by an application of a constant and a lambda abstraction, successively.

Contents

1.4. **The Context of the Problem and the Philosophical Implications.**

1.4.1. *The main problem and its motivation.* In the evolution of modern logic, any initiative that considered the formal language and its semantics as the basic starting point for mathematical activity overcome the problem of the *descriptive terms.* Gottlob Frege, Bertrand Russell and David Hilbert, all three made their own suggestions on how to understand, in mathematical theories, the sentences of the form

(1) "The F is G."

where the term "the F" is the definite description [Freg], [Whit], [Hilb]. So I think of sentences like "The smallest positive prime number is even." or "The greatest prime number is odd". Although it turned out that the interpretation of these sentences is essentially a task of quantification, as any language philosophical problem, this question has also not been answered without a doubt. The biggest problem is obviously that, in the absence of a single and existing F, we do not have any intuition about what this thing or these things may be like, whether it is G (they are Gs) or not. Note that, Russell in the *Principia*'s formal language used the term

$$(\iota x)\varphi$$

instead of "the F", however $(\iota x)\varphi$ is in fact a meta-notation, and I will come back later to answer the question why.

David Hilbert, while proposing to answer the question of descriptions, raised the problem of a term which – as a linguistic expression – is less natural than a definite description, but easier to handle for mathematical examinations. In the following, I would introduce Hilbert's epsion in a reader-friendly way, rather than his laconic introduction. In the sentence

(2) "An F, if there is any F, is G."

the condition "if there is any F" relieves us of having to think for anything definite when there are no Fs. 'An F, if there is any F', actually, is considered to be a single term, with a reference which is an F, if there is an F, and otherwise there is no constrained stipulation on what its reference would be. It follows from the First Epsilon Axiom that the above sentence may give the meaning of the epsilon symbol, but it has not been entirely clear for a long time. The traditional notation is ε_F or $(\varepsilon x)F$. Comparing the Russellian and the Hilbertian terms at an intuitive level, we find that, while the definite descriptions violate the law of bivalence, provided that the term 'the F' has a singular meaning, the use of the Hilbertian term, as a singular unit, is consistent in classical logic. Indeed, on the one hand, if there is no F, then the sentence '$\neg(\exists x)(x =$ the F)' is logically valid. However, it is a plausible assumption that the sentence 'the F = the F' is also a logically valid one, which raises at least the possibility of a semantic contradiction. On the other hand, if there is no F, then ε_F is also not an F, however '$(\exists x)(x = \varepsilon_F)$' is logically valid, as well as the sentence '$\varepsilon_F = \varepsilon_F$', and in this case, the semantic paradox seems to be resolved.

In *Grundlagen der Mathematik*, Hilbert, after having introduced the *epsilon operator* as a descriptor with the above meaning, declared a statement which is highly intuitive and at the same time a guideline for the future research:

> [...] there is no need to include the ε-symbol in the final deductive structure of logical-mathematical formalism. Rather, operating on the ε-symbol can be considered as a mere helper calculus, which is of considerable benefit to many meta-mathematical considerations. [Hilb, p. 13]

Based on Hilbert's statement, the following naïve conjectures can be set, as programmatic reformulations. On the one hand, as it is mentioned above a semantic conjecture arises:

> **Naïve Semantic Conjecture** – If a sentence containing epsilon terms is independent of the choice of the reference of the epsilon terms, then the sentence is equivalent to a plain quantified one.

On the other hand it can be restated as a syntactic (proof theoretical) conjecture:

> **Naïve Syntactic Conjecture** – When a sentence, containing epsilon terms, is logically equivalent to an epsilon free one, the role of the epsilon operator is to be a tool for quantification.

In this paper, I prove different forms of the these conjectures in different symbolic languages, focusing on whether there is a specific, explicit form of the above quantifications.

1.4.2. *The descriptor elimination program.* Modeling descriptions in formal languages and eliminating them by quantification can be considered as a research program started by Bertrand Russell in his (and Whitehead's) famous *Principia Mathematica* or for a wider audience in the paradigmatic article *On Denoting* [WhRu, p. 30-32], [Russ], [Whit]. In Russell's works, there are *definite and indefinite descriptions.* Following David Hilbert's proposal, I extended the kinds of descriptions by the epsilon symbol.

According to Michael Dummett's analysis, a plausible approach to give meaning to the natural language expressions is the *realist standpoint,* which can be summed up in Dummett's two premisses [Dumm, p. 325]. A realist should

(1) accept the principle of bivalence, and
(2) accept that the sentences have the same semantic structure, that they appear on their syntactic surface.

Russell's famous example shows that this two premisses lead to an apparent contradiction in the case of definite descriptions. Consider the sentence:

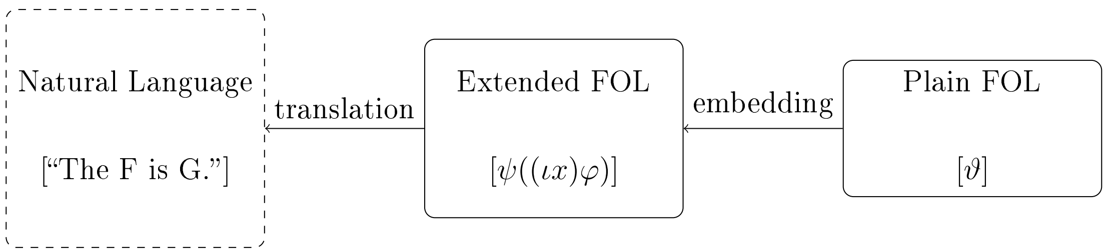
The present King of France is bold.

Consider the list of all bold people. Since, there is no present King of France, the present King of France does not appear in this list, hence, the sentence is false. But the negation "The present King of France is not bold." is also false, since the list of not bold people does not contain the present King of France either. The priciple of bivalence (the Law of Excluded Middle and the Law of Contradiction together) is in contradiction with the principle of *compositionality.* which is the assumption that the sentence, in the semantic value analysis, can be divided into two parts, the noun phrase "the present King of France" and the verb phrase "is bold". According to Russell, the definite descriptions, like "the present King of France", are not *proper names* (they are not taking part in the semantic analysis as singular terms) [Russ]. As Dummett said

> Under Russell's celebrated analysis, a definite description is not a genuine singular term at all, not even an integral semantic unite. When a sentence containing it is correctly analysed, it is seen as it express a proposition either true or false in every case, but no longer contains any term, or even any distinguishable constituent corresponding to the definite description. [Dumm, p. 325]

In short, the definite descriptions are not proper names, they only make sense in the sentence as a whole. They have no referential meaning, only *contextual.*

In order to describe the situation in a more accurate way let $\mathcal{L}$ be a first-order language and let $\mathcal{L}^{\iota}$ be the extension of $\mathcal{L}$ by the terms $(\iota x)\varphi$ for every formula φ. Furthermore, let F and G be monadic predicates in the natural languages so that their formal translation in $\mathcal{L}$ are the one-variable formulas $\varphi(x)$ and $\psi(x)$ respectively. Then the $\mathcal{L}^{\iota}$ formula $\psi((\iota x)\varphi)$ is the formal translation of "The F is G".



According to Russell's Theory of Descriptions (RTD for short) there is a FOL sentence ϑ which can be considered as the meaning (or the consistent reading) of $\psi((\iota x)\varphi)$ and this ϑ is the following

$$(\exists x)(\varphi(x) \wedge (\forall y)(\varphi(y) \to x = y) \wedge \psi(x)) \tag{3}$$

Despite the fact that the translation of the definite description "the F" is the closed term $(\iota x)\varphi$ and the translation of "is G" is $\psi(x)$, $(\iota x)\varphi$ is not contained in the Russellian meaning of $\psi((\iota x)\varphi)$ which is ϑ. The proposal solves the problem of the boldness of the present king of France by eliminating the description of the sentence and reformulates it in a FOL form.

The sentence (3) above is not the unique solution of the problem of sentences containing descriptions. The sentence "The present king of France is not bald." itself has two readings, as the formalization has itself two possible forms $(\neg\psi)((\iota x)\varphi)$ and $\neg(\psi((\iota x)\varphi))$, which lead to different ϑ-s. The common strategy is rather to reformulate the sentences into *quantified* sentences instead of using separate terms and their substitutions in variables. As Zvolensky puts it

> Initially, at issue was the meaning of a specific, rather narrow class of expressions, incomplete definite descriptions: are they devices

> of reference or of quantification? Subsequently, Russell's followers strived to provide a general framework for treating incompleteness phenomena exhibited by *all* devices of quantification, incomplete definite descriptions included. [Zvol, p. 1]

Therefore, the description elimination method is neither unique, nor capable of being completely defined. For example, it is not a small challenge to give the Russellian meaning of the sentence "The present king of France is not the present king of France".

1.4.3. *The quantificational reading and the Hilbertian conservative expansion.* There is a radical difference between the Russellian definite and indefinite description and the Hilbertian epsilon term. While Russellian descriptions can violate the principle of bivalence when occurring in a sentence, the Hilbertian terms results in no effects concerning the truth values. According to Hilbert's analysis, the definite description $(\iota x)(\varphi)$ has the following pre-suppsition

$$(\exists x)(\varphi(x) \wedge (\forall y)(\varphi(y) \rightarrow x = y))$$

and if it does not hold, then we are not in the position to use $(\iota x)\varphi$ in a consistent way. (Note that, RTD says that if the pre-supposition does not hold, then $\psi((\iota x)\varphi)$, for an atomic ψ, is false.) Hilbert and Bernays introduced their epsilon term by the following axiom

$$((\exists x)\varphi(x)) \rightarrow \varphi((\varepsilon x)\varphi)$$

or informally, if there is any φ then $(\varepsilon x)\varphi$ is a φ. As Slater pointed out, the pre-supposition for Hilbert's epsilon (letting us to use it in the language) is the following

$$(\exists x)((\exists y)(\varphi(y)) \rightarrow \varphi(x))$$

which is a classical first-order logical truth [Slat07]. Since, if there is a φ, then it is a φ, and if there is no φ then the first clause of the conditional is false and the sentence is true by the law of "from falsehood, anything follows". Kneebone called the epsilon term an indefinite description, but it is rather a *conditional indefinite description* and it formally models the following natural language description

an F, if there is any F at all.

[Knee] However, the proper semantic status of this term is ambiguous, since its motivation comes from a mathematical axiom and not from the usage of a natural language expression.

Comparing the two description operators (Russell's ι and Hilbert's ε) one can conclude on the one hand that there is no way to formalize a statement in the *same* language of the form

$$\psi((\iota x)\varphi) \leftrightarrow \vartheta$$

as a theorem expressing RTD provided that the logic is classical. Simply, the law of bivalence does not allow to make it happen. And indeed, the attempts to state the RTD in theorems of the form above are all modal or three valued logical examples [Ruzs][Fitt]. On the other hand, as I prove in Section 3 and Section 5, there is a possibility to state theorems like $\psi((\varepsilon x)\varphi) \leftrightarrow \vartheta$ in classical logic, under reasonable conditions and for epsilon invariant sentences. According to Theorem 6, ϑ is

$$(\,(\exists x)\varphi \wedge (\forall x)(\varphi \to \psi)\,) \vee (\,\neg(\exists x)\varphi \wedge (\forall x)\psi\,)$$

or in lambda language, where the sentence is formulated as $(\lambda x.P)((\varepsilon x)Q)$, according to Theorem 8, the corresponding formula is

$$(\,(\exists x)Q \wedge (\forall x)(Q \to P)\,) \vee (\,\neg(\exists x)Q \wedge (\forall x)P\,).$$

1.4.4. *A language philosophical approach.* On this basis, we are ready to examine the cases of attributive and non-attributive descriptions. However, these investigations belong rather to the philosophy of language and should take place within intensional or philosophical logic. The first-order language (FOL) expanded by the operator ε is possibly a good choice as an environment modelling a couple of formal linguistic and language philosophical phenomena concerning descriptions [Slat07], [Knee, p. 100]. As we have seen the term

$$(\varepsilon x)\varphi \tag{4}$$

where φ is a FOL formula and x is a variable, has the following intuitive meaning:

(5) „an F, if there is any F at all"

where predicate F is the intended meaning or the natural language translation of the formula φ. (Here, the notion of translation is due to Tarski. In the present case, the object-language is FOL and the meta-language is the natural language. (Cf. the notion of translation as applied in Convention T in [Tars], p. 188.) The intended meaning of the epsilon term shows that $(\varepsilon x)\varphi$ can be called a *conditional* *indefinite description,* since “an F” alone is an indefinite description, with the addition of the conditional clause “there is any F at all” it becomes a different linguistic entity with, perhaps, a different meaning. Obviously, I do not have to mention that the meaning of the phrase “an F” is itself a problematic one. Therefore, the problem of the semantic difference between “an F” and “an F, if there is any F at all” is also a tricky one. In the paper I am committed to the standpoint that these phrases have the same meaning.

In order to show an application of Hilbert’s symbol let me provide a formal reconstruction and analysis of the sentence

(6) The man drinking a martini is interesting-looking.

in FOL extended by ε (this extended language is denoted by FOL+ε). (The original sentence can be found in [Donn].) Since, FOL+ε does not contain definite descriptions, the phrase “the man drinking a martini” can be seen as a special case of the use of ε. A possible solution is to add a uniqueness clause to the formula ‘$\mathsf{man\ drinking\ a\ martini}(x)$’ (the formula in FOL+$\varepsilon$ expressing the natural language predicate “…is a man and drinks a martini”) [Slat09, p. 417].

$$(\varepsilon x)(\mathsf{man\ drinking\ a\ martini}(x)\ \&\ (\forall y)(\mathsf{man\ drinking\ a\ martini}(y)\ \equiv\ (x = y))$$

Let us denote the term above by

(7) $$(\varepsilon_{\mathrm{D}} x)\mathsf{man\ drinking\ a\ martini}(x)$$

Then the sentence (6) is formulated as follows:

(8) $$\mathsf{interesting\text{-}looking}((\varepsilon_{\mathrm{D}} x)(\mathsf{man\ drinking\ a\ martini}(x)))$$

Let me remark again, that the claim that the phrase “the man drinking a martini” can be expressed by $(\varepsilon_{\mathrm{D}} x)\mathsf{man\ drinking\ a\ martini}(x)$ is not an obvious one, however a possibly good enough working hypothesis. Without a man holding martini in his hand, the meaning of “the man drinking a martini” is as vague as the meaning of the phrase “the man drinking a martini, if anybody at all”.

Accepting the hypothesis above, by sentence (6) one can refer to the interesting-looking person in question, even if he holds a glass of water in his hand. In this case, the semantic value of the term (7) is a person – not drinking a martini – who seems to be interesting.

The problem of sentence (8) reminds one of Russell's Theory of Descriptions (RTD). In Russell's *On Denoting* or in [Whit] it is proposed that descriptions must not be treated as proper names, but as incomplete parts of quatified sentences.

> Thus we must either provide a denotation in cases in which it is at first sight absent, or we must abandon the view that the denotation is what is concerned in propositions which contain denoting phrases. [Russ, p. 484]
>
> According to the view I advocate, a denoting phrase is essentially *part* of a sentence, and does not, as like most single words, have any significance on its own account. [Russ, p. 488]

According to RTD, a description D, as a denoting phrase, is not interchangeable by an other individual name N which is identical to D, since D is meaningless in separation, and has only contextual meaning. (Russell in *On Denoting* gives a FOL reformulation for sentences of the form (6) [Russ].) But in the general case, when the natural language sentence contains more than one descriptions or a lot of logical operators the FOL reformulation can be carried out along different lines. One must mind the scope of logical operators and descriptions. Hence, in the general case RTD is appears to be a FOL reformulation program, in the spirit of the treatment of the simple case described in [Russ].

1.4.5. *Morning Star and King of France Tests.* With our results in Theorem 8, we stay within the realm of mathematical logic taken in the narrow sense.

The concluding facts can be stated in two claims:

(1) In the formal language $\mathcal{L}_{\lambda}^{\forall\varepsilon}$ (which is supposed to model the behaviour of descriptions) the (closed) term $(\varepsilon x)Q$ has *referential meaning* in the sense that a fixed model $(\mathfrak{M}, g)$ singles out an individual $[\![(\varepsilon x)Q]\!]^{(\mathfrak{M},g)} \in M$ for $(\varepsilon x)Q$ as semantic value.
(2) In some cases, when $(\varepsilon x)Q$ is part of a compound sentence $[(\lambda x)P]((\varepsilon x)Q)$, with all its components being epsilon-invariant, the $(\varepsilon x)Q$ has a *contextual meaning,* such that the sentence $[(\lambda x)P]((\varepsilon x)Q)$ has an equivalent epsilon-free reformulation using quantified expressions from the plain language $\mathcal{L}_{\lambda}^{\forall}$.

We do not intend to set up a weaker theory than Russell's Theory of Descriptions. A new theory must serve at least as many solutions as far as Russell's proposal was able to solve. An appropriate indicator is to look at the two problems that the Theory of Descriptions solved and examine what the new model porposes. The first one is the problem of Hesperus and Phosphorus (below it will be called Morning Star Test), the second one is the problem of the empty names (the King of France Test).

Morning Star Test. In 1905, Russell gave a FOL-based solution of *Frege's Puzzle* in terms of RTD, understandably, without mentioning the intensional tools of possible world semantics, which is a much later development. Here, I would like to show briefly that even the exposition of the puzzle is so widely criticized, that the RTD result of the test is rather irrelevant to us.

> "Gottlob thinks that the Morning Star is illuminated by the Sun."
> "The Evening Star is the Morning Star."
> —
> "Gottlob thinks that the Evening Star is illuminated by the Sun".
> (Cf. [Freg].)

First of all, I would like to point out that several scholars are committed to the assumption that the names such as "the Morning Star" or "the Evening Star" are understood tacitly as definite descriptions. For Russell, these names abbreviate descriptions, hence they are denoting phrases too [Dumm, p. 97]. The problem is that, according to Leiniz's Rule, since the Evening Star is the Morning Star, the two phrases are interchangeable. However, the above inference does not seem to be valid, since it is possible that Gottlob thinks that the Morning Star is illuminated by the Sun, but he does not necessarily know this fact about the Evening Star, even if in reality the two planets are the same, which is the case. Russell's solution was that the phrases "the Morning Star" and "the Evening Star" are not proper names, they only have contextual meanings, hence they are not interchangeable due to formal reasons [Russ], [Whit].

In the epsilon language $\mathcal{L}_{\lambda}^{\forall\varepsilon}$, the definite descriptions are proper names, they are manifested as epsilon terms on the object language level, hence the modelling in terms of the epsilon-language fails the Morning Star Test, and it does not explain the puzzle. Fortunately, hitherto, the Frege Puzzle and the semantic status of the expressions like "the Morning star" are not completely solved. If the phrase "the Morning star" is a rigid designator, as it is done in Kripke's proposal, then the Puzzle is solved. Here, temporarily, not having modal context, 'rigid' means that the model designates a single individual in one step, and does not select first a set, then a member of it, by a choice function. (Of course, it is a rough simplification. Picking an individual means direct reference, rigid means the term has the same semantic value along the possible worlds. What is more, the notion of 'rigid' above is understandable, but mathematically vague.) Then the puzzle only says that, if planet Venus is illuminated by the Sun, then planet Venus is illuminated by the Sun. According to Kripke's approach, the problematic case is the sentence "The Evening Star is the Morning Star". It is a necessary truth, but it may be problematic from an epistemological point of view [Krip, p. 102] (the whole story can be found in [Zvol]). For the epsilon model, the solution is the same. According to Monk, the closed epsilon terms are constants, therefore they are rigid designators in accordance with the Kripke doctrine. However, as Fitting pointed out, an epsilon term, being description-like, can neither be a constant, nor a variable. It is a complex flexible designator [Fitt]. Here, if "the Morning star" is a complex demonstrative (selected by a descriptive term in the actual world), then it is a rigid designator [Kapl]. Clearly, now, I do not have to deal with the modal context of epsilon terms, knowing that the highly applicable tool of demonstratives might make the modal approach much more complex, and might not add essentially more to the above consideration.

The King of France Test. Consider the following two sentences

> "The present King of France is bald."
> "The present King of France is not bald."

In order to determine the truth value of the first one, let us imagine the set of all bald people. Since the present King of France is not in this set, the first sentence is false. But, the same reasoning leads to the fact that the second sentence is false too. Which is a contradiction. Hence, the phrase "the present King of France" is not a proper name, it cannot have a meaning in isolation, rather it only has a contextual meaning and the sentences containing such phrases are quantified formulas. This is Russell's solution.

In the epsilon calculus the semantic values of the epsilon terms are defined in all cases. The two sentences above are unproblematic, they assign to the phrase "the present King of France" an existing individual as reference. And it is either bald or not bald. According to Theorem 8 of the present paper, sentences *may* possess contextual meaning too, where the truth value is also well-defined. Of course, the reference of “the present King of France” in the epsilon calculus is not the present King of France. Approaching the situation on a more formal level, let us consider the symbolic sentence

$$(\varepsilon x)(x \neq x) = (\varepsilon x)(x \neq x)$$

This is a sentence containing terms which are ill-defined as descriptions: $x \neq x$ is an empty predicate. However, the semantic value of $(\varepsilon x)(x \neq x)$, in a given model, is well-defined. Moreover, $(\varepsilon x)(x \neq x) = (\varepsilon x)(x \neq x)$ is an epsilon invariant sentence, since, it is true in any given epsilon semantics. And indeed, there are epsilon semantics (for example the Bourbaki group’s formal systems), where $(\varepsilon x)(x \neq x) = (\varepsilon x)(x \neq x)$ is syntactically identical to the sentence $(\forall x)(x = x)$. $(\varepsilon x)(x \neq x) = (\varepsilon x)(x \neq x)$ is an epsilon-invariant sentence, which has contextual meaning too: it is equivalent to the fact that every individual is identical to itself. The situation is very similar to the problem of the interesting-looking man holding a martini. In this case, the “the present King of France” is rather a person who is, in fact, bald, but not the present King of France, and $(\varepsilon x)(x \neq x)$ is an existing individual, which is identical to itself, but of course, it does not hold that it is not the same as itself.

