PARAMETER ESTIMATION IN ELECTROMAGNETIC DEVICES BY THE MULTILAYERED MEDIUM AND MODEL BASED APPROACH

PhD Dissertation

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KIVONAT

Az „érzékelő nélküli” (sensorless) irányelv elektromágneses beavatkozók esetében egy egyre elterjedtebb megközelítés a beavatkozót működtető rendszer költséghatékonyságának és megbízhatóságának növelésére. Lényegé, hogy normál működési üzem mellett az eszköz bizonyos fizikai paraméterei az eszköztől alkotott modell segítségével és alternatív be/kimeneti mennyiségek mérésével kerülnek meghatározásra, amely paramétereket egyébként nem, vagy csak dedikált külső érzékelővel lehetne mérni.

Kutatáson kezdeti célkitűzése az érzékelő nélküli irányelv vizsgálata volt lineáris elektromágneses beavatkozók esetére, azaz új módszerek kidolgozása a mozgórész (dugattyú) pozíciójának és sebességének, a külső terhelését és az eszköz termikus állapotának a meghatározására, becslesésére. Ezeket az eszközöket általában kapcsolási (relé) és ténfogatáram szabályozási (sele) célokra alkalmazzák, túlnyomó részt korlátozott erőforrású beágyazott rendszerekben. Ezért a megbízhatóság és pontosság mellett a létrejövő modellek és eljárások alacsony komplexitása és számítási igénye kules fontosságú szempontok voltak.

Kutatáson első részében a beágyazott rendszerek speciális erőforrás igényeinek megfelelő, alacsony komplexitású és számítási igényű módszerek kidolgozásával foglalkoztam a dugattyú pozíciójának, a beavatkozót terhelő külső erőhatásnak és a tekercselés elektromos ellenállásának a meghatározására. A termikus modellel szemben az ellenállás villamos modellen alapuló becslése számítástechnikailag gazdaságosabb, mindemellett képes információt szolgáltatni a beavatkozó belső termikus állapotáról, a tekercs átlagos hőmérsékletéről. Viszont a részletes hőmérsékleti eloszlás leírására és annak mélyreható vizsgálatára nem alkalmas. Ami a dugattyú pozíciójának érzékelő nélküli meghatározását illeti, a szakirodalomban számos módszer létezik, viszont a pozíció és a külső terhelés egyidejű becslése még egy nyitott terület. Tipikusan folyadékáramlási vagy nyomásszabályozási alkalmazásoknál a beavatkozót időben változó külső erőhatás érheti, ami befolyásolja a szükséges meghajtó tekercsáramot. Amennyiben a külső terhelésre egy becslés adható, lehetőség nyílik a külső erő- illetve nyomásérzékelők elhagyására a rendszerből és a beavatkozó meghajtásához szükséges áram minimalizálására. Ezáltal a rendszer költséghatékonysága és hatásfoka növekedhet. Fontos megjegyezni, hogy az érzékelő nélküli módszerek az elektromágneses beavatkozók meghajtásához legelterjedtebben használt PWM (impulzus szélesség modulációs) technikával kompatibilisnek kell lennie, és az eszközök az eredeti beavatkozó funkcióját maradéktalanul teljesítenie kell. Az ehhez a területhez kapcsolódó új, tudományos eredményeim [1], [4-5] és az I. tézis alatt találhatók.