2. Conservativity Connections in Intuitionistic Epsilon Logic

In this chapter, I intend to approach the proof theoretic meaning of the epsilon symbol using the system of natural deduction in a fragment of the intuitionistic predicate calculus. Here I only consider implication and extending it by the rules of existential quantification and some corresponding epsilon rules. As it is well-known, the intuitionistic logic is a conservative expansion of the implicational one, hence it does not seem to be an oversimplification if we consider only implication [Sore, p. 37]. I will take two connections under investigation. The first will be when I consider the logic of implication plus existential quantifier expanded by epsilon rules. The second will be when the implicational plus epsilon logic is expanded by the existential quantifier and its inference rules. I will prove that in the former case the expansion will not be conservative. This is supported by Bell's work, which proved that the Heyting Arithmetic becomes "classical" in the presence of epsilon rules [Bell]. Not surprisingly, the not derivable sentence is the (so called) *existential presupposition* of the classical epsilon terms:

$$(\exists x)((\exists x)\varphi(x) \to \varphi(x))$$

This sentence is true in classical logic, since there are two cases in that logic. If there is an element satisfying φ, then $(\exists x)\varphi(x)$ is true and for this existing x, it is true that $\varphi(x)$. Hence, $(\exists x)\varphi(x) \to \varphi(x)$ is also true. If there is no φ, then $(\exists x)\varphi(x)$ is false, hence for every x the formula $(\exists x)\varphi(x) \to \varphi(x)$ is true. However, in the intuitionistic logic $(\exists x)((\exists x)\varphi(x) \to \varphi(x))$ is not derivable, since (being a constructive logic) the Law of Excluded Middle does not generally hold. Indeed, according to the Brouwer–Heyting–Kolmogorov intuitionistic paradigm (the so called BHK Interpretation), an existential proposition is provable if and only if, there is a construction of an object (named by a term) and a proof which proves that the proposition holds for that object. Generally, there is no hope for a constructively given object, named by the term c, and a proof for the problem $(\exists x)\varphi(x) \to \varphi(c)$, since in the case, when $(\exists x)\varphi(x)$ is proved to be an impossible proposition, there is no such possible, constructed c. If a c could be constructed such that $(\exists x)\varphi(x) \to \varphi(c)$ could be proved, and if $(\exists x)\varphi(x)$ could lead to a contradiction, then – by "from contradiction everything follows", which is a valid inference rule in the intuitionistic logic – there could be a proof of $\varphi(c)$, which is a (constructive) proof of $(\exists x)\varphi(x)$, which is a contradiction.

Regarding the case, when the implicational epsilon logic is expanded to az existentialy quantified logic, according to one's natural conjecture the expansion will be conservative. This intuition is supported by the well-known fact that with the epsilon symbol in our hand, we are capable of expressing existential quantification.

It is worth noting that not only the shape of the derivation rules influences the result in a very sensitive way, but also the choice of the formal language. For example, in Bell's article, different formulas can be deduced if there are constants in the language or these constants are postulated to have different properties [Bell]. In the following structure, I will approach the proof-theoretic problems via Curry–Howard Isomorphism. I will construct a special *dependent typed lambda calculus,* which will be able to handle the proofs of the above derivation systems, such that the types will correspond to the formulas and terms of the FOL fragment mentioned above, and the proofs (or rather the codes of the proofs) will correspond to the expressions of the lambda expressions.

This approach is close to that described in [Troe, Ch. 2.2]. The problem arose in the second-order typed lambda calculus, and Bell's article [Bell93b] also deals with the conservatism of a kind of expansions. However, the latter article can only serve as motivation for the present work, since FOL is weaker than second-order type theory. It is also worth mentioning the work [Carl], however, being a type theoretical one, it also requires different inference rules for epsilon symbols than the present work, and in its subject matter is neither related to this chapter (conservativity) nor to the following (meaning of epsilon-invariant sentences).

2.1. **Formulas or the language of types.** First of all, I define the *language of implicational-existential logic with epsilon* as follows. Let us fix a *similarity type*

$$t = (\mathrm{Var}, (f_i)_{i\in I}, (r_j)_{j\in J}),$$

where $\mathrm{Var} = \{v_k \mid k \in \mathbf{N}\}$ is the countable set of variables, the f_i-s are the function signs with arity $o(f_i)$ and the r_j-s are the relation signs with arity $o(r_j)$, respectively. The set of all zero arity function symbols is denoted by Const.

Definition 1. Let $L_{\exists\varepsilon} = (\mathrm{Tm}(L_{\exists\varepsilon}), \mathrm{Fm}(L_{\exists\varepsilon}))$ be the recursively defined couple:

$$\mathrm{Tm}(L_{\exists\varepsilon}) ::= \mathrm{Var} \mid (f_i \underbrace{\mathrm{Tm}(L_{\exists\varepsilon}) \ldots \mathrm{Tm}(L_{\exists\varepsilon})}_{o(f_i)}) \mid ((\varepsilon\mathrm{Var})\mathrm{Fm}(L_{\exists\varepsilon}))$$

$$\mathrm{Fm}(L_{\exists\varepsilon}) ::= (r_j \underbrace{\mathrm{Tm}(L_{\exists\varepsilon}) \ldots \mathrm{Tm}(L_{\exists\varepsilon})}_{o(r_j)}) \mid (\mathrm{Fm}(L_{\exists\varepsilon}) \to \mathrm{Fm}(L_{\exists\varepsilon})) \mid ((\exists\mathrm{Var})\mathrm{Fm}(L_{\exists\varepsilon})).$$

Remark 1. This recursive definition operates simultaneously on the terms and the formulas, the two recursive sets refer to each other in the mutual way. For instance:

$$((\varepsilon x)r_1xy) \in \mathrm{Tm}(L_\varepsilon)$$

$$(r_1x((\varepsilon x)r_1xy)) \in \mathrm{Fm}(L_\varepsilon)$$

Both $\exists$ and ε in $(\exists x)\varphi$ and $(\varepsilon x)\varphi$ bind the occurrences of the variable x in the formula φ.

Then, two restriction of the language are defined, one without the existential quantifier, and one without the epsilon:

Definition 2. Let $L_\exists$ be the restriction $L_{\exists\varepsilon}\restriction_{\exists,\to}$ that is

$$\mathrm{Tm}(L_\exists) ::= \mathrm{Var} \mid (f_i \underbrace{\mathrm{Tm}(L_\exists) \ldots \mathrm{Tm}(L_\exists)}_{o(f_i)}))$$

$$\mathrm{Fm}(L_\exists) ::= (r_j \underbrace{\mathrm{Tm}(L_\exists) \ldots \mathrm{Tm}(L_\exists)}_{o(r_j)} \mid (\mathrm{Fm}(L_\exists) \to \mathrm{Fm}(L_\exists)) \mid ((\exists\mathrm{Var})\mathrm{Fm}(L_\exists))$$

and let L_ε be the restriction $L_{\exists\varepsilon}\restriction_{\varepsilon,\to}$ that is

$$\mathrm{Tm}(L_\varepsilon) ::= \mathrm{Var} \mid (f_i \underbrace{\mathrm{Tm}(L_\varepsilon) \ldots \mathrm{Tm}(L_\varepsilon)}_{o(f_i)}) \mid ((\varepsilon\mathrm{Var})\mathrm{Fm}(L_\varepsilon))$$

$$\mathrm{Fm}(L_\varepsilon) ::= (r_j \underbrace{\mathrm{Tm}(L_\varepsilon) \ldots \mathrm{Tm}(L_\varepsilon)}_{o(r_j)}) \mid (\mathrm{Fm}(L_\varepsilon) \to \mathrm{Fm}(L_\varepsilon)).$$

The notions of free and bound occurrences of variables, substitution ($\varphi[x/t]$), set of free variable ($\mathrm{FreeVar}(\varphi)$) are considered to be known.

2.2. **Language of proof expressions.** The next task is to define the language of the proof expressions. Similarly, there are two proof systems, one for the implicational-existential logic and one for the implicational-epsilon logic. We fix the infinite set V of proof variables (atomic proof terms). Then we will recursivelly define the language of the proof expressions.

Definition 3. Let Exp be the following recursively defined language:

$$\mathrm{Exp} ::= \mathrm{V} \mid (\mathrm{Exp}\mathrm{Exp}) \mid (\lambda \mathrm{V}.\mathrm{Exp}) \mid (\mathsf{pair}_{(\bullet \mathrm{Var})\mathrm{Fm}(L_{\exists\varepsilon})}(\mathrm{Exp}, \mathrm{Tm}(L_{\exists\varepsilon}))) \mid$$

$$\mid (\mathsf{ind}_{(\bullet \mathrm{Var})\mathrm{Fm}(L_{\exists\varepsilon}),\mathrm{Fm}(L_{\exists\varepsilon})}(\mathrm{Exp}, \mathrm{V}.\mathrm{Var}.\mathrm{Exp}))$$

where the index $\bullet$ has value $\exists$ or ε.

We say that in $\lambda u.P$ all occurrences of variable u in P are bound and in $\mathsf{ind}_{(\bullet x)\varphi,\psi}(P, u.x.Q)$ all occurrences of the variables u and x in Q are bound.

Furthermore, the resticted languages of proof expressions are also defined:

$$\mathrm{Exp}_\exists = \mathrm{Exp}\restriction_{\exists,\to}$$

$$\mathrm{Exp}_\varepsilon = \mathrm{Exp}\restriction_{\varepsilon,\to} .$$

Remark 2. The intuition behind these expressions is the following. If M and N are proofs, then

$$MN$$

will be the proof code of the result of a modus ponens obtained by applying M to N.

The expression

$$\lambda u.M$$

will be the code of the application of the deduction theorem, as an operation over proofs.

$\mathsf{pair}_{(\exists x)\varphi(x)}Mt$ will be the code of the constructive existential generalization. If t is a term and M is a code of proof proving that t is a φ, then the pairing operator

$$\mathsf{pair}_{(\exists x)\varphi(x)}(M, t)$$

will be the proof code of $(\exists x)\varphi(x)$. The name "pair" comes from the intuitions that in the BHK Interpretation of intuitionistic logic to establish a proof for an existential proposition it is needed an object and a proof proving that the object falls under the concept in question.

If M is the proof code of $(\exists x)\varphi(x)$ and N is a code of proof proving that φ entails ψ provided that x does not occur in ψ, then

$$\mathsf{ind}_{(\exists x)\varphi(x),\psi}(M, u.x.N)$$

is the code of the proof of ψ from $(\exists x)\varphi(x)$. Here u is proof variable possibly occurs in N, does not occur in $u.x.N$ as a free variable. Hence, ind is a variable binding operation just like λ binding u in $\lambda u.N$. The abbreviation "ind" is for induction, since this expression intends to code the application of the induction or recursion rule of the Σ-types which are generalizations of the type $(\exists x)\varphi(x)$ in general Type Theory.

2.3. **Context and derivability rules.** Let ι be a new sign not contained in the language of types, intended to mark intuitively by $t : \iota$ that term t denotes a constructively defined object.

Definition 4. A *context* is a finite function

$$\Gamma = \{(u_1, \varphi_1),\ \dots\ ,\ (u_n, \varphi_n),\ (t_1, \iota),\ \dots\ ,\ (t_m, \iota)\}$$

where $\{u_i\}_{i=1\dots n} \subseteq \mathrm{V}$, $\{\varphi_i\}_{i=1\dots n} \subseteq \mathrm{Fm}(L_{\varepsilon,\exists})$, and $\{t_i\}_{i=1\dots m} \subseteq \mathrm{Var} \cup \mathrm{Const}$. The set of all contexts is denoted by $\mathrm{Cntx}(L_{\varepsilon,\exists})$.

Contexts below are denoted by $\Gamma = \{u_1 \colon \varphi_1,\ \dots\ ,\ u_n \colon \varphi_n,\ t_1 \colon \iota,\ \dots\ ,\ t_m \colon \iota\}$. The element $(u, \varphi) \in \Gamma$ is called a *tag,* and we say that the tag $u : \varphi$ types the variable u by the formula φ. Here φ is also said to be a *premiss.*

Remark 3. Intuitively there are two *kinds* of types. The first one is the kind of *predicative types* whose members come form $\mathrm{Fm}(L)$. The other is the kind of *individual types*, which are the constructively defined elements of $\mathrm{Tm}(L)$ in a given context. We could mark the two *kinding judgements* separately, by $\Gamma \vdash \varphi \colon o$ (which intends to denote that φ is a predicative type) and $\Gamma \vdash t \colon \iota$ (which intends to denote that t is a constructively defined individual type), but since intuitively the former is the same as "φ is a wff", and the latter is not the same as "t is a wft", we do not so. It will be useful to introduce the only relevant *kinding judgement* $\Gamma \vdash t \colon \iota$ which means intuitively not just that t is a wft, but that the term t denotes a constructively defined object. Note that, "$\Gamma \vdash M \colon \varphi$" is also a judgement, but it is a *typing judgement.* The definition below is based on the inference rules of the existential quantification presented in the unpublished version of [SoreUn, p. 162].

Definition 5. The relation $\vdash$ on $\mathrm{Cntx}(L_{\varepsilon,\exists}) \times \mathrm{Exp} \times \mathrm{Fm}(L_{\varepsilon,\exists})$ is defined by the following recursive manner.

Variable typing rule. If $u \in \mathrm{Var}$, $\varphi \in \mathrm{Fm}(L_{\varepsilon,\exists})$ and $\Gamma \in \mathrm{Cntx}(L_{\varepsilon,\exists})$, then

$$\frac{}{\Gamma \cup \{(u : \varphi)\} \vdash u : \varphi}\,\text{var type}$$

Variable kinding rule. If $x \in \mathrm{V}$, $\Gamma \in \mathrm{Cntx}(L_{\varepsilon,\exists})$, then

$$\frac{}{\Gamma \cup \{(x : \iota)\} \vdash x : \iota}\text{var kind}$$

Constructive term kinding rule. If $t_1, \ldots, t_n \in \mathrm{Tm}(L_{\varepsilon,\exists})$, $\Gamma \in \mathrm{Cntx}(L_{\varepsilon,\exists})$, and f is a function symbol with arity n, then

$$\frac{\Gamma \vdash t_1 : \iota, \ldots, \Gamma \vdash t_n : \iota}{\Gamma \vdash f t_1 \ldots t_n : \iota}\text{term kind}$$

Constructive epsilon-term kinding rule. If $\varphi \in \mathrm{Fm}(L_{\varepsilon,\exists})$, $x \in \mathrm{V}$ and $\Gamma \in \mathrm{Cntx}(L_{\varepsilon,\exists})$, then

$$\frac{}{\Gamma \vdash (\varepsilon x)\varphi : \iota}\text{epsilon kind}$$

Implication introduction and elimination rules. If $u \in \mathrm{Var}$, $\varphi, \psi \in \mathrm{Fm}(L_{\varepsilon,\exists})$, $\Gamma \in \mathrm{Cntx}(L_{\varepsilon,\exists})$ and $P, Q \in \mathrm{Exp}$, then

$$\frac{\Gamma \cup \{u : \varphi\} \vdash P : \psi}{\Gamma \vdash \lambda u.P : \varphi \to \psi} \to \text{intro}$$

$$\frac{\Gamma \vdash P : \varphi \to \psi, \quad \Gamma \vdash Q : \varphi}{\Gamma \vdash PQ : \psi} \to \text{elim}$$

Existential quantifier introduction and elimination rules. If $u \in \mathrm{Var}$, $\varphi, \psi \in \mathrm{Fm}(L_{\varepsilon,\exists})$, $\Gamma \in \mathrm{Cntx}(L_{\varepsilon,\exists})$ and $P, Q \in \mathrm{Exp}$, then

$$\frac{\Gamma \vdash P : \varphi[x/t] \qquad\qquad \Gamma \vdash t : \iota}{\Gamma \vdash \mathsf{pair}_{(\exists x)\varphi}(P, t) : (\exists x)\varphi}\exists\text{intro}$$

$$\frac{\Gamma \vdash P : (\exists x)\varphi \qquad\qquad \Gamma \cup \{u : \varphi, x : \iota\} \vdash Q : \psi}{\Gamma \vdash \mathsf{ind}_{(\exists x)\varphi,\psi}(P, u.x.Q) : \psi}\exists\text{elim}$$

in the former $x \notin \mathrm{FreeVar}(t)$, and the later $x \notin \mathrm{FreeVar}(\mathrm{ran}(\Gamma) \cup \psi)$.

Epsilon introduction and elimination rules. If $u \in \mathrm{Var}$, $\varphi, \psi \in \mathrm{Fm}(L_{\varepsilon,\exists})$, $\Gamma \in \mathrm{Cntx}(L_{\varepsilon,\exists})$ and $P, Q \in \mathrm{Exp}$, then

$$\frac{\Gamma \vdash P : \varphi[x/t] \qquad\qquad \Gamma \vdash t : \iota}{\Gamma \vdash \mathsf{pair}_{(\varepsilon x)\varphi}(P, t) : \varphi[x/(\varepsilon x)\varphi]}\varepsilon\text{intro}$$

$$\frac{\Gamma \vdash P : \varphi[x/(\varepsilon x)\varphi] \qquad \Gamma \cup \{u : \varphi, x : \iota\} \vdash Q : \psi}{\Gamma \vdash \mathsf{ind}_{(\varepsilon x)\varphi,\psi}(P, u.x.Q) : \psi}\varepsilon\text{elim}$$

in the former $x \notin \text{FreeVar}(t)$, and the later $x \notin \text{FreeVar}(\text{ran}(\Gamma) \cup \psi)$.

Remark 4. In the $\to$intro rule, the $u.$ in $\lambda u.P$ denotes that u codes the "dischargeable premiss" φ, that is the rule, after it is performed, cancels the $u : \varphi$ tag from the context. It is also the case concerning the εelim and $\exists$elim rule. In proof expression $\mathsf{ind}_{(\exists x)\varphi,\psi}(P, u.x.Q)$ the notation $u.x.$ denotes that u and x code the "dischargeable premisses" φ and ι, respectively. The only distinction is that the tag $u : \varphi$ serves as a possible typing judgement while $x : \iota$ a kinding judgement.

Remark 5. If the proposition

$$\Gamma \vdash M : \varphi$$

holds for a formula φ, then we say that φ types the expression M in the context Γ, or M is a proof of φ depending on Γ, and $\Gamma \vdash M : \varphi$ is said to be a "typing judgement". If the proposition

$$\Gamma \vdash t : \iota$$

holds for the term t, then we say that t denotes an object of the constructive kind. $\Gamma \vdash t : \iota$ is said to be a "kinding judgement".

Notation 1. The rules above define the proof theory of the *intuitionistic implicational existential epsilon-logic* over the language $\text{Fm}(L_{\varepsilon,\exists})$ and proof expressions Exp, and it is denoted by

$$\text{IL}^{\to}_{\exists\varepsilon}$$

Notation 2. The set of rules

$$\{\text{var type}, \text{var kind}, \text{term}, \to \text{intro}, \to \text{elim}, \exists\text{intro}, \exists\text{elim}\}$$

over the language $\text{Fm}(L_{\exists})$ and proof expressions $\text{Exp}_{\exists}$ consists the proof theory of intuitionistic implicational existential logic and it is denoted by

$$\text{IL}^{\to}_{\exists}$$

Notation 3. The set of rules

$$\{\text{var type}, \text{var kind}, \text{term}, \text{epsilon kind}, \to \text{intro}, \to \text{elim}, \varepsilon\text{intro}, \varepsilon\text{elim}\}$$

over the language $\mathrm{Fm}(L_\varepsilon)$ and proof expressions Exp_ε consists the proof theory of intuitionistic implicational epsilon logic and it is denoted by

$$\mathrm{IL}_\varepsilon^\to$$

We have already mentioned in the introduction that, when a logic containing an epsilon symbol is intensional. This merely seeks to satisfy the requirement the statement of the first epsilon axiom is valid in the system. Indeed the logic above is an intensional epsilon logic.

Theorem 1. The derivability system $\mathrm{IL}_{\exists\varepsilon}^\to$ is an intensional epsilon logic: if $\varphi \in \mathrm{Fm}(L_{\varepsilon,\exists})$, then

$$\vdash_{\mathrm{IL}_{\exists\varepsilon}^\to} ((\exists x)\varphi) \to \varphi[x/(\varepsilon x)\varphi].$$

Proof.

$$\dfrac{\dfrac{}{\{v : (\exists x)\varphi\} \vdash v : (\exists x)\varphi} \qquad \dfrac{\dfrac{}{\{v : (\exists x)\varphi, u : \varphi, x : \iota\} \vdash u : \varphi[x/x]} \qquad \dfrac{}{\{v : (\exists x)\varphi, u : \varphi, x : \iota\} \vdash x : \iota}}{\{v : (\exists x)\varphi, u : \varphi, x : \iota\} \vdash \mathsf{pair}_{(\varepsilon x)\varphi}(u, x) : \varphi[x/(\varepsilon x)\varphi]}}{\dfrac{\{v : (\exists x)\varphi\} \vdash \mathsf{ind}_{(\exists x)\varphi,\varphi[x/(\varepsilon x)\varphi]}(v, u.x.\mathsf{pair}_{(\varepsilon x)\varphi}(u, x)) : \varphi[x/(\varepsilon x)\varphi]}{\vdash \lambda v.\mathsf{ind}_{(\exists x)\varphi,\varphi[x/(\varepsilon x)\varphi]}(v, u.x.\mathsf{pair}_{(\varepsilon x)\varphi}(u, x)) : ((\exists x)\varphi) \to \varphi[x/(\varepsilon x)\varphi]}}$$

□

Remark 6. Note that the concrete syntax tree above is also a *proof tree.* The leaves are all discharged by the →intro rule or the ∃elim rules.

2.4. **Normalization and intuitionisticaly invalid formulas.** A property of the elimination and introduction rules of intuitionistic logic – called *harmonic* by Michael Dummett – is suitable for cutting superfluous inferences from derivations [Dummett]. At a point in the derivation where an elimination rule immediately follows an introduction rule is called an *inversion.* A task of proof theory is to produce proofs in which there are no inversions. These are called *normal proofs* and the problem can be traced back to the idea of Gerhard Gentzen, who felt with excellent intuition about this property of natural derivation systems. As Gentzen claims

> [...] an introduction rule gives, so to say, a definition of the constant in question, [...] an elimination rule is only a consequence of the corresponding introduction rule, which may be expressed somewhat as follows: at an inference by an elimination rule, we are allowed to "use" only what the principal sign of the major premiss "means" according to the introduction rule for this sign [Gentzen, p. 189].

The normalization theorem of intuitionistic logic was first proved by Dag Prawitz in general, elaborating Gentzen's conjecture in detail. Before him, of course, many representatives of the theory of proof achieved special results, from Gödel to Schütte. According to Prawitz, the main conjecture is the folowing:

> *Inversion principle.* If a formula can be derived from a set of formulas, then there is also a proof of it from the formula set where non of the introduction rules are followed immediately by an elimination rule of the same logical operator.

The elimination of the inversions form a proof are called *reductions.* Concerning the proof systems above the three inversion situations are the following. If a $\to$ introduction rule is followed by a modus ponens, then the reduction itself is the so called beta reduction of typed lambda calculus:

$$\frac{\dfrac{\Gamma \cup \{u : \varphi\} \vdash P : \psi}{\Gamma \vdash \lambda u.P : \boxed{\varphi \to \psi}} \qquad \Gamma \vdash Q : \varphi}{\Gamma \vdash (\lambda u.P)Q : \psi} \to_\beta \Gamma \vdash P[u/Q] : \psi$$

or just looking at the proof expressions:

$$(\lambda u.P)Q \to_\beta P[u/Q].$$

The reduction rule for the existential quantifier is:

$$\frac{\dfrac{\Gamma \vdash P : \varphi[x/t] \qquad \Gamma \vdash t : \iota}{\Gamma \vdash \mathsf{pair}_{(\exists x)\varphi}(P, t) : \boxed{(\exists x)\varphi}} \qquad \Gamma \cup \{x : \iota, u : \varphi\} \vdash Q : \psi}{\Gamma \vdash \mathsf{ind}_{(\exists x)\varphi,\psi}(\mathsf{pair}_{(\exists x)\varphi}(P, t), u.x.Q) : \psi} \to_\exists \Gamma \vdash Q[u/P][x/t] : \psi$$

or encapsulated into the proof expression:

$$\mathsf{ind}_{(\exists x)\varphi,\psi}(\mathsf{pair}_{(\exists x)\varphi}(P, t), u.x.Q) \to_\exists Q[u/P][x/t].$$

Remark 7. Note that the $\boxed{\text{boxed}}$ formulas above contain one more logical operator ($\to$ or $\exists$) than the surrounding formulas. For this reason, they are called local peaks, and the proofs of Normalization Theorems are usually done by reducting them into a lower level.

Definition 6. Proof expressions of the form

$$(\lambda u.P)Q$$

and

$$\mathsf{ind}_{(\exists x)\varphi,\psi}(\mathsf{pair}_{(\exists x)\varphi}(P,t),u.x.Q)$$

are called *redexes.* If they are typed in the context Γ, and the type of the expression $\lambda u.P$ in $(\lambda u.P)Q$ (or $\mathsf{pair}_{(\exists x)\varphi}(P,t)$ in $\mathsf{ind}_{(\exists x)\varphi,\psi}(\mathsf{pair}_{(\exists x)\varphi}(P,t),u.x.Q)$) has d logical operators in it, then d is called the *redex degree* of the it. The redex degree of the redex R is denoted by $\deg(R)$.
A proof expression N is called a *normal,* if it does not contain any subexpression which is a redex.

Theorem 2. The system $\mathrm{IL}_{\exists}^{\rightarrow}$ is normalizable: for every $M \in \mathrm{Exp}_{\exists}$, $\Gamma \in \mathrm{Cntx}(L_{\exists})$, $\varphi \in \mathrm{Fm}(L_{\exists})$ such that $\Gamma \vdash M : \varphi$, there is a normal expression $N \in \mathrm{Exp}_{\exists}$ such that $\Gamma \vdash M : \varphi$. (Cf.: [Prawitz, p. 50], [Sore, p. 68], [Troe, p. 182])

Proof. The standard way of the proof goes by the so called right-most algorithm, i.e. to eliminate recursively all redexes in the proof terms from the right hand side. In order to achieve an equivalent proof term without redexes, we have to declare two other kinds of redexes and reduction rule. The first one is the so called premutation rule (π rule), the second is the simplification rule (σ rule). Let P be any proof term, containing the variable v at only one character position, then $P[v/Q]$ is also a proof term if Q is a proof term. Now, consider the rules

$$P[v/\mathsf{ind}_{(\exists x)\varphi,\psi}(M,u.x.N)] \rightarrow_{\pi} \mathsf{ind}_{(\exists x)\varphi,\psi}(M,u.x.P[v/N])]$$

and with u, x not free in N

$$\mathsf{ind}_{(\exists x)\varphi,\psi}(M,u.x.N) \rightarrow_{\sigma} N$$

These rules cut the superfluous detours in proof terms just as in the cases of the beta rules above. Let the maximal degree of the subredexes in M be d. Then there are at most d levels of distinct sets of redexes with different degrees.

Rightmost Normalization Algorithm

Input: term M with type α.

Output: (normal) term M'.

let $M' = M$
let Maxred(M') be the set of all redexes in M' of the type α, where α contains maximal number from the operator set $\{\rightarrow, \exists\}$ in it (maximal redexes)
while Maxred(M') contains any element **do**
  let R be the rightmost maximal redex among Maxred(M') in M'
  let M'' be the reductum of M' at R
  let M' be M''
end while
let M be M'

The number of redexes during the algorithm can be bounded by a function $n(l(M), d)$ of the length $l(M)$ of the original expression M and the maximal degree d. From right to left it is performed a reduction to each redex. This either decreases the number of redexes in that level or finaly eliminate all of the redexes on that level. The algorithm halts in less then $n(l(M), d)$ steps. The upper estimation of $n(l(M), d)$ gives a hyperexponential magnitude. (Cf. [Fortune].) □

It is an intuitive observation that a compound formula from the empty premiss set can only be deduced if *the last step* of its normal proof *was an application of an introduction rule.*

Lemma 1. Let N be a normal proof of $\varphi \rightarrow \psi$ or $(\exists x)\varphi$ depends upon $\Gamma = \emptyset$ in $\mathrm{IL}_{\exists}^{\rightarrow}$. Then N is of the form $\lambda u.P$ or $\mathsf{pair}_{(\exists x)\varphi}(P, t)$, respectively, with expressions t, P and variables u, x.

Proof. This is a consequence of a very long but easily provable theorem on the Form of Normal Deductions and its consequence, the Subformula Theorem. The textbook proof (from [Troe, p. 187] or [Prawitz, p. 53]) can be repeated without any difficulty in this case as well, since the the system described above is an inessential variation of the intuitionistic logic with $\rightarrow$ and $\exists$. The only difference is that in the introduction rule of $\exists$ there is one more branch among the premisses containing only individual expressions (not proof terms) corresponding terms in intuitionistic FOL. According to the Subformula Theorem [Troe, p. 188] or [Prawitz, p. 55], all the types of the subterms in the normal term N' in the sequence $\Gamma \vdash N' : \vartheta$ belong to the subtypes of the elements of $\Gamma \cup \{\vartheta\}$. Hence, all the types of subterms in N are in the subexpressions of $\varphi \rightarrow \psi$, $(\exists x)\varphi$. Since Γ is empty, the only types on the leaves in the proof tree are the types discharged by introducion rules. At the root-node, all the premisses have to be discharged, and in a normal proof an elimination rule never follows an introduction rule, hence the last inference rule in the proof tree is an introduction rule. ☐

As a consequence, it turns out that in $\mathrm{IL}_{\exists}^{\to}$ the formula expressing the *existential pre-supposition* of the epsilon terms cannot be generaly deduced. It seems to me that it has such proof which is very similar to the consistency proof of the intuitionistic predicate logic, where an atomic formula was shown to be not derivable form the empty set of premisses.

Remark 8. Note that, if $\Gamma = \{\varphi_1, \ldots, \varphi_n\} \subseteq \mathrm{Fm}(L_\exists)$ and $\varphi \in \mathrm{Fm}(L_\exists)$ then $\Gamma \underset{\mathrm{IL}_{\exists}^{\to}}{\vdash} \varphi$ denotes that there is an $M \in \mathrm{Exp}_\exists$ and variables $u_1, \ldots, u_n$ such that

$$\{u_1 : \varphi_1, \ldots, u_n : \varphi_n\} \underset{\mathrm{IL}_{\exists}^{\to}}{\vdash} M : \varphi.$$

Proposition 1.

$$\underset{\mathrm{IL}_{\exists}^{\to}}{\nvdash} (\exists x)(((\exists x)\varphi) \to \varphi)$$

(Cf.: folklore.)

Proof. Consider an arbitrary formula ψ. Suppose that there is an $M \in \mathrm{Exp}_\exists$ such that

$$\vdash M : (\exists x)\psi.$$

According to Lemma 1, there is a $t \in \mathrm{Tm}(L_{\varepsilon,\exists})$ and an $N \in \mathrm{Exp}_\exists$ such that $M = \mathsf{pair}_{(\exists x)\psi}(N, t)$ and

$$\frac{\vdash N : \psi[x/t] \qquad\qquad \vdash t : \iota}{\vdash \mathsf{pair}_{(\exists x)\psi}(N, t) : (\exists x)\psi}$$

that is there is a term t such that $\vdash t : \iota$, but here the left hand side of $\vdash$ is empty, hence, there is no term t for which the kinding judgement $\vdash t : \iota$ holds. As a consequence, the formula $(\exists x)(((\exists x)\varphi) \to \varphi)$ is also not derivable in system $\mathrm{IL}_{\exists}^{\to}$. ☐

Remark 9. The question arises as to whether the $\mathrm{IL}_{\exists\varepsilon}^{\to}$ expansion has a normalization property. I guess yes, but with this method I don't see how a positive answer could be given. The Hilbertian epsilon is not a logical operator, i.e. during formula formation, the point where the epsilon symbol would appear cannot be reached. Thus, in the succession of the rules of epsilon introduction and elimination, no local peak is formed, the elimination of which would control the induction does not arise. The system of natural deduction may not even be a good choice to decide this.

2.5. **Conservativity properties.** The following consideration shows that if the existential quantification rules in $\mathrm{IL}^{\rightarrow}_{\exists\varepsilon}$ are read as that of the epsilon's, then they can be derived in $\mathrm{IL}^{\rightarrow}_{\varepsilon}$. In order to formally grasp this fact, we need to introduce a generalized concept of conservativity. As we know, in the Hilbert's style deduction systems, the axiom system $X \cup Y$ over the language $\mathcal{L} \cup \mathcal{K}$ is a conservative extension of the axiom system X over the language $\mathcal{L}$, if for every sentence $\varphi \in \mathcal{L}$ the claim $X \cup Y \vdash \varphi$ yields $X \vdash \varphi$. This will be generalized for the natural deduction sytems above.

Theorem 3. $\mathrm{IL}^{\rightarrow}_{\exists\varepsilon}$ is conservative over $\mathrm{IL}^{\rightarrow}_{\varepsilon}$:

$$\text{if } \{\varphi\} \cup \Gamma \subseteq \mathrm{Fm}(L_\varepsilon) \text{ and } \Gamma \underset{\mathrm{IL}^{\rightarrow}_{\exists\varepsilon}}{\vdash} \varphi, \text{ then } \Gamma \underset{\mathrm{IL}^{\rightarrow}_{\varepsilon}}{\vdash} \varphi.$$

Proof. Let the translation function

$$f : \mathrm{Exp} \cup \mathrm{Fm}(L_{\exists\varepsilon}) \cup \mathrm{Tm}(L_{\exists\varepsilon}) \to \mathrm{Exp}_\varepsilon \cup \mathrm{Fm}(L_\varepsilon) \cup \mathrm{Tm}(L_\varepsilon)$$

be the following recursively defined language-epimorphism:

$$\begin{aligned} f(e) &= e, \quad e \in \mathrm{Exp}_\varepsilon \cup \mathrm{Fm}(L_\varepsilon) \cup \mathrm{Tm}(L_\varepsilon) \\ f((\exists x)\varphi) &= f(\varphi)[x/(\varepsilon x) f(\varphi)] \\ f(\mathsf{pair}_{(\exists x)\varphi}(P,t)) &= \mathsf{pair}_{(\varepsilon x) f(\varphi)}(f(P), f(t)) \\ f(\mathsf{ind}_{(\exists x)\varphi,\psi}(P, u.Q)) &= \mathsf{ind}_{(\varepsilon x) f(\varphi), f(\psi)}(f(P), u.f(Q)) \end{aligned}$$

Let $\varphi_1, \ldots, \varphi_n, \varphi \in \mathrm{Fm}(L_{\exists\varepsilon})$.

Lemma.
If $\{u_1 : \varphi_1, \ldots, u_n : \varphi_n\} \underset{\mathrm{IL}^{\rightarrow}_{\exists\varepsilon}}{\vdash} M : \varphi$, then $\{u_1 : f(\varphi_1), \ldots, u_n : f(\varphi_n)\} \underset{\mathrm{IL}^{\rightarrow}_{\varepsilon}}{\vdash} f(M) : f(\varphi)$
Indeed. The proof goes by induction of the structure of M. The only relevant cases are that of the existential quantification rules. But, these are defined exactly so that the values of the translation function f satisfy the epsilon inference rules.
As a consequence, if $\varphi_1, \ldots, \varphi_n, \varphi \in \mathrm{Fm}(L_\varepsilon)$ and $\{u_1 : \varphi_1, \ldots, u_n : \varphi_n\} \underset{\mathrm{IL}^{\rightarrow}_{\exists\varepsilon}}{\vdash} M : \varphi$ holds, then $\{u_1 : \varphi_1, \ldots, u_n : \varphi_n\} \underset{\mathrm{IL}^{\rightarrow}_{\exists\varepsilon}}{\vdash} f(M) : \varphi$ holds too. □

Remark 10. This means that by adding the rules of existential quantification to the intuitionistic (implicational) epsilon logic, the former does not change the proof-theoretic meaning of epsilon terms. What was not provable in the former epsilon language, then will not be provable in the larger system.

The counterpart of the theorem does not hold.

Theorem 4. $\mathrm{IL}^{\rightarrow}_{\exists\varepsilon}$ is not conservative over $\mathrm{IL}^{\rightarrow}_{\exists}$.

Proof. As we have seen, $\underset{\mathrm{IL}^{\rightarrow}_{\exists}}{\nvdash} (\exists x)(((\exists x)\varphi) \rightarrow \varphi)$. However, $\underset{\mathrm{IL}^{\rightarrow}_{\exists\varepsilon}}{\vdash} (\exists x)(((\exists x)\varphi) \rightarrow \varphi)$ holds.

Indeed. By intensionality (axiom) we know that there is a proof expression P in the language of $\mathrm{IL}^{\rightarrow}_{\exists\varepsilon}$ such that

$$\underset{\mathrm{IL}^{\rightarrow}_{\exists\varepsilon}}{\vdash} P : ((\exists x)\varphi) \rightarrow \varphi[x/(\varepsilon x)\varphi]$$

then,

$$\frac{\vdash P : ((\exists x)\varphi) \rightarrow \varphi[x/(\varepsilon x)\varphi] \qquad \dfrac{}{\vdash (\varepsilon x)\varphi : \iota}\,\text{epsilon kind}}{\vdash \mathsf{pair}_{(\exists x)(((\exists x)\varphi)\rightarrow\varphi)}(P, (\varepsilon x)\varphi) : (\exists x)(((\exists x)\varphi) \rightarrow \varphi)}$$

□

Remark 11. In this construction, the rules of system $\mathrm{IL}^{\rightarrow}_{\exists}$ have been chosen so weak that the cardinal formula $(\exists x)(((\exists x)\varphi) \rightarrow \varphi)$ cannot be deduced, and the rules of system $\mathrm{IL}^{\rightarrow}_{\exists\varepsilon}$ have been chosen so strong that the formula in question can be deduced. This does not have to be the case. For example, in Heyting Arithmetic, different assumptions about the epsilon symbol guarantee the derivability of different formulas [Bell]. Nevertheless, it is well supported by Bell's article also, namely that requiring epsilon axioms pushes the system in the direction of classical logic. For example, all De Morgan rules or even the Law of Excluded Middle can be deduced from them.

Remark 12. We know that one of Hilbert's intentions with the epsilon symbol was to incorporate the Axiom of Choice into logic through the appropriate choice of language. Therefore, it is not counterintuitive to assume that in this logic the epsilon terms are always given "constructively", as the epsilon kind rule states so. Not in the sense that their construction would be given, but in the sense that they are such that we can refer to objects by them without any restriction.

Remark 13. Hilbert argued that the epsilon symbol is nothing more than an aid to the quantification. This would mean that by expanding the predicate calculus with epsilon rules, we get a conservative expansion. In light of the above, however, it is clear that this is not true for the intuitionistic logic. On the contrary. In this case, the existential quantifier qualifies as an aid to the pure epsilon logic.

3. The Intensional Semantics

It is known that an epsilon-invariant sentence has a first-order reformulation, although it is not in an explicit form, since, the proof uses the non-constructive interpolation theorem. I make an attempt to describe the explicit meaning of sentences containing epsilon-terms, adopting the strong assumption of their first-order reformulability. I will prove that, if a monadic predicate is syntactically independent from an epsilon-term and if the sentence obtained by substituting the variable of the predicate with the epsilon-term is epsilon-invariant, then the sentence has an explicit first-order reformulation (Theorem 6). This section is based on the research that I did in [Moln2013].

3.1. **The proposed solution.** In the Tarskian sense, a notion is *logical* if it is a permutation-invariant operations such as quantification, identity, substitution of variables, and so on [Tars]. As opposed to Russell's descriptor, Hilbert's operator does not satisfy the permutation-invariant property. The sentence

$$\vartheta((\varepsilon x)\varphi)$$

is not necessarily purely logical, since generally, it does not have a plain first-order reformulation. Our goal is to find the first-order reformulation of the sentence $\vartheta((\varepsilon x)\varphi)$ and its meaning, provided the sentence $\vartheta((\varepsilon x)\varphi)$ is purely logical.

Our hypothesis is that, if there is any purely logical meaning of $\vartheta[(\varepsilon x)\varphi/x]$, then it must be a weak form of the purely logical meaning of $\vartheta[(\iota x)\varphi/x]$ proposed by Russell. Since, we follow Tarski, on "purely logical" we mean plain first-order reformulation, actually it will be enough to work with a certain type of epsilon-invariance. As it is known, if $\vartheta[(\varepsilon x)\varphi/x]$ is epsilon-invariant then, according to Caicedo, Blass and Gurevich (Theorem 5), it has a first-order reformulation. But, is there an explicit reformulation of $\vartheta[(\varepsilon x)\varphi/x]$? To answer the question, let us consider the sentence 'The F is G.' and recall Neale's version of Russell's proposal [Neal, p. 21].

'There is an F, there is at most one F, and *every* F is G.'

Since, we think of epsilon-terms as not necessarily definite descriptions, first, let us erase the uniqueness clause:

'There is an F, and every F is G.'

The schema obtained just now is obviously not applicable to the epsilon-operator, since there are sentences containing epsilon-terms which are true, while the existence formula associated to the epsilon-term is false. For example

$$(\varepsilon x)(x \neq x) = (\varepsilon x)(x \neq x)$$

is true, however,

$$(\exists x)(x \neq x)$$

is false. Similar effects occur when the property in question is true for every individuals, thus it seems sound to express our proposal in the following form

(M) 'There is an F and every F is G, or there is no F, however everything is G.'

In Theorem 6 (Sec. 3.6) we will conclude that (M) can be considered to be the meta-language translation of the formal sentence $\vartheta[(\varepsilon x)\varphi/x]$, if ϑ and φ both are independent from $(\varepsilon x)\varphi$ in the sense of Definition 20 (Sec. 3.6).

Note that, while the Theory of Descriptions is a part of linguistics and philosophy, exploring the model-theoretical meaning of sentences in the epsilon-language is a task of mathematical logic.

3.2. **Syntax.** In the literature there are two versions of epsilon-languages, for later purposes it is convenient to consider both of them. Let $\mathtt{t} = (r_i, f_j, c_k)_{ijk}$ be a similarity type and let $\mathcal{L}_\exists$ be the usual first-order language generated with respect to the operators $\neg, \vee, \exists, \mathtt{t}$. $\mathcal{L}_\exists$ can be extended to the language

$$\mathcal{L}_{\exists,\varepsilon}$$

by adding the terms $(\varepsilon x)\varphi$ to the class Term($\mathcal{L}_\exists$), where x is any variable, and φ is any formula. If φ is a one-variable formula of the variable x, then $(\varepsilon x)\varphi$ is called the epsilon-term *associated to* φ. Sometimes, $\mathcal{L}_{\exists,\varepsilon}$ is called *the language of the predicate calculus with epsilon* and it is accurately defined, for example, in [Zach, The Epsilon Calculus: Syntax, Sec. 2] or in [Monk, The Hilbert ε-operator, p. 481], aside from the operator **O**.

Another epsilon-language is obtained when one extends the language of *the elementary calculus* by adding the epsilon-terms. The elementary calculus is the first order logical calculus without quantifiers and the rules of quantification. It is denoted by EC. If $\mathcal{L}$ denotes the first-order language of type $\mathtt{t}$ without quantifiers, then let

$$\mathcal{L}_\varepsilon$$

be the extension of $\mathcal{L}$ by the epsilon-terms. It is clear that $\mathcal{L}_\varepsilon$ is the freely generated language with respect to the operators $\neg, \vee, \varepsilon, \mathtt{t}$ and it is called *the language of the elementary calculus with epsilon.* A more formal definition of $\mathcal{L}_\varepsilon$ can be found in [Zach, Sec. 2].

3.3. **Calculus.** A featured connection arises between $\mathcal{L}_{\exists,\varepsilon}$ and $\mathcal{L}_\varepsilon$ when we consider the axioms of the epsilon-calculus.

Definition 7 (PC_ε)**.** Let PC_ε be the deductive system of predicate logic with equality in the language $\mathcal{L}_{\exists,\varepsilon}$ with the following *critical axioms*

$$\text{(CA)} \qquad \varphi[t/x] \to \varphi[(\varepsilon x)\varphi/x]$$

where $[./x]$ is the operation of substitution, x is a variable, $t \in \mathrm{Term}(\mathcal{L}_{\exists,\varepsilon})$ is any term, and $\varphi \in \mathrm{Form}(\mathcal{L}_{\exists,\varepsilon})$ is a formula (t is free for x in φ).

Remark 14. The critical axioms imply the ones below

$$\underset{\mathrm{PC}_\varepsilon}{\vdash} \quad (\exists x)\varphi \;\leftrightarrow\; \varphi[(\varepsilon x)\varphi/x], \qquad \underset{\mathrm{PC}_\varepsilon}{\vdash} \quad (\forall x)\varphi \;\leftrightarrow\; \varphi[(\varepsilon x)\neg\varphi/x]$$

Definition 8 (EC_ε)**.** *The elementary calculus with epsilon* in the language $\mathcal{L}_\varepsilon$ (thus without explicit quantifiers), is the deductive system of the predicate logic (EC)—including the equality rules and without the quantifiers—extended by the schema (CA) and the following (Gen) as axioms

$$\text{(Gen)} \qquad \varphi \to \varphi[(\varepsilon x)\neg\varphi/x]$$

where x does not occur in φ. It is denoted by EC_ε.

Remark 15. Since $(\forall x)\varphi$ is defined to be $\varphi[(\varepsilon x)\neg\varphi/x]$ in EC_ε, after a time, (Gen) becomes the usual $\varphi \to (\forall x)\varphi$ schema.

Now, let

$$(.)^{*\varepsilon} : \mathcal{L}_{\exists,\varepsilon} \to \mathcal{L}_\varepsilon$$

be the language homomorphism such that $(.)^{*\varepsilon}$ preserves $\neg, \vee, \varepsilon, \mathtt{t}$ and the following correspondence

$$((\exists x)\varphi)^{*\varepsilon} = \varphi^{*\varepsilon}[(\varepsilon x)\varphi^{*\varepsilon}/x].$$

Proposition 2 (Moser–Zach)**.** $(.)^{*\varepsilon}$ is a $\mathrm{PC}_\varepsilon \to \mathrm{EC}_\varepsilon$ embedding which preserves the provability [Zach, The Embedding Lemma, Sec. 4].

Remark 16. Hence, it is clear that there is a canonical monomorphism

$$(.)^{\varepsilon} : \mathcal{L}_\exists \to \mathcal{L}_\varepsilon$$

which is invariant with respect to the operators $\neg, \vee, \mathtt{t}$ and the correspondence $((\exists x)\varphi)^{\varepsilon} = \varphi^{\varepsilon}[(\varepsilon x)\varphi^{\varepsilon}/x]$. This leads to the fact that PC can also be embedded into EC_ε.

Remark 17. Note that Moser and Zach's result above is an exact form of the fact that was proved by Bourbaki in an informal way. (In their work, the French group of mathematicians built the predicate calculus onto the epsilon-language without any explicit quantifiers.) Nevertheless, the embedding result has long been a part of the mathematicians' folklore.

3.4. **Intensional Semantics.** The first model-theoretic semantics goes back to Günter Asser. A complete description of the semantics can be found in [Asse] and [Ahre]. The latter paper contains a remarkable completeness proof in the context of automated theorem proving. In the following, we describe the *intensional* and the well-known *extensional semantics* briefly.

Remark 18. The authors of Gundlagen did not give any formal semantics to explain their calculus. It is not really surprising since, at that time, the model-theoretical semantics were not developed well enough. Just for that, Hilbert's terms were the objects which filled the lack of semantic reference and played the role of the members of the universe of a future canonical model. Indeed, if one compares the epsilon-terms to the Henkin witnesses, one can find that the Henkin witness c of the valid sentence $(\exists x)\varphi(x)$ corresponds to the epsilon-term associated to $\varphi(x)$, since they both satisfy the predicate $\varphi(x)$. Henkin witnesses are mentioned in the proof of the Compactness Theorem or the Omitting Types Theorem of FOL. If a formula $(\exists x)\varphi(x)$ holds and for a term c, then formula $\varphi(c)$ holds too, and c is a Henkin witness for the existential formula $(\exists x)\varphi(x)$ [Hodg, p. 265, p. 334]. Hence, the formulas

$$(\exists x)\varphi(x) \to \varphi(c) \quad \text{and} \quad (\exists x)\varphi(x) \to \varphi((\varepsilon x)\varphi(x))$$

hold. Using the above property of the epsilon-terms, Ackermann, Hilbert and Bernays were able to prove the consistency of some certain formal theories.

Epsilon-terms are closely related to Skolem functions, hence we will define the $\mathcal{L}_\varepsilon$-structures by special Skolem expansions of the usual first-order models, which is pointed out in [Monk, The Hilbert ε-operator, p. 481] and in [Mints, Sec. 2.: Quantifier-Free Extensions of Formulas and ε-Theorems)]. By Skolem expansions, we mean models of a first-order language extended by Skolem symbols, where the interpretations of Skolem symbols are Skolem functions (the definitions can be found in [Monk, 11.33-36, pp 211-2]).

Fact 1. Let us add the Skolem function symbol $S_{(\exists x)\varphi}$ to the language $\mathcal{L}_\exists$ for every existential formula $(\exists x)\varphi$. If $\mathcal{L}'_\exists$ denotes the Skolem extension of $\mathcal{L}_\exists$, then there is a *canonical language monomorphism*

$$(.)^* : \mathcal{L}_{\exists,\varepsilon} \to \mathcal{L}'_\exists$$

which sends the epsilon-terms to the Skolem expressions, i.e.

$$((\varepsilon x)\varphi)^* = S_{(\exists x)\varphi} v_1 \dots v_k$$

where $k = \max\{i \mid v_i \in \text{FreeVar}((\exists x)\varphi)\}$. (Cf. [Monk, Def. 29.23, p. 481].)

Remark 19. The first problem of constructing a semantics for the epsilon-terms turns up at this point. It is clear that, in the Skolem expansion $\mathfrak{N}$

$$((S_{(\exists x)\varphi} v_1 \dots v_k)[t/v_i])^{\mathfrak{N}}[a] = (S_{(\exists x)\varphi} v_1 \dots v_k)^{\mathfrak{N}}[a_i^n],$$

where term t is free for v_i in $(\exists x)\varphi$, $a \in {}^\omega N$, $n = t^{\mathfrak{N}}[a]$, and the sequence $(a \setminus \{(i, a_i)\}) \cup \{(i, n)\}$ is denoted by a_i^n. But, for us, the Skolem terms (as the $(.)^*$-images of the epsilon-terms) are needed to show the following property

$$\text{(Sub)} \qquad (S_{(\exists x)\varphi[t/v_i]} v_1 \dots v_k)^{\mathfrak{N}}[a] = (S_{(\exists x)\varphi} v_1 \dots v_k)^{\mathfrak{N}}[a_i^n].$$

Property (Sub) is the same as substitutivity in [Ahre, Def. 5] and this one is so important that we redefine it in the context of Skolem functions.

Definition 9. Let $\mathfrak{M}$ be a first-order model. The Skolem expansion $\mathfrak{N}$ of $\mathfrak{M}$ is *substitutive* (*substitutive epsilon-stucture*), if for every $(\exists x)\varphi$ and $a \in {}^{\omega}M$

$$(S_{(\exists x)\varphi[t/v_i]}v_1 \dots v_k)^{\mathfrak{N}}[a] = (S_{(\exists x)\varphi}v_1 \dots v_k)^{\mathfrak{N}}[a_i^{t^{\mathfrak{N}}[a]}]$$

for every term t such that t is free for v_i in $(\exists x)\varphi$. (Cf. [Ahre, Def. 5].)

Remark 20. Note that, Ahrendt and Giese introduced several types of epsilon-structures. The structures expanded by Skolem functions and the substitutive Skolem expansions correspond to the *intensional and substitutive structures* respectively (see [Ahre, Def. 4,5]).

The solution of the problem of substitutivity is to introduce epsilon-matrices.

Definition 10 (Epsilon-matrix)**.** Let us suppose that the epsilon-term t has an occurrence in the formula φ. If this occurrence of t is also an occurrence of t in another epsilon-term s occurring in φ, then it is said to be *interior* in φ, otherwise it is an *exterior* occurrence of t in φ. The epsilon-term

$$(\varepsilon x)\psi(x, y_1, \dots, y_n)$$

is *a matrix of the epsilon-term* $(\varepsilon x)\varphi$ if

(1) $y_1, ..., y_n \notin \mathrm{Var}((\varepsilon x)\varphi)$,
(2) $\psi(x, y_1, \dots, y_n)$ is an $(n+1)$-variable formula, and
(3) the distinct epsilon-terms $t_1, \dots, t_k$ have exterior occurrence in φ such that

$$(\varepsilon x)\varphi = (\varepsilon x)\psi(x, y_1, \dots, y_n)[t_1/y_1, \dots, t_n/y_n].$$

(Cf. [Mints, p. 138]).

Remark 21. It is known that for $(\varepsilon x)\varphi$ there are unique $\psi(x, y_1, \dots, y_n), t_1, \dots, t_n$ with the above three properties. Here, "unique" means that up to the change of variables $y_1, \dots, y_n$ and the simultaneous change of x in $(\varepsilon x)\varphi$ and $(\varepsilon x)\psi(x, y_1, \dots, y_n)$ (Cf. [Mose, Sec. 3, p. 21]). The relation '$(\varepsilon x)\varphi$ and $(\varepsilon y)\vartheta$ have the same epsilon-matrix' is an equivalence relation between the epsilon-terms.

Definition 11. The equivalence class containing $(\varepsilon x)\varphi$ is denoted by

$$\mathrm{mat}((\varepsilon x)\varphi)$$

and the set of all equivalence classes is denoted by

$$\mathrm{Mat}.$$

Remark 22. Note that $\mathrm{mat}((\varepsilon x)\varphi)$ is substitution-invariant in the sense that, if $(\varepsilon x)\psi$ is a matrix of $(\varepsilon x)\varphi$ and t is an epsilon-term such that $x \notin \mathrm{BoundVar}(t)$, then

$$(\varepsilon x)\psi[t/v] \in \mathrm{mat}((\varepsilon x)\varphi).$$

The definition below is due to Moser and Zach, and it is presented at the 17th Computer Science Logic Workshop and 8th Kurt Gödel Colloquium (Vienna, 2003).

Definition 12 (Model)**.** Let t be a similarity type and $\mathfrak{M}$ be a first-order model of type t. Let $\mathcal{P}(M)$ be the set of all subsets of M and let ${}^{[\omega]}M$ be the set of finite sequences in M. *A model of the language* $\mathcal{L}_\varepsilon$ (or an $\mathcal{L}_\varepsilon$*-structure*) is a pair $(\mathfrak{M}, f)$ where the function f satisfies the property

$$f : \mathrm{Mat} \times \mathcal{P}(M) \times {}^{<\omega}M \longrightarrow M,$$

$$f\left(\mathrm{mat}((\varepsilon v_i)\varphi), S, (a_1, \ldots, a_n)\right) \in S, \quad \text{if } S \in \mathcal{P}(M) \setminus \{\varnothing\}.$$

We will denote the class of all such models of an epsilon-language $\mathcal{L}_\varepsilon$ by $\mathsf{Int}(\mathcal{L}_\varepsilon)$.

Remark 23. In order to define the satisfaction in $(\mathfrak{M}, f)$, we will consider Monk's observation and we will use the correspondence $(.)^* : \mathcal{L}_{\exists,\varepsilon} \to \mathcal{L}'_\exists$ between the language $\mathcal{L}_{\exists,\varepsilon}$ and the Skolem expansion $\mathcal{L}'_\exists$ of the language $\mathcal{L}_\exists$ (Cf. [Monk, Prop. 29.24, p. 482]). Note that $\mathcal{L}_\varepsilon \subseteq \mathcal{L}_{\exists,\varepsilon}$ and $(.)^*$ sends the epsilon-terms of $\mathcal{L}_\varepsilon$ also to Skolem terms.

Definition 13 (Satisfaction)**.** Let $(\mathfrak{M}, f)$ be a model of the epsilon language $\mathcal{L}_\varepsilon$. First we define the Skolem expansion $\mathfrak{M}_f$ of $\mathcal{L}_\exists$. $\mathfrak{M}$ is a reduct of $\mathfrak{M}_f$ i.e. $\mathfrak{M}_f \restriction \mathcal{L}_\exists = \mathfrak{M}$ and the interpretations of the Skolem terms are as follows. If $a \in {}^{\omega}M$, then

$$\left(\mathrm{S}_{(\exists v_i)\varphi} v_1 \ldots v_k\right)^{\mathfrak{M}_f}[a] =$$
$$= f\left(\mathrm{mat}((\varepsilon v_i)\varphi), \{m \in M \mid \mathfrak{M}_f \models \varphi[a_i^m]\}, (t_1^{\mathfrak{M}_f}[a], \ldots, t_n^{\mathfrak{M}_f}[a])\right)$$

where $t_1, \ldots, t_n$ are obtained by representing the epsilon-term $(\varepsilon v_i)\varphi$ in its matrix form: $(\varepsilon v_i)\varphi = (\varepsilon v_i)\psi(v_i, t_1, \ldots, t_n)$.

Finally, the meanings of the terms t and formulas φ of $\mathcal{L}_\varepsilon$ under the valuation a in $(\mathfrak{M}, f)$ are defined to be

$$t^{(\mathfrak{M},f)}[a] = t^{*\mathfrak{M}_f}[a] \quad \text{and} \quad (\mathfrak{M}, f) \models \varphi[a] \text{ iff } \mathfrak{M}_f \models \varphi^*[a].$$

An immediate consequence of the usual Substitution Lemma is that every model $(\mathfrak{M}, f)$ can be constructed from a substitutive Skolem expansion and vice versa. We state and prove the Substitution Lemma for the epsilon-logic.

Proposition 3 (Substitution Lemma)**.** Let $(\mathfrak{M}, f)$ be a model of the epsilon-language $\mathcal{L}_\varepsilon$, $\varphi \in \mathrm{Form}(\mathcal{L}_\varepsilon)$, $t, s \in \mathrm{Term}(\mathcal{L}_\varepsilon)$, $k \in \omega$, $a \in {}^\omega M$, $u = t^{(\mathfrak{M},f)}[a]$ and t is free for v_k in φ and s. Then

$$(\mathfrak{M}, f) \models \varphi[t/v_k][a] \quad \text{iff} \quad (\mathfrak{M}, f) \models \varphi[a^u_k], \qquad (s[t/v_k])^{(\mathfrak{M},f)}[a] = s^{(\mathfrak{M},f)}[a^u_k].$$

[Moln2013]

Proof. By structural induction. The only non-trivial case is the induction step with the epsilon-terms. Suppose that t is an epsilon-term and $s = (\varepsilon v_i)\varphi = (\varepsilon v_i)\psi(v_i, t_1, \dots, t_n)$ with its matrix $(\varepsilon v_i)\psi(v_i, y_1, \dots, y_n)$. Since v_i is bound in s, let us assume without the loss of generality that $v_i \neq v_k$. Thus, it follows that $s[t/v_k] = (\varepsilon v_i)(\varphi[t/v_k])$. Note that if t occurs in $\psi(v_i, t_1, \dots, t_n)[t/v_k]$, then it is an interior occurrence, therefore $(\varepsilon v_i)\psi(v_i, t_1, \dots, t_n)[t/v_k] \in \mathrm{mat}((\varepsilon v_i)\varphi)$. Furthermore, by the induction hypothesis

$$t_1[t/v_k]^{(\mathfrak{M},f)}[a] = t_1^{(\mathfrak{M},f)}[a^u_k], \ \dots, \ t_n[t/v_k]^{(\mathfrak{M},f)}[a] = t_n^{(\mathfrak{M},f)}[a^u_k].$$

For short, let $\mu = \mathrm{mat}((\varepsilon v_i)\varphi)$ and $\mu' = \mathrm{mat}((\varepsilon v_i)\psi(v_i, t_1, \dots, t_n)[t/v_k])$. As we know $\mu = \mu'$, thus

$$((\varepsilon v_i)\varphi[t/v_k])^{(\mathfrak{M},f)}[a] =$$

$$\begin{aligned}
&= f\left(\mu', \{m \in M \mid (\mathfrak{M}, f) \models \varphi[t/v_k][a^m_i]\}, (t_1[t/v_k]^{(\mathfrak{M},f)}[a], \dots, t_n[t/v_k]^{(\mathfrak{M},f)}[a])\right)\\
&= f\Big(\mu', \{m \in M \mid (\mathfrak{M}, f) \models \varphi[a^{m\ u}_{i\ k}]\}, (t_1^{(\mathfrak{M},f)}[a^u_k], \dots, t_n^{(\mathfrak{M},f)}[a^u_k])\Big)\\
&= f\Big(\mu, \{m \in M \mid (\mathfrak{M}, f) \models \varphi[a^{u\ m}_{k\ i}]\}, (t_1^{(\mathfrak{M},f)}[a^u_k], \dots, t_n^{(\mathfrak{M},f)}[a^u_k])\Big)\\
&= s^{(\mathfrak{M},f)}[a^u_k].
\end{aligned}$$

□

Here, we do not prove the soundness and completeness property in a straightforward way. Instead, we recall Ahrendt and Giese's construction and we show that the semantics above is the substitutive semantics described in [Ahre].

Consequence 1 (Molnár, 2013)**.** The substitutive Skolem expansions of the first-order model $\mathfrak{M}$ defined in Definition 9 are the Skolem expansions $\mathfrak{M}_f$ defined in Definition 13. [Moln2013]

Proof. The Substitution Lemma states that every $\mathfrak{M}_f$ is substitutive. Let us consider a substitutive Skolem expansion $\mathfrak{N}$ of $\mathfrak{M}$. We construct a function f such that $\mathfrak{N} = \mathfrak{M}_f$. We show that, for an epsilon-term $(\varepsilon x)\varphi$, $(S_{(\exists x)\varphi} v_1 \dots v_k)^{\mathfrak{N}}$ is determined by the Skolem function of the matrix of $(\varepsilon x)\varphi$. Let $(\varepsilon x)\psi(x, t_1, \dots, t_k)$ be the matrix representation of $(\varepsilon x)\varphi$. By substitutivity

$$(S_{(\exists x)\varphi} v_1 \dots v_n)^{\mathfrak{N}}[a] = (S_{(\exists x)\psi(x,t_1,\dots,t_k)} v_1 \dots v_n)^{\mathfrak{N}}[a]$$
$$= (S_{(\exists x)\psi(x,v_{i_1},\dots,v_{i_k})} v_1 \dots v_n)^{\mathfrak{N}} \left[a_{i_1}^{t_1^{\mathfrak{N}}[a]} \dots {}_{i_k}^{t_k^{\mathfrak{N}}[a]}\right]$$

Now, write v_{i_0} instead of x. Let $f : \mathrm{Mat} \times \mathcal{P}(M) \times {}^{<\omega}M \longrightarrow M$ be the following

$$f\,(\mathrm{mat}((\varepsilon v_{i_0})\psi(v_{i_0}, v_{i_1}, \dots, v_{i_k})), S, b) =$$
$$= (S_{(\exists v_{i_0})\psi(v_{i_0},v_{i_1},\dots,v_{i_k})} v_1 \dots v_n)^{\mathfrak{N}} \left[a_{i_1}^{t_1^{\mathfrak{N}}[a]} \dots {}_{i_k}^{t_k^{\mathfrak{N}}[a]}\right]$$

for every matrix $(\varepsilon v_{i_0})\psi(v_{i_0}, v_{i_1}, \dots, v_{i_k})$ and for every (S, b) of the form

$$\left(\left\{m \in M \mid \mathfrak{N} \models \psi(v_{i_0}, v_{i_1}, \dots, v_{i_k}) \left[a_{i_0}^{m}{}_{i_1}^{t_1^{\mathfrak{N}}[a]} \dots {}_{i_k}^{t_k^{\mathfrak{N}}[a]}\right]\right\}, (t_1^{\mathfrak{N}}[a], \dots, t_k^{\mathfrak{N}}[a])\right)$$

where $t_1, \dots, t_k$ are arbitrary terms and a is arbitrary valuation, and a simple choice function otherwise. Finally, by induction it can be shown that f is well-defined and $\mathfrak{M}_f = \mathfrak{N}$. □

Remark 24. The soundness and completeness hold, i.e. for every set of sentences $\Gamma \cup \{\varphi\} \subseteq \mathrm{Sent}(\mathcal{L}_\varepsilon)$

$$\Gamma \underset{\mathrm{EC}_\varepsilon}{\vdash} \varphi \quad \text{iff} \quad \Gamma \underset{\mathrm{Int}}{\models} \varphi.$$

It is shown in [Ahre, Thm. 4, 5] or it can be shown by the Henkin construction of canonic term model.

The *validity* concerning classes can be defined as follows.

Definition 14. Let φ be a sentence in the language $\mathcal{L}_\varepsilon$ of similarity type t. If K is a class of first-order models of type t, then

$$\mathsf{K} \underset{\mathrm{Int}}{\models} \varphi \quad \overset{\text{def}}{\Longleftrightarrow} \quad (\mathfrak{M}, f) \models \varphi \text{ for every } (\mathfrak{M}, f) \in \mathsf{Int}(\mathcal{L}_\varepsilon) \text{ such that } \mathfrak{M} \in \mathsf{K}.$$

3.5. **Extensional Semantics.** The semantics above is the *intensional semantics*, however, it is not an intensional system in the sense of modal logic, and it is not only intensional but also substitutive in the sense of [Ahre, Def. 5]. The term was introduced in contrast to the following class of epsilon-models.

Definition 15 (Extensional models)**.** Let $\mathcal{L}_\varepsilon$ be the language of the elementary calculus with epsilon.

$$\mathsf{Ext}(\mathcal{L}_\varepsilon) = \{(\mathfrak{M}, f) \in \mathsf{Int}(\mathcal{L}_\varepsilon) \mid (\forall t_1, t_2 \in \mathrm{Mat})(\forall s_1, s_2 \in {}^{<\omega}M)\ f(t_1, ., s_1) = f(t_2, ., s_2)\ \}.$$

(Cf. [Monk, Def. 29.23, p. 481] and [Ahre, Def. 6].) The members of $\mathsf{Ext}(\mathcal{L}_\varepsilon)$ are called the *extensional models* or the *dependent choice structures*.

Remark 25. The reason for the name is that the reference of an epsilon-term $(\varepsilon x)\varphi$ depends only on the extension of φ.

Definition 16 ((EEC$_\varepsilon$))**.** On the *extensional epsilon-calculus*, denoted by EEC$_\varepsilon$, we mean EC$_\varepsilon$ with the following axiom schema

$$(\forall x)(\varphi \leftrightarrow \psi)\ \rightarrow\ (\varepsilon x)\varphi = (\varepsilon x)\psi.$$

Remark 26. The extensional semantics is sound and complete with respect to the above calculus [Ahre, Sec. 4.2], that is

$$\Gamma \underset{\mathrm{EEC}_\varepsilon}{\vdash} \varphi \quad \text{iff} \quad \Gamma \underset{\mathrm{Ext}}{\models} \varphi$$

for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Sent}(\mathcal{L}_\varepsilon)$. Here $\Gamma \underset{\mathrm{Ext}}{\models} \varphi$ denotes that the sentence φ is valid in every extensional model $(\mathfrak{M}, f)$ which is a model of the set of sentences Γ.

Definition 17 (Epsilon-invariant formula)**.** The formula $\varphi \in \mathrm{Form}(\mathcal{L}_\varepsilon)$ is *epsilon-invariant* if for all $(\mathfrak{M}, f), (\mathfrak{M}, g) \in \mathsf{Ext}(\mathcal{L}_\varepsilon)$ and for all $a \in {}^{\omega}M$

$$(\mathfrak{M}, f) \models \varphi[a] \quad \text{iff} \quad (\mathfrak{M}, g) \models \varphi[a].$$

Blass and Gurevich proved the following theorem concerning epsilon-invariance.

Theorem 5 (Blass–Gurevich–Caicedo)**.** If a formula $\varphi \in \mathrm{Form}(\mathcal{L}_\varepsilon)$ is epsilon-invariant, then there is a first-order (epsilon-free) formula $\psi \in \mathrm{Form}(\mathcal{L}_\exists)$ such that

$$\varphi^{(\mathfrak{M},f)} = \psi^{\mathfrak{M}}$$

holds for all $(\mathfrak{M}, f) \in \mathsf{Ext}(\mathcal{L}_\varepsilon)$. (Cf. [Blas, Prop. 3.2])

Remark 27. Its proof first presented in [Caic], but [Blas] claims before the theorem (Prop. 3.2.) that the proposition is a folklore, and "it is mentioned in [Caic] without a reference".

Remark 28. In a sense, the theorem states that the epsilon-invariant formulas are the formulas in which the ε operators are used solely for quantification. However, following their proof there is hardly any chance to find the first-order formula ψ above, since it uses Craig's non-constructive interpolation theorem.

Epsilon-invariance might be introduced in a more specific way.

Definition 18 (Epsilon-invariance over a class)**.** The formula $\varphi \in \mathrm{Form}(\mathcal{L}_\varepsilon)$ is *epsilon-invariant over the class* K *of first-order models*, if for every model $\mathfrak{M} \in \mathsf{K}$, for every $a \in {}^{\omega}M$ and for all choice functions f and g such that $(\mathfrak{M}, f), (\mathfrak{M}, g) \in \mathsf{Ext}(\mathcal{L}_\varepsilon)$

$$(\mathfrak{M}, f) \models \varphi[a] \quad \text{iff} \quad (\mathfrak{M}, g) \models \varphi[a]$$

holds. (Cf. [Otto, Def. 1], [Moln2013])

Remark 29. The concept above is crucial in [Otto], where it is shown that epsilon-languages are more expressive than first-order languages over the class of all finite models.

The following proposition is a simple fact about the epsilon-invariance and the intensionally valid sentences.

Proposition 4 (Molnár 2013)**.** Let K be a set of first-order models and let $\varphi \in \mathrm{Sent}(\mathcal{L}_\varepsilon)$. If $\mathsf{K} \underset{\mathrm{Int}}{\models} \varphi$, then φ is epsilon-invariant over the class K.

Proof. Let $\mathfrak{M} \in \mathsf{K}$, $a \in {}^{\omega}M$ and let f and g be such that $(\mathfrak{M}, f), (\mathfrak{M}, g) \in \mathsf{Ext}(\mathcal{L}_\varepsilon)$. $(\mathfrak{M}, f) \models \varphi[a]$ and $(\mathfrak{M}, g) \models \varphi[a]$ holds, since f and g, as extensional choice functions, are also intensional choice functions. Hence, φ is epsilon-invariant over K. □

3.6. **Proofs.** Let us recall sentence (M) from Section 3.1 and let us introduce a notation for its first-order form.

Definition 19. Let

$$\mathsf{InvSub}(\varphi, \vartheta)$$

denotes the formula

$$(\,(\exists x)\varphi \wedge (\forall x)(\varphi \to \vartheta)\,) \vee (\,\neg(\exists x)\varphi \wedge (\forall x)\vartheta\,).$$

Our aim is to show, step by step, a meta-equivalence like

$$\vdash \vartheta[(\varepsilon x)\varphi/x] \quad \text{iff} \quad \vdash \mathsf{InvSub}(\varphi, \vartheta)$$

without semantic conditions.

First, note that, in the epsilon-calculus, $\mathsf{InvSub}(\varphi,\vartheta)$ implies $\vartheta[(\varepsilon x)\varphi/x]$ without any assumptions.

Proposition 5 (Molnár 2013)**.** If φ and ϑ are monadic formulas of the variable x, then

$$\underset{\mathrm{EC}\varepsilon}{\vdash} \vartheta[(\varepsilon x)\varphi/x] \;\leftarrow\; \mathsf{InvSub}(\varphi,\vartheta)$$

Proof. First, let us suppose $(\exists x)\varphi$ and $(\forall x)(\varphi \to \vartheta)$. By the rules of epsilon-calculus, we have

$$\{(\exists x)\varphi\} \underset{\mathrm{EC}\varepsilon}{\vdash} \varphi[(\varepsilon x)\varphi/x] \quad \text{and}$$
$$\{(\exists x)\varphi, (\forall x)(\varphi \to \vartheta)\} \underset{\mathrm{EC}\varepsilon}{\vdash} \vartheta[(\varepsilon x)\varphi/x].$$

This yields $\underset{\mathrm{EC}\varepsilon}{\vdash} ((\exists x)\varphi \wedge (\forall x)(\varphi \to \vartheta)) \;\to\; \vartheta[(\varepsilon x)\varphi/x]$.
Second, suppose $\neg(\exists x)\varphi$ and $(\forall x)\vartheta$. Then

$$\{(\forall x)\vartheta\} \underset{\mathrm{EC}\varepsilon}{\vdash} \vartheta[(\varepsilon x)\varphi/x]$$

implies $\underset{\mathrm{EC}\varepsilon}{\vdash} (\neg(\exists x)\varphi \wedge (\forall x)\vartheta) \;\to\; \vartheta[(\varepsilon x)\varphi/x]$. Using the method of proof by cases it follows that the proposition holds. □

In order to prove a couple of meta-equivalences between $\vartheta[(\varepsilon x)\varphi/x]$ and $\mathsf{InvSub}(\varphi,\vartheta)$ we need some new syntactic and semantic notions. In the epsilon-language the terms and formulas are defined simultaneously and they are called *well-formed expressions*, or *wf expressions* for short. Every wf expression α has at least one *wf expression construction* $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, where $\alpha_n = \alpha$ and the wf expression α_i is generated by the previous ones by the operators $\neg, \vee, \varepsilon, \mathsf{t}$ following the inductive definition of the terms and formulas (for every $1 \leq i \leq n$).

Definition 20. The wf expression α *omits the set* $\mathrm{mat}((\varepsilon x)\varphi)$ if α has a wf expression construction $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ such that

$$\{\alpha_i \mid 1 \leq i \leq n\} \cap \mathrm{mat}((\varepsilon x)\varphi) = \emptyset.$$

[Moln2013]

Definition 21. Let K be a class of first-order models of type t. The formula ϑ is *epsilon-invariant in* $(\varepsilon x)\varphi$ *over* K, if for all models $\mathfrak{M} \in \mathsf{K}$ and for every choice function f, g such that $(\mathfrak{M}, f), (\mathfrak{M}, g) \in \mathsf{Int}(\mathcal{L}_\varepsilon)$, $f = g$ on the set

$(\mathrm{Mat} \setminus \{\mathrm{mat}((\varepsilon x)\varphi)\}) \times \mathcal{P}(M) \times {}^{<\omega}M$ and for every $a \in {}^{\omega}M$

$$(\mathfrak{M}, f) \models \vartheta[a] \text{ iff } (\mathfrak{M}, g) \models \vartheta[a].$$

[Moln2013]

The former concept 'an expression omitting the set $\mathrm{mat}((\varepsilon x)\varphi)$' is the same as 'the matrix does not occur in an expression' which is defined in [Mose]. Here, we prefer an exact definition that recall the structure of a wf expression by mentioning the construction of the wf expression. The latter concept is a weak version of epsilon-invariance. Clearly, if a formula is epsilon-invariant over a class K, then it is epsilon-invariant in every epsilon-term over the class K. Now we continue with a lemma.

Lemma 2. Let the wf expression α omit the set $\mathrm{mat}((\varepsilon x)\varphi)$ and let $(\mathfrak{M}, f), (\mathfrak{M}, g) \in \mathsf{Int}(\mathcal{L}_\varepsilon)$. If $f(m, S, s) = g(m, S, s)$ for every $m \in \mathrm{Mat} \setminus \{\mathrm{mat}((\varepsilon x)\varphi)\}$, $S \in \mathcal{P}(M)$, and $s \in {}^{<\omega}M$, then

$$\alpha^{(\mathfrak{M},f)} = \alpha^{(\mathfrak{M},g)}.$$

Proof. The proof goes by structural induction. The crucial case is that of the epsilon-terms. Let $(\varepsilon v_i)\vartheta$ be an epsilon-term.

$$\begin{aligned} ((\varepsilon v_i)\vartheta)^{(\mathfrak{M},f)}[a] &= f\left(\mathrm{mat}((\varepsilon v_i)\vartheta), \{u \in M \mid (\mathfrak{M}, f) \models \vartheta[a_i^u]\}, s\right) \\ &= g\left(\mathrm{mat}((\varepsilon v_i)\vartheta), \{u \in M \mid (\mathfrak{M}, f) \models \vartheta[a_i^u]\}, s\right) \qquad [*] \\ &= ((\varepsilon v_i)\vartheta)^{(\mathfrak{M},g)}[a]. \end{aligned}$$

If the formula ϑ above is epsilon-free, then step * is obviously valid, if ϑ is not epsilon-free, then in step * we can use the induction hypothesis. □

Now we prove the formula $\vartheta[(\varepsilon x)\varphi/x] \leftrightarrow \mathsf{InvSub}(\varphi, \vartheta)$ in a given model. The phrase "ψ is epsilon-invariant in $(\varepsilon x)\varphi$ over the model $\mathfrak{M}$" means ψ is epsilon-invariant in $(\varepsilon x)\varphi$ over the class $\{(\mathfrak{M}, f) \in \mathsf{Int}(\mathcal{L}_\varepsilon) \mid f \in \mathrm{pr}_2\, \mathsf{Int}(\mathcal{L}_\varepsilon)\}$ with the *fixed* model $\mathfrak{M}$.

Proposition 6 (Molnár 2013)**.** Let φ and ϑ be monadic predicates of the variable x. If the formulas ϑ and φ omit the set $\mathrm{mat}((\varepsilon x)\varphi)$, and $\vartheta[(\varepsilon x)\varphi/x]$ is epsilon-invariant in $(\varepsilon x)\varphi$ over the model $\mathfrak{M}$, then for every f such that $(\mathfrak{M}, f) \in \mathsf{Int}(\mathcal{L}_\varepsilon)$

$$(\mathfrak{M}, f) \models \vartheta[(\varepsilon x)\varphi/x] \to \mathsf{InvSub}(\varphi, \vartheta)$$

holds.

Proof. Let us suppose that $(\mathfrak{M}, f) \models \vartheta[(\varepsilon x)\varphi/x]$. At first, we will prove that

$$(\mathfrak{M}, f) \models (\forall x)\neg\varphi \text{ implies } (\mathfrak{M}, f) \models (\forall x)\vartheta,$$

then we will prove that

$$(\mathfrak{M}, f) \models (\exists x)\varphi \text{ implies } (\mathfrak{M}, f) \models (\forall x)(\varphi \longrightarrow \vartheta).$$

(1) Let us suppose that $(\mathfrak{M}, f) \models (\forall x)\neg\varphi$. It is needed to prove that $(\mathfrak{M}, f) \models (\forall x)\vartheta$, which is equivalent to

$$(\mathfrak{M}, f) \models \vartheta[(\varepsilon x)(\neg\vartheta)/x].$$

It is clear that $(\mathfrak{M}, f) \models (\forall x)\neg\varphi$ implies $\varphi^{(\mathfrak{M},f)} = \varnothing$, since, for every valuation a

$$\begin{aligned}(\mathfrak{M}, f) \models ((\forall x)\neg\varphi)[a] &\text{ iff } (\mathfrak{M}, f) \models (\neg\varphi[(\varepsilon x)\varphi/x])[a] \\ &\text{ iff } (\mathfrak{M}, f) \models (\neg\varphi)\left[((\varepsilon x)\varphi)^{(\mathfrak{M},f)}[a]\right] \quad \text{[Sub. Lem.]} \\ &\text{ iff } (\mathfrak{M}, f) \not\models \varphi\left[((\varepsilon x)\varphi)^{(\mathfrak{M},f)}[a]\right] \\ &\text{ iff } ((\varepsilon x)\varphi)^{(\mathfrak{M},f)}[a] \notin \varphi^{(\mathfrak{M},f)}[a] \\ &\text{ iff } \varphi^{(\mathfrak{M},f)} = \varnothing.\end{aligned}$$

Let $(\varepsilon x)\psi(x, t_1, \ldots, t_n) = (\varepsilon x)\varphi$ where $(\varepsilon x)\psi(x, w_1, \ldots, w_n)$ is the matrix of $(\varepsilon x)\varphi$ and $t_1, \ldots, t_n$ are its exterior epsilon-terms. Let $b = ((\varepsilon x)(\neg\vartheta))^{(\mathfrak{M},f)}$ and let g be such that $(\mathfrak{M}, g) \in \mathsf{Int}(\mathcal{L}_\varepsilon)$, and for every $(m, S, s) \in \mathrm{Mat} \times \mathcal{P}(M) \times {}^{<\omega}M$,

$$g(m, S, s) = \begin{cases} b & \text{if } (m, S, s) = (\mathrm{mat}((\varepsilon x)\varphi), \emptyset, (t_1^{(\mathfrak{M},f)}, \ldots, t_n^{(\mathfrak{M},f)})) \\ f(m, S, s) & \text{otherwise}\end{cases}$$

By the previous lemma,

$$\begin{aligned}((\varepsilon x)\varphi)^{(\mathfrak{M},g)} &= g(\mathrm{mat}((\varepsilon x)\varphi), \varphi^{(\mathfrak{M},g)}, (t_1^{(\mathfrak{M},g)}, \ldots, t_n^{(\mathfrak{M},g)})) \\ &= g(\mathrm{mat}((\varepsilon x)\varphi), \varphi^{(\mathfrak{M},f)}, (t_1^{(\mathfrak{M},f)}, \ldots, t_n^{(\mathfrak{M},f)})) \qquad \text{[prev. lem.]} \\ &= g(\mathrm{mat}((\varepsilon x)\varphi), \emptyset, (t_1^{(\mathfrak{M},f)}, \ldots, t_n^{(\mathfrak{M},f)})) \\ &= b.\end{aligned}$$

Therefore

$$\begin{array}{rlll}
(\mathfrak{M}, f) \models \vartheta[(\varepsilon x)\varphi/x] & \text{iff} & (\mathfrak{M}, g) \models \vartheta[(\varepsilon x)\varphi/x] & \text{[eps.-inv.]} \\
& \text{iff} & ((\varepsilon x)\varphi)^{(\mathfrak{M},g)} \in \vartheta^{(\mathfrak{M},g)} & \text{[Sub. Lem.]} \\
& \text{iff} & b \in \vartheta^{(\mathfrak{M},g)} & \text{[def. of } g\text{]} \\
& \text{iff} & b \in \vartheta^{(\mathfrak{M},f)} & \text{[prev. lem.]} \\
& \text{iff} & ((\varepsilon x)(\neg\vartheta))^{(\mathfrak{M},f)} \in \vartheta^{(\mathfrak{M},f)} & \text{[def. of } b\text{]} \\
& \text{iff} & (\mathfrak{M}, f) \models \vartheta[(\varepsilon x)(\neg\vartheta)/x] & \text{[Sub. Lem.]}
\end{array}$$

Hence, $(\mathfrak{M}, f) \models (\forall x)\vartheta$.

(2) Let us suppose that $(\mathfrak{M}, f) \models (\exists x)\varphi$. We will show that

$$(\mathfrak{M}, f) \models (\forall x)(\varphi \longrightarrow \vartheta)$$

which is equivalent to

$$(\mathfrak{M}, f) \models \varphi[(\varepsilon x)(\neg(\varphi \to \vartheta))/x] \longrightarrow \vartheta[(\varepsilon x)(\neg(\varphi \to \vartheta))/x].$$

For the sake of simplicity, let us denote $(\varepsilon x)(\neg(\varphi \to \vartheta))$ by t. Let us suppose that

$$(\mathfrak{M}, f) \models \varphi[t/x].$$

Now, we let $b = t^{(\mathfrak{M},f)}$ and let g be the following function: for every $(m, S, s) \in \mathrm{Mat} \times \mathcal{P}(M) \times {}^{<\omega}M$,

$$g(m, S, s) = \begin{cases} b & \text{if } (m, S, s) = (\mathrm{mat}((\varepsilon x)\varphi), \varphi^{(\mathfrak{M},f)}, (t_1^{(\mathfrak{M},f)}, \dots, t_n^{(\mathfrak{M},f)})) \\ f(m, S, s) & \text{otherwise} \end{cases}$$

Hence, $((\varepsilon x)\varphi)^{(\mathfrak{M},g)} = b$. By the Substitution Lemma and the assumption, it is clear that $b \in \varphi^{(\mathfrak{M},f)}$. All these imply the following

$$\begin{aligned}
(\mathfrak{M}, f) \models \vartheta[(\varepsilon x)\varphi/x] \quad &\text{iff} \quad (\mathfrak{M}, g) \models \vartheta[(\varepsilon x)\varphi/x] && \text{[eps.-inv.]}\\
&\text{iff} \quad ((\varepsilon x)\varphi)^{(\mathfrak{M},g)} \in \vartheta^{(\mathfrak{M},g)} && \text{[Sub. Lem.]}\\
&\text{iff} \quad b \in \vartheta^{(\mathfrak{M},g)} && \text{[def. of } g\text{]}\\
&\text{iff} \quad b \in \vartheta^{(\mathfrak{M},f)} && \text{[prev. lem.]}\\
&\text{iff} \quad t^{(\mathfrak{M},f)} \in \vartheta^{(\mathfrak{M},f)} && \text{[def. of } b\text{]}\\
&\text{iff} \quad (\mathfrak{M}, f) \models \vartheta[t/x] && \text{[Sub. Lem.]}
\end{aligned}$$

Hence, $(\mathfrak{M}, f) \models (\varphi \longrightarrow \vartheta)[t/x]$ and finally

$$(\mathfrak{M}, f) \models (\forall x)(\varphi \longrightarrow \vartheta)$$

holds. □

To do the next step, we omit the epsilon-invariance condition by turning to the meta-level.

Corollary 1. Let the monadic formulas φ and ϑ of the variable x omit the set $\mathrm{mat}((\varepsilon x)\varphi)$, and let K be a class of first-order models.

(1) If $\vartheta[(\varepsilon x)\varphi/x]$ is epsilon-invariant in $(\varepsilon x)\varphi$ over K, then

$$\mathsf{K} \underset{\mathrm{Int}}{\models} \vartheta[(\varepsilon x)\varphi/x] \leftrightarrow \mathsf{InvSub}(\varphi, \vartheta).$$

(2)

$$\mathsf{K} \underset{\mathrm{Int}}{\models} \vartheta[(\varepsilon x)\varphi/x] \quad \text{iff} \quad \mathsf{K} \underset{\mathrm{Int}}{\models} \mathsf{InvSub}(\varphi, \vartheta).$$

Proof. (1) is implied by Proposition 6. (2) By Proposition 4, $\mathsf{K} \underset{\mathrm{Int}}{\models} \vartheta[(\varepsilon x)\varphi/x]$ implies that $\vartheta[(\varepsilon x)\varphi/x]$ is epsilon-invariant in $(\varepsilon x)\varphi$ over K. Hence, $\mathsf{K} \underset{\mathrm{Int}}{\models} \mathsf{InvSub}(\varphi, \vartheta)$ follows from (1). The opposite direction holds by Proposition 5. □

Finally, let us introduce a notation and prove the main theorem. Let Γ be a set of epsilon-invariant sentences. According to the Blass–Gurevich–Caicedo Theorem, for every $\varphi \in \Gamma$ there is a first-order reformulation ψ preserving the validity. Let Γ' be the set of all such reformulations of the members of Γ and let $\mathsf{Mod}^{\mathrm{FO}}(\Gamma)$ be the class of all first-order models of Γ'. Note that, for every choice function f sending elements into M, if $\mathfrak{M} \in \mathsf{Mod}^{\mathrm{FO}}(\Gamma)$, then $(\mathfrak{M}, f) \models \Gamma$.

Theorem 6 (Molnár 2013)**.** If the monadic formulas φ and ϑ of the variable x omit the set $\mathrm{mat}((\varepsilon x)\varphi)$ and Γ consists of epsilon-invariant sentences, then

$$\Gamma \underset{\mathrm{EC}\varepsilon}{\vdash} \vartheta[(\varepsilon x)\varphi/x] \quad \text{iff} \quad \Gamma \underset{\mathrm{EC}\varepsilon}{\vdash} \mathsf{InvSub}(\varphi, \vartheta).$$

Proof. Let us set $\mathsf{K} = \mathrm{Mod}^{\mathrm{FO}}(\Gamma)$, use Corollary 1.2, and apply the completeness property. □

If we consider plain first-order sentences for the elements of Γ, then Theorem 6 becomes a non-model-theoretic (actually syntactic) proposition on the meaning of epsilon-substitutions.

4. The Extensional Semantics and the Algebraic Approach

One of the benefit properties implied by the extensionality axiom of Hilbert's epsilon calculus is that the calculus becomes complete with respect to the choice structures as semantics. Another implication of the axiom, discussed in the section, is that an algebra is generated over the universe of the canonical model of a theory, which is isomorphic to a quotient algebra of the Lindenbaum–Tarski algebra of the theory. Especially, in the case of Boolean or monadic algebras, the canonical model of the theory of a finite model is isomorphic to the algebra induced by the axiom of extensionality.

4.1. **Canonical models of extensional epsilon calculi.** In order to establish an appropriate environment for the algebraic approach, we deal with the canonical model of a complete and consistent theory in an epsilon language [Hodg, p. 18]. Let $\mathtt{t} = (r_i, f_j, c_k)_{(i,j,k)}$ be a similarity type and let $\mathcal{L}_\varepsilon$ be the freely generated language with respect to the operations $\neg, \vee, \varepsilon, \mathtt{t}$. Let $\Gamma \subseteq \mathrm{Sent}(\mathcal{L}_\varepsilon)$ be a complete and consistent set of sentences in the extensional epsilon calculus. The pair $(\mathfrak{M}, f) \in \mathrm{Ext}(\mathcal{L}_\varepsilon)$ which is going to be defined, will be a *choice structure* or an *extesional model* ([Monk, p. 481] or Section 3 above). Consider the set of epsilon terms of the single-variable formulae

$$\mathrm{Eps}_1 = \{(\varepsilon v_i)\varphi \mid v_i \in \mathrm{Var}(\mathcal{L}_\varepsilon) \quad \text{and} \quad \varphi \in \mathrm{Fm}_{v_i}(\mathcal{L}_\varepsilon)\}$$

and the equivalence relation $=_\Gamma$ over the set Eps_1 as follows

$$t_1 =_\Gamma t_2 \quad \Longleftrightarrow \quad \Gamma \vdash t_1 = t_2$$

for all $t_1, t_2 \in \mathrm{Tm}(\mathcal{L}_\varepsilon)$. Let the universe of the model $\mathfrak{M}$ be the set

$$M = \mathrm{Eps}_1/_{=_\Gamma}$$

and let the interpretations of the relation, function and constant sings respectively be

$$\begin{array}{rcl} (a_1/_{=_\Gamma}, ..., a_l/_{=_\Gamma}) \in r_i^{\mathfrak{M}} & \Leftrightarrow & \Gamma \vdash r_i(a_1/v_1, ..., a_l/v_l) \\ f_j^{\mathfrak{M}}(a_1/_{=_\Gamma}, ..., a_m/_{=_\Gamma}) & = & (\varepsilon v_0)(v_0 = f_j(a_1/v_1, ..., a_m/v_m)/_{=_\Gamma} \\ c_k^{\mathfrak{M}} & = & (\varepsilon v_0)(v_0 = c_k)/_{=_\Gamma} \end{array}$$

The interpretations are well-defined, since they are independent of the choice of the representants. Then – as it is well known from [Monk] – the axiom of extensionality and the axiom of transfinity allow us to define the *canonical model.* Furthermore, this is a unique structure in the sense of the following proposition.

For the sake of simplicity we set

$$\varphi(v_i)^\Gamma = \{(t/_{=_\Gamma}) \in M \mid \Gamma \vdash \varphi[t/v_i]\}$$

where $\varphi \in \mathrm{Fm}_{v_i}(\mathcal{L}_\varepsilon)$.

Proposition 7 (Molnár, 2011)**.** Let M, $\mathfrak{M}$, Γ be as above.

(a) For all $\varphi, \psi \in \mathrm{Fm}_{v_i}(\mathcal{L}_\varepsilon)$

$$\text{if } \varphi(v_i)^\Gamma = \psi(v_i)^\Gamma \text{ then } ((\varepsilon v_i)\varphi)/_{=_\Gamma} \;=\; ((\varepsilon v_i)\psi)/_{=_\Gamma}$$

(b) There is a choice function f such that for all $\varphi \in \mathrm{Fm}(\mathcal{L}_\varepsilon)$, $t \in \mathrm{Tm}(\mathcal{L}_\varepsilon)$ and valuation $a = (a_1/_{=_\Gamma}, a_2/_{=_\Gamma}, \ldots)$ of $\mathfrak{M}$

$$(\mathfrak{M}, f) \models \varphi[a] \quad \text{iff} \quad \Gamma \vdash \varphi[a_1/v_1, \ldots, a_n/v_n]$$

and

$$\Gamma \vdash t^{\mathfrak{M}f}[a] = t[a_1/v_1, \ldots, a_n/v_n]$$

moreover, on the set

$$\{\{(t/_{=_\Gamma}) \in M \mid \Gamma \vdash \varphi[t/v_i]\} \mid \varphi \in \mathrm{Fm}_{v_i}(\mathcal{L}_\varepsilon)\}$$

f is unique [Moln2011].

Proof. (a) Let $s \mathrel{\mathring{=}} (\varepsilon v_i)\neg(\varphi \leftrightarrow \psi)$. Since $s \in \mathrm{Eps}_1$ then by $\varphi(v_i)^\Gamma = \psi(v_i)^\Gamma$ we have $\Gamma \vdash \varphi[s/v_i] \leftrightarrow \psi[s/v_i]$. According to the definition of the universal quantifier

$$(\forall v_i)(\varphi \leftrightarrow \psi) \mathrel{\mathring{=}} (\varphi \leftrightarrow \psi)[(\varepsilon v_i)\neg(\varphi \leftrightarrow \psi)/v_i]$$

therefore

$$\Gamma \vdash \varphi[s/v_i] \leftrightarrow \psi[s/v_i] \quad \text{iff} \quad \Gamma \vdash (\varphi \leftrightarrow \psi)[s/v_i] \quad \text{iff} \quad \Gamma \vdash (\forall v_i)(\varphi \leftrightarrow \psi)$$

The formula $(\forall v_i)(\varphi \leftrightarrow \psi) \to (\varepsilon v_i)\varphi = (\varepsilon v_i)\psi$ is an instance of the *axiom of extensionality* hence

$$\Gamma \vdash (\varepsilon v_i)\varphi = (\varepsilon v_i)\psi$$

holds and obviously $((\varepsilon v_i)\varphi)/_{=_\Gamma} \;=\; ((\varepsilon v_i)\psi)/_{=_\Gamma}$.

(b) Let f_* be a choice function such that $f_*(S) \in S$ if $S \in \mathrm{Sb}(M) \setminus \{\emptyset\}$ and $f_*(\emptyset) \in M$. Let $f : \mathrm{Sb}(M) \to M$ be the function

$$f(S) = \begin{cases} ((\varepsilon v_i)\varphi)/_{=_\Gamma} & \text{if } S = \{(t/_{=_\Gamma}) \in M \mid \Gamma \vdash \varphi[t/v_i]\} \text{ for some } \varphi \in \mathrm{Fm}_{v_i}(\mathcal{L}_\varepsilon) \\ f_*(S) & \text{otherwise} \end{cases}$$

If $S \in \mathrm{Sb}(M) \setminus \{\emptyset\}$ then $S = \{(t/_{=_\Gamma}) \in M \mid \Gamma \vdash \varphi[t/v_i]\}$ for some $\varphi \in \mathrm{Fm}_{v_i}(\mathcal{L}_\varepsilon)$. By the axiom of transfinity, we have $\vdash \varphi[t/v_i] \to \varphi[(\varepsilon v_i)\varphi/v_i]$, hence $(\varepsilon v_i)\varphi/_{=_\Gamma} \in S$, f is also a choice function and $(\mathfrak{M}, f)$ is a choice structure in the language $\mathcal{L}_\varepsilon$. The uniqueness can easily be shown by structural induction. □

Definition 22. We define the *canonical injection* η of the complete and consistent theory Γ. Let $(\mathfrak{N}, g)$ be a model of Γ and let us denote the canonical model of Γ by $\mathfrak{Can}(\Gamma)$ and its universe by $\mathrm{Can}(\Gamma)$ then the canonical injection is

$$\eta : \mathrm{Can}(\Gamma) \to N,\ \eta((\varepsilon v_i)\varphi)^{\mathfrak{Can}\,\Gamma} = ((\varepsilon v_i)\varphi)^{\mathfrak{N}g}$$

Proposition 8. If $(\mathfrak{N}, g)$ is a choice structure then the canonical injection η : $\mathrm{CanTh}(\mathfrak{N}, g) \to N$ is an elementary embedding from $\mathfrak{Can}\mathrm{Th}(\mathfrak{N}, g)$ to $(\mathfrak{N}, g)$.

Proof. η is a well-defined injection. Indeed, let us denote $\mathrm{Th}(\mathfrak{N}, g)$ by Γ and let $t/_{=_\Gamma}$, $s/_{=_\Gamma} \in \mathrm{Can}\,\mathfrak{N}g$. We check the definition of $\mathfrak{Can}\,\mathfrak{N}g$,

$$t^{\mathfrak{Can}\mathrm{Th}(\mathfrak{N},g)} = s^{\mathfrak{Can}\mathrm{Th}(\mathfrak{N},g)} \quad \text{iff} \quad \mathrm{Th}(\mathfrak{N}, g) \vdash t = s$$

holds, therefore if $t/_{=_\Gamma} = s/_{=_\Gamma}$, then $t^{\mathfrak{N}g} = s^{\mathfrak{N}g}$. Conversly, if $t^{\mathfrak{N}g} \neq s^{\mathfrak{N}g}$, then $(\mathfrak{N}, g) \models t \neq s$ and $\mathrm{Th}(\mathfrak{N}, g) \vdash t \neq s$.

η is an elementary embedding. By definition, let $\varphi \in \mathrm{Fm}(\mathcal{L}_\varepsilon)$ and $a = (a_1/=_\Gamma, a_2/=_\Gamma, ...) \in \mathrm{Val}(\mathfrak{Can}\mathrm{Th}(\mathfrak{N}, g))$, then

$$\begin{aligned} \mathfrak{Can}\mathrm{Th}(\mathfrak{N}, g) \models \varphi[a] \quad &\text{iff} \quad \mathrm{Th}(\mathfrak{N}, g) \vdash \varphi[a_1, ..., a_n/x_1, ..., x_n] \\ &\text{iff} \quad (\mathfrak{N}, g) \models \varphi[a_1, ..., a_n/x_1, ..., x_n] \\ &\text{iff} \quad (\mathfrak{N}, g) \models \varphi[\eta \circ a] \end{aligned}$$

□

If $(\mathfrak{N}_1, g_1)$ and $(\mathfrak{N}_2, g_2)$ are $\mathcal{L}_\varepsilon$-models and $h : N_1 \to N_2$ is a homomorhism such that

$$\mathrm{Ran}(h \restriction \mathrm{Eps}_1^{\mathfrak{N}_1 g_1}) \subseteq \mathrm{Eps}_1^{\mathfrak{N}_2 g_2}$$

then let us define the function $h^* : \mathrm{CanTh}(\mathfrak{N}_1, g_1) \to \mathrm{CanTh}(\mathfrak{N}_2, g_2)$ by the relation

$$h^*(t^{\mathfrak{CanTh}(\mathfrak{N}_1,g_1)}) = s^{\mathfrak{CanTh}(\mathfrak{N}_2,g_2)} \quad \text{whenever} \quad h(t^{\mathfrak{N}_1 g_1}) = s^{\mathfrak{N}_2 g_2}$$

Proposition 9 (Molnár 2011)**.** Let $(\mathfrak{N}_1, g_1)$ and $(\mathfrak{N}_2, g_2)$ be $\mathcal{L}_\varepsilon$-models and let $h : (\mathfrak{N}_1, g_1) \to (\mathfrak{N}_2, g_2)$ be a homomorphism such that

$$\mathrm{Ran}(h \restriction \mathrm{Eps}_1^{\mathfrak{N}_1 g_1}) \subseteq \mathrm{Eps}_1^{\mathfrak{N}_2 g_2}.$$

Then the canonical injection η is a natural transformation in the sense that the following diagram commutes

$$\begin{array}{ccc} \mathrm{CanTh}(\mathfrak{N}_1, g_1) & \xrightarrow{h^*} & \mathrm{CanTh}(\mathfrak{N}_2, g_2) \\ \eta_1 \big\downarrow & & \big\downarrow \eta_2 \\ N_1 & \xrightarrow{\quad h \quad} & N_2 \end{array}$$

[Moln2011]

Proof. By the definition of relation h^* it follows that $h^* = \eta_1 \circ h \circ \eta_2^{-1}$, where $\eta_i : \mathrm{CanTh}(\mathfrak{N}_i, g_i) \to N_i$ is the canonical injection. □

4.2. **Canonical term algebras in extensional calculi.** The axiom of extensionality induces an algebra over the universe of the canonical model. Let $(\mathfrak{M}, f)$ be an $\mathcal{L}_\varepsilon$-model then $\mathbf{LT}\,\mathfrak{M}f$ denotes the Lindenbaum–Tarski algebra of the formulae with the universe

$$\mathrm{LT}\,\mathfrak{M}f = \mathrm{Form}(\mathcal{L}_\varepsilon)\big/_{\mathfrak{M}\models.\leftrightarrow.}$$

The next proposition describes the connection between the neat-1-reduct $\mathbf{Nr}_1\,\mathbf{LT}\,\mathfrak{M}f$ of the cylindric algebra $\mathbf{LT}\,\mathfrak{M}f$ and the universe of the canonical model.

If $(\mathfrak{M}, f) \in \mathrm{Mod}(\mathcal{L}_\varepsilon)$ then the function

$$\Phi : \mathrm{Nr}_1\,\mathbf{LT}\,\mathfrak{M}f \to \mathrm{Can}\,\mathfrak{M}f, \quad \Phi(\varphi\big/_{\mathfrak{M}\models.\leftrightarrow.}) = ((\varepsilon v_0)\varphi)\big/_{=_{\mathrm{Th}\,\mathfrak{M}f}}$$

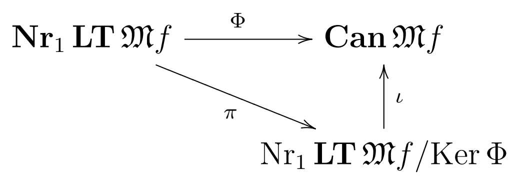
is a *surjection.* Indeed, let $((\varepsilon v)\varphi)\big/_{=_{\mathrm{Th}\,\mathfrak{M}f}} \in \mathrm{Can}\,\mathfrak{M}f$ where $\varphi(v) \in \mathrm{Form}(\mathcal{L}_\varepsilon)$. We can assume that v_0 is free for v in $\varphi(v)$. $\varphi(v_0)\big/_{\mathfrak{M}\models.\leftrightarrow.} \in \mathrm{Nr}_1\,\mathbf{LT}\,(\mathfrak{M}, f)$ and $\mathfrak{M} \models (\varepsilon v)\varphi(v) = (\varepsilon v_0)\varphi(v_0)$.

Definition 23. Let $\mathbf{Can}\,(\mathfrak{M}, f)$ denotes the monadic algebra generated by Φ.

Proposition 10 (Molnár, 2011)**.**

$$\mathbf{Nr}_1\, \mathbf{LT}\, \mathfrak{M}f/\mathrm{Ker}\,\Phi \cong \mathbf{Can}\,\mathfrak{M}f$$

Proof. By the First Isomorphism Theorem there exists an isomorphism ι such that, the following diagram commutes



where π is the canonical projection. □

According to the following fact, the canonical term algebra $\mathbf{Can}(\mathfrak{M}, f)$ (as a monadic algebra) is rich [Halm, p. 77.]. Moreover, the cylindrification c_0 of $\mathbf{Can}\,\mathfrak{M}f$ is a Boolean homomorphism from the Boolean reduct $\mathbf{Can}\,\mathfrak{M}f \restriction \mathcal{L}_\varepsilon^{\mathrm{BA}}$ onto the trivial Boolean algebra $\mathbf{2}$.

Fact 2 (Molnár, 2011)**.** If $(\mathfrak{M}, f)$ is a model, then $\mathbf{Nr}_0\mathbf{Can}\,\mathfrak{M}f \restriction \mathcal{L}_\varepsilon^{\mathrm{BA}} \cong \mathbf{2}$.

4.3. **Canonical term algebras of** BA**s and** CA_1**s.** We know that a Boolean algebra is isomorphic to its second dual, through the Stone-correspondence. The question is what can be said about the relationship with its first dual. More precisely, how does a Boolean algebra as a model relate to the Lindenbau–Tarski Algebra of the given model. When the model $(\mathfrak{M}, f)$ is a finite Boolean algebra, thenthere is a close algebraic relationship between the cylindric set algebra of $(\mathfrak{M}, f)$ and the canonical model $\mathfrak{Can}\,\mathfrak{M}f$. The theorem below describes the main connection.

Theorem 7 (Molnár, 2011)**.** Let $(\mathfrak{B}, g) \in \mathsf{BA}$ and $(\mathfrak{M}, f) \in \mathsf{CA}_1$ such that $\mathfrak{Can}\,\mathfrak{B}g$ and $\mathfrak{Can}\,\mathfrak{M}f$ are finite algebras. Then

(1) $\mathfrak{Can}\,\mathfrak{B}g \cong \mathbf{Can}\,\mathfrak{B}g \restriction \mathcal{L}_\varepsilon^{\mathrm{BA}}$
(2) $\mathfrak{Can}\,\mathfrak{M}f \cong \mathbf{Can}\,\mathfrak{M}f$ iff $\mathbf{Nr}_0\,\mathfrak{Can}\,\mathfrak{M}f \cong \mathbf{2}$

Proof. (1) By definition, universes of $\mathfrak{Can}\,\mathfrak{B}f$ and $\mathbf{Can}\,\mathfrak{B}f \restriction \mathcal{L}_\varepsilon^{\mathrm{BA}}$ are the same. It is known that all finite Boolean algebras with the same cardinality are isomorphic. Cf. [Monk, Corollary 9.32., p. 152.]. Hence,

$$\mathfrak{Can}\,\mathfrak{M}f \restriction \mathcal{L}_\varepsilon^{\mathrm{BA}} \cong \mathbf{Can}\,\mathfrak{M}f \restriction \mathcal{L}_\varepsilon^{\mathrm{BA}}$$

(2) According to (1) the Boolean reducts of $\mathfrak{Can}\,\mathfrak{M}f$ and $\mathbf{Can}\,\mathfrak{M}f$ are isomorphic. By Proposition 2, the cylindrification c_0 is a Boolean homomorphism. Hence, $\mathbf{Nr}_0\,\mathfrak{Can}\,\mathfrak{M}f \cong \mathbf{Nr}_0\,\mathbf{Can}\,\mathfrak{M}f$ iff $\mathbf{Nr}_0\,\mathfrak{Can}\,\mathfrak{M}f \cong \mathbf{2}$. □

As an application, we prove two sufficient conditions concerning the existence of the algebraic connection between the cylindric set algebra of the model and the canonical model.

Proposition 11 (Molnár, 2011)**.** Let $(\mathfrak{B}, g) \in \mathsf{BA}$ and $(\mathfrak{M}, f) \in \mathsf{CA}_1$.

(1) If $(\mathfrak{B}, g)$ is finite, then

$$\mathbf{Cs}\,(\mathfrak{B}, g) \restriction \mathcal{L}_{\varepsilon}^{\mathrm{BA}} / \mathrm{Ker}\,\Phi \cong \mathfrak{Can}\,\mathfrak{B}g$$

(2) If $(\mathfrak{M}, f) \restriction \mathcal{L}_{\varepsilon}^{\mathrm{BA}}$ is finite and $\mathbf{Nr}_0(\mathfrak{M}, f) \cong \mathbf{2}$, then

$$\mathbf{Cs}\,(\mathfrak{M}, f) / \mathrm{Ker}\,\Phi \cong \mathfrak{Can}\,\mathfrak{M}f$$

where $\Phi : \mathrm{Cs}\,(\mathfrak{M}, f) \to \mathrm{Can}\,\mathfrak{M}f, \quad \Phi(\varphi^{\mathfrak{M}f}) = ((\varepsilon v_0)\varphi)\big/_{=_{\mathrm{Th}\,\mathfrak{M}f}}$.

Proof. It is clear that $\mathbf{Cs}\,(\mathfrak{M}, f) \cong \mathbf{LT}\,\mathfrak{M}f$. Finiteness implies that, the canonical model is isomorphic to a subalgebra of $(\mathfrak{M}, f)$ which is a power set algebra [Kopp, p. 31, CH. 1, Corollary 2.7.]. Hence, the canonical model is finite and by Proposition 10 and Theorem 7 it follows that the mentioned quotient algebra is isomorphic to the canonical model. □

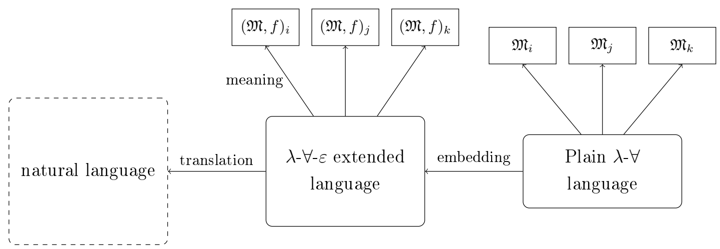
5. Epsilon Logic in Simply Typed Lambda Calculus

This section is about the lambda calculus representation of the extensional semantics of the epsilon calculus. Here I have supplemented the work [Moln2017] with generalizations of the results.

5.1. **Introduction.** For FOL+ε, the quantification reformulation problem has been positively answered in [Moln2013], however with the application of a lot of technical conditions. When one changes FOL to lambda calculus the picture becomes much more clear. The point is that, in FOL the substitution $\psi[x/(\varepsilon x)\varphi]$ is only a meta-language operation, but in the lambda-calculus it is encoded into the object-language via the application MN, where M is an expression of the lambda-language and N is an epsilon-term of the form $(\varepsilon x)P$.

5.1.1. *Hilbert's epsilon and the lambda operator.* In Subsection 5.2, a syntax and semantics will be given for the epsilon symbol in the context of typed lambda calculus (TL). The syntactic notions will be the well-known ones, but in the definitions different way will be followed, based on labeled, ordered trees. Since, by the Curry–Howard Correspondence, TL is closely related to the proof theory of the natural deduction system of propositional logic, we make use of the possibility to define the TL notions of TL syntax the same style as proofs. The form of the definitions will fit this doctrine and a tree-based method will be applied.

In Subsection 5.3, it will be seen that in TL the result can be reached much more faster than in FOL. There is no need to refer to the *intensional and substitutional epsilon semantics.* Note that, Ahrendt and Giese introduced several types of epsilon semantics. See [Ahre, Def. 4,5]. In [Moln2013] the substitutional semantics was applied. Now, in TL the extensional semantics (see [Moln2013, p.821.] or [Monk, Def. 29.23, p. 481]) will be enough. The strategy will be the following. The typed lambda language extended by Hilbert's epsilon ($\mathcal{L}_\lambda^{\forall\varepsilon}$) will be considered as a formal model of the fragment of the natural language containing descriptions. Then, if it is possible, the epsilon expressions will be eliminated and the sentences containing them will be mapped, in an explicit way, to the epsilon-free quantified reduct $\mathcal{L}_\lambda^{\forall}$ of $\mathcal{L}_\lambda^{\forall\varepsilon}$. The plain lambda language reformulation will keep the logical truth in the model. Giving Montague-semantics to the extended language and to the plain epsilon-free language as models (the $(\mathfrak{M}, f)$-s and the $\mathfrak{M}$-s below, respectively) the construction will be unproblematic.



Chain of fragments. The natural language, the λ-∀-ε expressible fragment and the λ-∀ expressible fragment.

In Section 1.4, it was pointed out that the result is not less effective than the RTD proposed by Russell.

5.2. **Syntax and Semantics of Typed Lambda System with Epsilon.**

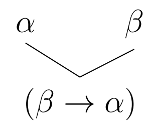
5.2.1. *Syntax.* For building the syntax a tree-based method is chosen (parsing or construction trees), which is much more transparent than the old-fashioned character sequence technique. One thing to note is that here the trees grop upward, as those used by by linguists in Combinatory Categorial Grammar, or, what the main motivation is, in proof theory of the style used in natural deduction.

The definitions below are basically combinations of the well-known ones from [Troe] Sec. 1. and from [Sore].

The so called *typeability relation* ($\vdash$) is a pure syntactic relation that joins the expressions of the lambda calculus to types with respect to a fixed set of typed variables called *context.* The relation $\vdash$ plays a fundamental role in the Curry–Howard Isomorphism, which links the lambda expressions to proof trees of the natural deduction system of the implicational logic.

Definition 24. The *language of types* is the tuple $\mathcal{L}_{\mathrm{Typ}} = \langle \iota, o, \rightarrow, (,) \rangle$. The set of its strings $\mathrm{Srt}(\mathcal{L}_{\mathrm{Typ}})$ contains the finite sequences of the characters from $\{\iota, o, \rightarrow, (,)\}$. A *construction tree* Π of the string $\gamma \in \mathrm{Srt}(\mathcal{L}_{\mathrm{Typ}})$ is a finite, labeled, ordered tree such that the labels of Π are from $\mathrm{Srt}(\mathcal{L}_{\mathrm{Typ}})$ and

(1) the labels of the leaves of Π come from the set $\{\iota, o\}$,
(2) the branch nodes of Π (these are not leaves) and their labels are of the form



(3) the root of Π is γ.

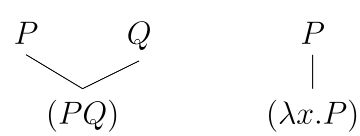
Remark 30. Types serve as categories for the expressions of the language. Type ι denotes the category of individual names, type o denotes the category of sentences (statements). Compound types are the function types or functor-categories. For example, $\iota \to o$ is the functor category of functors which send a name to a sentences, such as the "prime number" predicate. $\iota \to (o \to 0))$ is the category of functors which send a name and a sentence to a sentence, such as "…believes in statement…" structure category.

If there is a tree Π such that Π is a construction tree of $\alpha \in \mathrm{Srt}(\mathcal{L}_{\mathrm{Typ}})$, then α is said to be a *type*. The set of all types in $\mathcal{L}_{\mathrm{Typ}}$ is denoted by Typ. (Cf. [Troe, p. 9] Def. 1.2.1, [Troe, p. 7] Def. 1.1.7.)

Note that the construction tree of a type is unique. The construction tree of the type α is denoted by $\mathrm{Tree}(\alpha)$. The reference to brackets $(,)$ is avoided when a type α is well-known and its construction tree can be completely reconstructed without them.

Definition 25. A *lambda language* is a tuple $\mathcal{L}_\lambda = \langle V, C, (,), ., \lambda\rangle$, where V is an infinite and C is non-empty set and V is disjoint to C. $\mathrm{Srt}(\mathcal{L}_\lambda)$ contains the finite sequences from $V \cup C \cup \{\lambda, (,), .\}$. A *construction tree* Π of the $M \in \mathrm{Srt}(\mathcal{L}_\lambda)$ is a finite, labeled, ordered tree such that the labels of Π are from $\mathrm{Srt}(\mathcal{L}_\lambda)$ and

(1) the labels of the leaves of Π come from the set $V \cup C$,
(2) the branch nodes of Π and their labels are of the form



(3) the root of Π is M.

If there is a tree Π such that Π is a construction tree of $M \in \mathrm{Srt}(\mathcal{L}_\lambda)$, then M is said to be an *expression* in $\mathcal{L}_\lambda$. The set of all expression in $\mathcal{L}_\lambda$ is denoted by $\mathrm{Exp}(\mathcal{L}_\lambda)$.
The elements of V are called the variables of $\mathcal{L}_\lambda$ and V is denoted by $\mathrm{Var}(\mathcal{L}_\lambda)$.

The elements of C are the constants of $\mathcal{L}_\lambda$ and C is denoted by $\mathrm{Const}(\mathcal{L}_\lambda)$. (Cf. [Troe, p. 9] Def. 1.2.2, [Troe, p. 7] Def. 1.1.7.)

Note that the construction tree of an expression is unique. The construction tree of the expression M is denoted by $\mathrm{Tree}(M)$. The *height* of $\mathrm{Tree}(M)$ is defined by the well-known manner and is denoted by $|\mathrm{Tree}(M)|$.
Referring to brackets $(,)$ is avoided when an expression M is known and its construction tree can be uniquely reconstructed without them.

Definition 26. Let $\langle V, C, (,), ., \lambda\rangle$ be a lambda language. The tuple $\mathcal{L}_\lambda = \langle V, C, (,), ., \lambda, Z\rangle$ is a *typed lambda language,* if $Z : C \to \mathrm{Typ}$. The function Z is denoted by $\mathrm{CnstTp}(\mathcal{L}_\lambda)$.

For a typed lambda language $\mathcal{L}_\lambda$ the sets of variables, expressions, contexts etc. defined and denoted by the same manner as for a lambda languages.

Definition 27. Let $\mathcal{L}_\lambda$ be a typed lambda language. A *context* is a finite function $\Gamma = \{(x_1, \varphi_1),\ \dots\ ,\ (x_n, \varphi_n)\}$ where $\{x_i\}_{i=1\dots n} \subseteq \mathrm{Var}(\mathcal{L}_\lambda)$, $\{\varphi_i\}_{i=1\dots n} \subseteq \mathrm{Typ}$. The set of all contexts is denoted by $\mathrm{Cont}(\mathcal{L}_\lambda)$.

Definition 28. Let $\mathcal{L}_\lambda$ be a typed lambda language. The *typeability relation*

$$\vdash \subseteq \mathrm{Cont}(\mathcal{L}_\lambda) \times \mathrm{Exp}(\mathcal{L}_\lambda) \times \mathrm{Typ}$$

is defined as follows.

$$\Gamma \vdash M : \varphi$$

will be defined for every context $\Gamma \in \mathrm{Cont}(\mathcal{L}_\lambda)$, expression $M \in \mathrm{Exp}(\mathcal{L}_\lambda)$ and type φ by recursion on the height of the construction tree of the M expressions as follows.

(1) Let $|\mathrm{Tree}(M)| = 0$.
   - (a) If $c \in \mathrm{Const}(\mathcal{L}_\lambda)$ and Γ is a context, then $\Gamma \vdash c : \varphi$, if $\varphi = \mathrm{CnstTp}(\mathcal{L}_\lambda)(c)$.
   - (b) If $x \in \mathrm{Var}(\mathcal{L}_\lambda)$ and Γ is a context, then $\Gamma \vdash x : \varphi$, if $(x, \varphi) \in \Gamma$.

(2) Let us suppose that $n > 0$ and for every context Δ, type ψ and expression N with $|\mathrm{Tree}(N)| < n$, the relation $\Delta \vdash N : \psi$ is defined. Let Γ be a context, φ a type and M an expression such that $|\mathrm{Tree}(M)| = n$.
   - (a) If $M = PQ$, then $\Gamma \vdash M : \varphi$, if $\Gamma \vdash Q : \beta$ and $\Gamma \vdash P : \beta \to \alpha$ and $\varphi = \alpha$.

(b) If $M = \lambda x.P$. Then $\Gamma \vdash M : \varphi$, if $\Delta \vdash P : \alpha$, $\varphi = \beta \to \alpha$ and $\Gamma = \Delta \setminus \{(x, \beta)\}$. (Cf. [Sore], p. 41, def. 3.1.1.)

For some examples, see [Troe, p. 10], or later in the present paper.

5.2.2. *Montague-semantics.*

Definition 29. Let $M \neq \emptyset$. By recursion on $|\mathrm{Tree}(\varphi)|$, the domain set $D_M(\varphi)$ of type $\varphi \in \mathrm{Typ}$ is defined as follows.

(1) $D_M(o) = \{\mathsf{T}, \mathsf{F}\}$, $\qquad D_M(\iota) = M$

(2) If $D_M(\alpha)$ and $D_M(\beta)$ is defined earlier, then

$$D_M(\alpha(\beta)) = {}^{D_M(\beta)}D_M(\alpha)$$

where ${}^{D_M(\beta)}D_M(\alpha)$ is the function set $\{f : D_M(\beta) \to D_M(\alpha),\, b \mapsto f(b)\}$.

If M is fixed, then $D(\varphi)$ is written instead.

Definition 30. If $M \neq \emptyset$, $\mathcal{L}_\lambda$ is a lambda-language and $(\Xi : \Gamma)$ is a context, then a function $a : \mathrm{Var}(\mathcal{L}_\lambda) \to \cup_{\varphi \in \mathrm{Typ}} D_M(\varphi)$ is an *pre-assignment* of the variables. The pre-assignment a is an *assignment of the type* Γ, if for every variable $x \in \mathrm{dom}\,\Gamma$, $a(x) \in D_M(\alpha)$ whenever $(x, \alpha) \in \Gamma$.

Definition 31. Let $\mathcal{L}_\lambda$ be a typed lambda-language, $M \neq \emptyset$. The tuple $\mathfrak{M} = \langle M, \mathrm{Ip}^{\mathfrak{M}} \rangle$ is a *model* over the language $\mathcal{L}_\lambda$, if $\mathrm{Ip}^{\mathfrak{M}} : \mathrm{CnstTp}(\mathcal{L}_\lambda) \to \cup_{\varphi \in \mathrm{Typ}} D_M(\varphi)$ such that $\mathrm{Ip}^{\mathfrak{M}}(c) \in D_M(\mathrm{CnstTp}(\mathcal{L}_\lambda)(c))$.

Definition 32. Let $\mathcal{L}_\lambda$ be a typed lambda-language, $\mathfrak{M} = \langle M, \mathrm{Ip}^{\mathfrak{M}} \rangle$ a model over the language $\mathcal{L}_\lambda$, Γ a context and a an assignment of type Γ. Suppose that for $N \in \mathrm{Exp}(\mathcal{L}_\lambda)$ there is a type φ such that $\Gamma \vdash N : \varphi$. By recursion on $|\mathrm{Tree}(N)|$ the semantic value $[\![N]\!]^{\mathfrak{M}}_a$ in context Γ is defined as follows.

(1) If $N = c \in \mathrm{Const}(\mathcal{L}_\lambda)$, then

$$[\![c]\!]^{\mathfrak{M}}_a = \mathrm{Ip}^{\mathfrak{M}}(c).$$

(2) If $N = x \in \mathrm{Var}(\mathcal{L}_\lambda)$, then

$$[\![x]\!]^{\mathfrak{M}}_a = a(x).$$

(3) If $N = PQ$, then

$$[\![PQ]\!]^{\mathfrak{M}}_a = [\![P]\!]^{\mathfrak{M}}_a([\![Q]\!]^{\mathfrak{M}}_a).$$

(4) Suppose $N = \lambda x.P$ and
let the assignment $a[x \to \xi]$ be the following:

$$a[x \to \xi](y) = a(y) \text{ for every variable } y \neq x, \text{ and } a(x) = \xi.$$

Then

$$[\![\lambda x.P]\!]_a^{\mathfrak{M}} : D(\alpha) \to D(\beta) \; ; \xi \mapsto [\![P]\!]_{a[x \to \xi]}^{\mathfrak{M}}$$

where $(x, \alpha) \in \Gamma$ and $\Gamma \vdash P : \beta$.

Remark 31. Note that, if N is not typeable in a context Γ, i.e. there is no type φ such that

$$\Gamma \vdash N : \varphi$$

then N has no semantic value in an assignment of type of the context. For example, let the type of the constant c be $\iota \to o$ and the context $\Gamma = \{x : o\}$. Then the expression cx is not typeable from the context $\{x : o\}$, since in cx the right and side expression must be an expression of the type ι. However, cx is a well-defined expression, it has no "semantic value" in the context $\{x : o\}$.

5.2.3. *Logical and epsilon expansions.* The logical operators will be defined as constants of certain types. If $\mathcal{L}_\lambda$ is a typed lambda language and $\mathfrak{M}$ a model of it, then the *logical expansion* is the expansion with the following constants $\neg$, $\vee$, $\forall$ and their interpretations:

(1) $Z(\neg_\alpha) = (\alpha \to o) \to (\alpha \to o)$

$$\mathrm{Ip}^{\mathfrak{M},\neg}(\neg_\alpha) : \; D(\alpha \to o) \to D(\alpha \to o); \; f \mapsto (g : D(\alpha) \to \{\mathsf{T}, \mathsf{F}\}, \; a \mapsto \overline{f(a)}),$$

where $\overline{()}$ is the two valued negation: $\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}$. The constant $\neg_\emptyset$ with $Z(\neg_\emptyset) = o \to o$ is also defined similarly.

(2) $Z(\vee_\alpha) = (\alpha \to o) \to ((\alpha \to o) \to (\alpha \to o))$

$$\mathrm{Ip}^{\mathfrak{M},\vee}(\vee_\alpha) : D(\alpha \to o) \to (D(\alpha \to o) \to D(\alpha \to o)); \; (f, g) \mapsto f + g$$

where $+$ is the two valued disjunction taken pointwise. The constant $\vee_\emptyset$ with $Z(\vee_\emptyset) = o \to (o \to o)$ is also defined similarly.

(3) $Z(\forall_\alpha) = (\alpha \to o) \to o$

$$\mathrm{Ip}^{\mathfrak{M},\forall}(\forall_\alpha) : D(\alpha \to o) \to \{\mathsf{T}, \mathsf{F}\}; \; f \mapsto \begin{cases} \mathsf{T}, & f = (D(\alpha) \to \{\mathsf{T}, \mathsf{F}\}; \, a \mapsto \mathsf{T}) \\ \mathsf{F}, & \text{otherwise} \end{cases}$$

The constant $\forall_\emptyset$ with $Z(\forall_\emptyset) = o \to o$ is the identity function.

The following is called the *plain expansion*

$$\mathcal{L}_\lambda^\forall \text{ with } \mathrm{Const}(\mathcal{L}_\lambda^\forall) = \mathrm{Const}(\mathcal{L}_\lambda) \cup \{\neg_\alpha, \vee_\alpha, \forall_\alpha\}_{\alpha \in \mathrm{Typ} \cup \{\emptyset\}}$$

The constant below is the representation of the epsilon symbol. The following function g is a *"choice function"*:

$$g : \{(\alpha, S) \mid \alpha \in \mathrm{Typ}, S \subseteq D(\alpha)\} \to \bigcup_{\alpha \in \mathrm{Typ}} D(\alpha)$$

if it has the property

$$\begin{cases} g(\alpha)(S) \in S, & \text{if } S \neq \emptyset \\ g(\alpha)(S) \in D(\alpha), & \text{if } S = \emptyset \end{cases}.$$

Then $Z(\varepsilon_\alpha) = (\alpha \to o) \to \alpha$

$$\mathrm{Ip}^{(\mathfrak{M},g)}(\varepsilon_\alpha) : D(\alpha \to o) \to D(\alpha);\ f \mapsto g(\alpha)(\{a \in D(\alpha) \mid f(a) = \mathsf{T}\})$$

The *epsilon expansion* $\mathcal{L}_\lambda^{\forall\varepsilon}$ of the logical extension $\mathcal{L}_\lambda^\forall$ is where

$$\mathrm{Const}(\mathcal{L}_\lambda^{\forall\varepsilon}) = \mathrm{Const}(\mathcal{L}_\lambda) \cup \{\neg_\alpha, \vee_\alpha, \forall_\alpha, \varepsilon_\alpha\}.$$

If $\mathfrak{M}$ is a model of $\mathcal{L}_\lambda^\forall$, then $(\mathfrak{M}, g)$ will denote the model of the $\mathcal{L}_\lambda^{\forall\varepsilon}$ expansion with a choice function g described above. Actually, epsilon-terms are a special kind of Skolem functions; it is pointed out in [Monk, The Hilbert ε-operator, p. 481] and in [Mints, Sec. 2.: Quantifier-Free Extensions of Formulas and ε-Theorems)].
Some further notations will also be used, here the subscripts are considered to be known

$$P \mathbin{\dot\vee} Q = \vee PQ, \qquad P \to Q = (\neg P) \mathbin{\dot\vee} Q, \qquad P \& Q = \neg((\neg P) \mathbin{\dot\vee} (\neg Q))$$

$$(\forall x)P = \forall(\lambda x.P)$$

$$(\varepsilon x)P = \varepsilon(\lambda x.P).$$

For further purposes the language $\mathcal{L}_\lambda^{\forall\varepsilon=}$ using *identity* of individuals is also introduced and the meaning of $=$ is defined as
$Z(=_\alpha) = \alpha \to (\alpha \to o)$

$$\mathrm{Ip}^{\mathfrak{M},=}(=_\alpha) : D(\alpha) \to (D(\alpha) \to \{\mathsf{T}, \mathsf{F}\}),\ (x, y) \mapsto \begin{cases} \mathsf{T}, & \text{if } x = y \\ \mathsf{F}, & \text{otherwise} \end{cases}$$

A further notation is the following:

$$(P \underset{\cdot}{=}_\alpha Q) = (=_\alpha PQ)$$

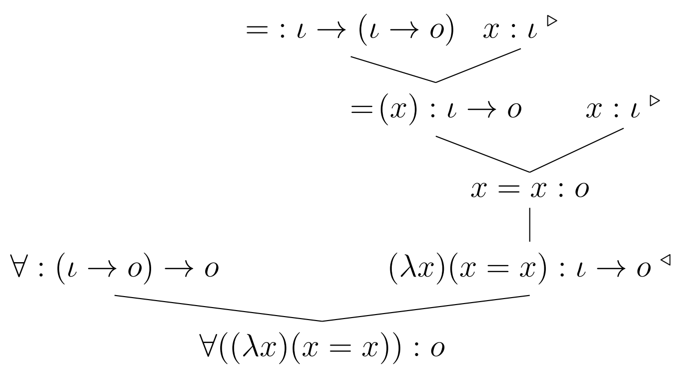
5.2.4. *Examples.*

Proposition 12. Let x be a variable and $(\mathfrak{M}, g)$ be a model over the language $\mathcal{L}_\lambda^{\forall\varepsilon=}$. Then

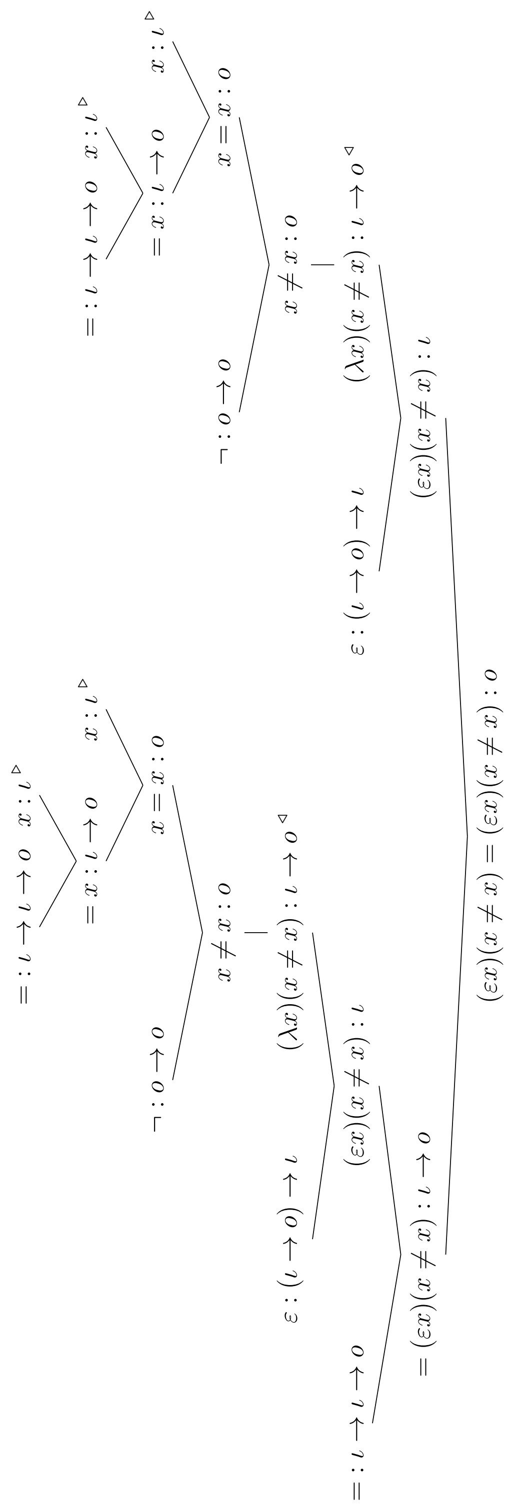
(1) $\vdash (\forall_\iota x)(x \underset{\cdot}{=}_\iota x) : o$
(2) $\vdash (\varepsilon_\iota x)(x \underset{\cdot}{\neq}_\iota x) \underset{\cdot}{=}_\iota (\varepsilon x)(x \underset{\cdot}{\neq}_\iota x) : o$
(3) $[\![(\forall_\iota x)(x \underset{\cdot}{=}_\iota x)]\!]^{(\mathfrak{M},g)} = [\![(\varepsilon_\iota x)(x \underset{\cdot}{\neq}_\iota x) \underset{\cdot}{=} (\varepsilon_\iota x)(x \underset{\cdot}{\neq}_\iota x)]\!]^{(\mathfrak{M},g)} = \mathsf{T}$

Note, that the $\vdash$ judgement is not the derivability, but the typeability. Below, the $\triangleright$ signs are used to denote the dischargeable premisses.

Proof. (1)



According to part (2b) of definition of the typeability, both the $x : \iota$-s are discharged by the node $(\lambda x)([=(x)](x)) : \iota \to o$, i.e. $(x : \iota)$ can be canceled from the context, which is now an empty set. Note that, the use of the labels $^{\triangleleft}$ and $^{\triangleright}$ is completely unnecessary. The variable x in the leaves marks the "dischargeable premisses" and the symbol (λx) marks the node discharging the premisses labeled by the free variable x, after which x becomes a bound variable.


$= : \iota \to \iota \to o$
$x : \iota^{\vartriangle}$
$= x : \iota \to o$
$x : \iota^{\vartriangle}$
$x = x : o$
$\neg : o \to o$
$x \neq x : o$
$(\lambda x)(x \neq x) : \iota \to o^{\triangledown}$
$\varepsilon : (\iota \to o) \to \iota$
$(\varepsilon x)(x \neq x) : \iota$
$= : \iota \to \iota \to o$
$x : \iota^{\vartriangle}$
$= x : \iota \to o$
$x : \iota^{\vartriangle}$
$x = x : o$
$\neg : o \to o$
$x \neq x : o$
$(\lambda x)(x \neq x) : \iota \to o^{\triangledown}$
$\varepsilon : (\iota \to o) \to \iota$
$(\varepsilon x)(x \neq x) : \iota$
$= (\varepsilon x)(x \neq x) : \iota \to o$
$(\varepsilon x)(x \neq x) = (\varepsilon x)(x \neq x) : o$

(2)

(3)

$$\begin{aligned}
[\![(\forall x)(x=x)]\!]_a^{(\mathfrak{M},g)} &= [\![\forall(\lambda x)(x=x)]\!]_a^{(\mathfrak{M},g)} \\
&= [\![\forall]\!]_a^{(\mathfrak{M},g)}([\![(\lambda x)(x=x)]\!]^{(\mathfrak{M},g)})_a \\
&= [\![\forall]\!]_a^{(\mathfrak{M},g)}(\xi \mapsto [\![=(x)(x)]\!]_{a[x\to\xi]}^{(\mathfrak{M},g)}) \\
&= [\![\forall]\!]_a^{(\mathfrak{M},g)}(\xi \mapsto [\![=(x)]\!]_{a[x\to\xi]}^{(\mathfrak{M},g)}(\xi)) \\
&= [\![\forall]\!]_a^{(\mathfrak{M},g)}(\xi \mapsto [\![=]\!]_{a[x\to\xi]}^{(\mathfrak{M},g)}(\xi)(\xi)) \\
&= [\![\forall]\!]_a^{(\mathfrak{M},g)}(\xi \mapsto \mathsf{T}) \\
&= \mathsf{T}
\end{aligned}$$

The second expression's semantic value is trivial:

$$\begin{aligned}
[\![(\varepsilon x)(x\neq x)=(\varepsilon x)(x\neq x)]\!]_a^{(\mathfrak{M},g)} &= [\![=]\!]_a^{(\mathfrak{M},g)}([\![(\varepsilon x)(x\neq x)]\!]_a^{(\mathfrak{M},g)})([\![(\varepsilon x)(x\neq x)]\!]_a^{(\mathfrak{M},g)}) \\
&= \mathsf{T}
\end{aligned}$$

below, we determine it:

$$\begin{aligned}
[\![(\varepsilon x)(x\neq x)]\!]_a^{(\mathfrak{M},g)} &= [\![\varepsilon((\lambda x)(x\neq x))]\!]_a^{(\mathfrak{M},g)} \\
&= [\![\varepsilon]\!]_a^{(\mathfrak{M},g)}([\![(\lambda x)(x\neq x)]\!]_a^{(\mathfrak{M},g)}) \\
&= [\![\varepsilon]\!]_a^{(\mathfrak{M},g)}(\xi \mapsto [\![x\neq x]\!]_{a[x\to\xi]}^{(\mathfrak{M},g)}) \\
&= g(\iota)(\{\xi\in M \mid [\![x\neq x]\!]_{a[x\to\xi]}^{(\mathfrak{M},g)}=\mathsf{T}\}) \\
&= g(\iota)(\{\xi\in M \mid [\![\neg]\!]_{a[x\to\xi]}^{(\mathfrak{M},g)}([\![=]\!]_{a[x\to\xi]}^{(\mathfrak{M},g)}(\xi)(\xi))=\mathsf{T}\}) = g(\iota)(\emptyset)
\end{aligned}$$

□

5.2.5. *Epsilon-invariant expressions.*

Definition 33. Let $N \in \mathrm{Exp}(\mathcal{L}_\lambda^{\forall\varepsilon})$ be such that for a context Γ the relation $\Gamma \vdash N : \varphi$ holds for a type φ and let $\mathfrak{M}$ be a $\mathcal{L}_\lambda^{\forall}$ model. N is said to be *epsilon-invariant over the model* $\mathfrak{M}$, if for every assignment a of type Γ and choice functions g_1, g_2 over $\mathfrak{M}$ it holds that

$$[\![N]\!]_a^{(\mathfrak{M},g_1)} = [\![N]\!]_a^{(\mathfrak{M},g_2)}.$$

The notion above is a symbolic formulation of the intuitive term "epsilon-independent". In FOL this concept was applied to show that "epsilon-independent" sentences can be reformulated into an epsilon-free one, provided the sentence is independent over *every* model [Blas].

5.3. **Epsilon and application.** Suppose that expression T is in the long normal form with a sentential final type i.e.

$$\Gamma \vdash \lambda x_1 \dots \lambda x_n . P : \alpha_1 \to \dots \to \alpha_n \to o$$

where $P = \lambda x_1 \dots \lambda x_n . P$. Furthermore, let $R_1, \dots, R_n$ be typed by so that

$$\Gamma \vdash R_1 : \alpha_1 \to o, \dots, \Gamma \vdash R_n : \alpha_n \to o$$

In the following, I will prove that, if all the non-epsilon-term constituents of

$$(\lambda x_1 \dots \lambda x_n . T)((\varepsilon x_1) R_1) \dots ((\varepsilon x_n) R_n)$$

are epsilon-invariant, then it has an explicite plain quantificational form. First we start with the $n = 1$ case, then we turn to the more complex cases.

Theorem 8. Let $P, R \in \mathrm{Exp}(\mathcal{L}_\lambda^{\forall\varepsilon})$, $\mathfrak{M}$ be a model of $\mathcal{L}_\lambda^{\forall}$, Γ a context, $\Gamma \vdash \lambda x.P : \alpha \to o$, $\Gamma \vdash \lambda x.R : \alpha \to o$, furthermore, let $(\lambda x.P)(\varepsilon_\alpha x)R$, P and R be epsilon-invariant over the model $\mathfrak{M}$. Then for every assignment a of type Γ and choice function g over $\mathfrak{M}$:

$$[\![(\lambda x.P)(\varepsilon_\alpha x)R]\!]_a^{(\mathfrak{M},g)} = [\![((\forall x)(\neg R) \,\&\, (\forall x)P) \vee (((\exists x)R) \,\&\, (\forall x)(R \to P))]\!]_a^{(\mathfrak{M},g)}.$$

Proof. (1) Let the right hand side be T. First case: $[\![((\forall x)(\neg R)\&(\forall x)P)]\!]_a^{(\mathfrak{M},g)} = \mathsf{T}$. Then $[\![(\forall x)P)]\!]_a^{(\mathfrak{M},g)} = \mathsf{T}$ holds and let $m = [\![(\varepsilon_\alpha x)R]\!]_a^{(\mathfrak{M},g)} \in M$. Hence, by definition

$$\mathsf{T} = [\![(\forall x)P)]\!]_a^{(\mathfrak{M},g)} = [\![\forall((\lambda x)P)]\!]_a^{(\mathfrak{M},g)}$$

that is

$$[\![(\lambda x)P]\!]_a^{(\mathfrak{M},g)} = \left(\xi \mapsto [\![P]\!]_{a[x\to\xi]}^{(\mathfrak{M},g)}\right) \equiv \mathsf{T}.$$

Hence

$$[\![(\lambda x.P)(\varepsilon_\alpha x)R]\!]_a^{(\mathfrak{M},g)} = [\![\lambda x.P]\!]_a^{(\mathfrak{M},g)}(m) = [\![P]\!]_{a[x\to m]}^{(\mathfrak{M},g)} = \mathsf{T}.$$

Second case: $[\![(((\exists x)R)\&(\forall x)(R \to P))]\!]_a^{(\mathfrak{M},g)} = \mathsf{T}$. Then

$$[\![(\lambda x)\neg R]\!]_a^{(\mathfrak{M},g)} = \left(\xi \mapsto [\![\neg R]\!]_{a[x\to\xi]}^{(\mathfrak{M},g)}\right) \not\equiv \mathsf{T}$$

hence for a $\xi \in M$ $[\![R]\!]^{(\mathfrak{M},g)}_{a[x\to\xi]} = \mathsf{T}$. Therefore, if $[\![\varepsilon((\lambda x)R)]\!]^{(\mathfrak{M},g)}_a = m$ then $[\![(\lambda x)R]\!]^{(\mathfrak{M},g)}_a(m) = \mathsf{T}$. But from $[\![(\forall x)(R \to P))]\!]^{(\mathfrak{M},g)}_a = \mathsf{T}$ it follows that $[\![P]\!]^{(\mathfrak{M},g)}_{a[x\to m]} = \mathsf{T}$, since $[\![R]\!]^{(\mathfrak{M},g)}_{a[x\to m]} = \mathsf{T}$. Hence, $[\![\lambda x.P]\!]^{(\mathfrak{M},g)}_a(m) = [\![(\lambda x.P)(\varepsilon_\alpha x)R]\!]^{(\mathfrak{M},g)}_a = \mathsf{T}$.
(2) Suppose the left hand side is T. First case: let $[\![((\forall x)(\neg R)]\!]^{(\mathfrak{M},g)}_a = \mathsf{T}$, $m \in D(\alpha)$ arbitrary and g' is the choice function such that $g'(\alpha)(\emptyset) = m$. Hence, by the epsilon-invariance of P and $(\lambda x.P)(\varepsilon_\alpha x)R$ it follows that

$$\mathsf{T} = [\![(\lambda x.P)(\varepsilon_\alpha x)R]\!]^{(\mathfrak{M},g)}_a = [\![(\lambda x.P)(\varepsilon_\alpha x)R]\!]^{(\mathfrak{M},g')}_a = [\![P]\!]^{(\mathfrak{M},g')}_{a[x\to m]} = [\![P]\!]^{(\mathfrak{M},g)}_{a[x\to m]}$$

therefore $[\![(\forall x)P]\!]^{(\mathfrak{M},g)}_a = \mathsf{T}$. Second case: let $[\![(\exists x)R]\!]^{(\mathfrak{M},g)}_a = \mathsf{T}$, $m \in M$ arbitrary such that $[\![R]\!]^{(\mathfrak{M},g)}_{a[x\to m]} = \mathsf{T}$ and g' is the choice function such that

$$g'(\alpha)(\{\xi \in M \mid [\![R]\!]^{(\mathfrak{M},g)}_{a[x\to\xi]} = \mathsf{T}\}) = m.$$

Then by the epsilon-invariance of P, R and $(\lambda x.P)(\varepsilon_\alpha x)R$ it follows that

$$\mathsf{T} = [\![(\lambda x.P)(\varepsilon_\alpha x)R]\!]^{(\mathfrak{M},g)}_a = [\![(\lambda x.P)(\varepsilon_\alpha x)R]\!]^{(\mathfrak{M},g')}_a = [\![P]\!]^{(\mathfrak{M},g')}_{a[x\to m]} = [\![P]\!]^{(\mathfrak{M},g)}_{a[x\to m]}$$

for every m such that $[\![R]\!]^{(\mathfrak{M},g)}_{a[x\to m]} = \mathsf{T}$. Hence, $[\![(\forall x)(R \to P)]\!]^{(\mathfrak{M},g)}_a = \mathsf{T}$ □

The situation is similar to Russell's Theory of Descriptions. This approach not provides a single formula for how sentences containing epsilon terms can be interpreted, but gives rise the opportunity to find out what such sentences mean under several conditions. An example of this is a sentence in which the epsilon expressions contain only one variable. But first let us define a logical expression which will be equivalent to a such complex sentence.

$$\mathsf{cases}(\lambda x_1 \ldots \lambda x_n.P, R_1, \ldots, R_n) =$$

$$= \bigvee_{(e_1,\ldots,e_n)\in\{0,1\}^n} (\mathsf{Q}^{e_1}x_1)R_1 \,\&\, \ldots \,\&\, (\mathsf{Q}^{e_n}x_n)R_n \,\&\, ((\forall x_1)\ldots(\forall x_n)(R_1^{e_1} \,\&\, \ldots \,\&\, R_n^{e_n}) \to P)$$

where

$$\mathsf{Q}^0 = \neg\exists$$

$$\mathsf{Q}^1 = \exists$$

$$R_i^0 = (\forall z)(z = z)$$

$$R_i^1 = R_i$$

where z is a (fresh) variable (or R_i^0 could be any true sentence).

Proposition 13. Let $\mathfrak{M}$ be a model of $\mathcal{L}_\lambda^\forall$, Γ be a context, $\Gamma \vdash \lambda x_1 \ldots \lambda x_n.P : \alpha_1 \to \cdots \to \alpha_n \to o$, $\Gamma \vdash \lambda x_1.R_1 : \alpha_1 \to o, \ldots, \Gamma \vdash \lambda x_n.R_n : \alpha_n \to o$, and $\mathrm{FV}(R_i) \subseteq \{x_i\}$. Suppose that all of the non-epsilon term constituents of $(\lambda x_1 \ldots \lambda x_n.P)(\varepsilon x_1)R_1 \ldots (\varepsilon x_n)R_n$ is epsilon-invariant over the model $\mathfrak{M}$. Then for every assignment a of type Γ and choice function g over $\mathfrak{M}$:

$$[\![(\lambda x_1 \ldots \lambda x_n.P)(\varepsilon x_1)R_1 \ldots (\varepsilon x_n)R_n]\!]_a^{(\mathfrak{M},g)} = [\![\mathsf{cases}(\lambda x_1 \ldots \lambda x_n.P, R_1, \ldots, R_n)]\!]_a^{(\mathfrak{M},g)}$$

Proof. The proof goes by induction on n. The base case is valid because of the previous theorem. Consider the sentence

$$(\lambda x_0 \lambda x_1 \ldots \lambda x_n.P)(\varepsilon x_0)R_0(\varepsilon x_1)R_1 \ldots (\varepsilon x_n)R_n$$

By the induction hypothesis

$$[\![(\lambda x_0 \lambda x_1 \ldots \lambda x_n.P)(\varepsilon x_0)R_0(\varepsilon x_1)R_1 \ldots (\varepsilon x_n)R_n]\!]_a^{(\mathfrak{M},g)} =$$

$$= [\![(\lambda x_n.\mathsf{cases}(\lambda x_0 \ldots \lambda x_{n-1}.P, R_0, \ldots, R_{n-1}))(\varepsilon x_n)R_n]\!]_a^{(\mathfrak{M},g)} =$$

by again the previous theorem

$$= [\![((\forall x_n)(\neg R_n) \,\&\, (\forall x_n)\mathsf{cases}(\lambda x_0 \ldots \lambda x_{n-1}.P, R_0, \ldots, R_{n-1}))\vee$$

$$\vee(((\exists x_n)R_n) \,\&\, (\forall x_n)(R_n \to \mathsf{cases}(\lambda x_0 \ldots \lambda x_{n-1}.P, R_0, \ldots, R_{n-1}))]\!]_a^{(\mathfrak{M},g)} =$$

then, by routine calculation, using the definition of the interpretation of logical expressions, we obtain that

$$= [\![\mathsf{cases}(\lambda x_0 \ldots \lambda x_n.P, R_0, \ldots, R_n)]\!]_a^{(\mathfrak{M},g)}$$

□

Remark 32. For example, for $n = 2$ it holds that

$$[\![(\lambda x_1 \lambda x_2.P)(\varepsilon x_1)R_1(\varepsilon x_2)R_2]\!]_a^{(\mathfrak{M},g)} =$$

$$= [\![(\neg(\exists x_1)R_1 \,\&\, \neg(\exists x_2)R_2 \,\&\, (\forall x_1)(\forall x_2)P)\vee$$

$$\vee(\neg(\exists x_1)R_1 \,\&\, (\exists x_2)R_2 \,\&\, (\forall x_1)(\forall x_2)R_2 \to P)\vee$$

$$\vee((\exists x_1)R_1 \,\&\, \neg(\exists x_2)R_2 \,\&\, (\forall x_1)(\forall x_2)R_1 \to P)\vee$$

$$\vee((\exists x_1)R_1 \,\&\, (\exists x_2)R_2 \,\&\, (\forall x_1)(\forall x_2)R_1 \,\&\, R_2 \to P)]\!]_a^{(\mathfrak{M},g)}$$

Note, that, if R_1 or R_2 contain more then one variable, then the simplification cannot be done, but it could be also a quantified sentence, however a complicated one.

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