Normál működés közben az elektromágneses beavatkozók paraméterei megváltozhatnak. Mivel az érzékelő nélküli eljárás a vizsgált eszköztől alkotott modellen alapszik, a modell folyamatos frissítése és a paraméterek nyomon követése fontos a pontos és megbízható működés (becslés) érdekében. Ez szükségeséhez a vizsgált eszköz termikus állapotának részletes ismeretét, mivel a beavatkozótól alkotott modell paraméterei, kiváltéképp a tekercs elektromos ellenállása, a hőmérséklettől nagymértékben függhetnek. Emellett egy átfogó hőmérsékleti modell segítségével nyújt minden tervezési, méretezési (optimalizálási) és diagnosztikai (forró-pontok meghatározása) feladatoknál. Az előbb említett okok miatt kutatáson második részében egy, a lineáris elektromágneses beavatkozók vizsgálatára alkalmas, átfogó termikus modell létrehozásával foglalkoztam. A hőmérsékleti modell megalkotásában fontos szempont volt a szélsőkörü alkalmazhatóság és a hőmérsékleti tér minél általánosabb leírása; ezért a hőmérsékleti eloszlásnak (diffúziós egyenlet) az analitikus kifejezését kerestem. Az analitikus megoldások
széleskörű betekintést nyújtanak a végbemenő fizikai folyamatokba és a paraméterek közötti összefüggésekbe, ami tervezési és optimalizálási feladatoknál előnyös tulajdonság. Emellett többnyire kompakt, zárt alakú megoldások, így alkalmazásuk az erőforrásban hiányos beágyazott rendszerekben kedvező. A hőmérsékleti modell létrehozása során a szakirodalomban a többrétegű struktúráként nevezett koncepciót (multilayered medium) alkalmaztam. A modellválasztást az indokolja, hogy a lineáris elektromágneses beavatkozó, kiváltéképp a belső tekercselése, jó közelítéssel felfogható egy sugárirányban rétegelt szerkezetként. Az előbb ismertetett kutatási probléma röviden úgy foglalható össze és a szakirodalomban úgy szerepel, hogy a diffúziós egyenlet analitikus megoldása többrétegű struktúrákban. Kiemelném, hogy az így vizsgált problémakör általános jellegű, emiatt az elért eredmények minden olyan területen érvényesek, amelyek a diffúziós egyenleting leírhatók, így az elektrosztatikára és hővezetésre egyaránt. Emellett a lineáris elektromágneses beavatkozók modellezésén túl egy szélesebb kutatási és alkalmazási körben is hasznosíthatók, ahol a többrétegű struktúra megközelítés helytálló, például kompozit anyagok vizsgálatára. Az ehez a területhez kapcsolódó új, tudományos eredményeim [di2-3] és II.-III. tézisek alatt találhatók.
THE SENSORLESS PRINCIPLE IS BECOMING MORE AND MORE POPULAR FOR IMPROVING THE COST EFFECTIVENESS AND RELIABILITY OF SYSTEMS WHICH UTILIZE ELECTROMAGNETIC ACTUATORS. THE CONCEPT IS THAT CERTAIN PHYSICAL PARAMETERS OF THE DEVICE, WHICH PARAMETERS OTHERWISE COULD NOT BE MEASURED OR COULD BE MEASURED WITH DEDICATED COSTLY SENSORS, ARE ESTIMATED UNDER NORMAL OPERATION BY USING A MODEL OF THE ELECTROMAGNETIC DEVICE AND MEASURING ITS ALTERNATIVE INPUT AND OUTPUT QUANTITIES. THE MAIN FOCUS OF MY RESEARCH WAS INITIALLY ON STUDYING THE SENSORLESS PRINCIPLE FOR LINEAR ELECTROMAGNETIC ACTUATORS, I.E., CREATING IMPROVED METHODS FOR ESTIMATING THE POSITION OF ITS MOVING PART (SPOOL), ITS EXTERNAL LOAD AND THERMAL STATE. TYPICAL FIELDS OF APPLICATIONS OF SUCH DEVICES INCLUDE SWITCHING (CONTACTOR) AND FLOW CONTROLLING (VALVE) PURPOSES IN EMBEDDED SYSTEMS, WHICH SYSTEMS HAVE STRICT LIMITATIONS IN THE COMPUTATIONAL RESOURCES. THEREFORE, THE LOW COMPLEXITY AND COMPUTATIONAL LOAD, WITH ACCURACY AND RELIABILITY, WERE KEY REQUIREMENTS TO THE NEW MODELS AND METHODS.


solutions are usually closed-form, compact solutions; therefore, preferable in embedded applications which have strict resource limitations. Considering the modeling approach, the so-called multilayered medium approach was applied for creating the thermal model because linear electromagnetic devices, especially the coil, can be considered as a multilayered medium that is layered along the radial direction. This part of my research can be succinctly summarized as the analytical solution of the diffusion (heat) equation in multilayered media. It has to be highlighted that because of the generality of the studied problem, my results are applicable to every field which is governed by the diffusion equation, e.g. to electrostatics and heat conduction; and the results are applicable not only to the modeling of electromagnetic actuators but to a wider field of research and application, where the multilayered medium approach holds, e.g., for the analysis of composite materials. The results of my research, which correspond to the analytical solutions of multilayered heat diffusion, are to be found in [di2-3] and in the Theses II.-III. in Section VI.1.
LIST OF SYMBOLS

List of Symbols for Sections II-III

$L$ Inductance
$R$ Electrical resistance
$U$ Voltage
$i$ Electrical current
$t$ Time
$T$ Temperature; and time period of a PWM cycle
$\Psi$ Flux linkage
$x$ Position
$H$ Magnetic field intensity
$B$ Magnetic flux density
$l$ Length
$N$ Number of turns
$\mu$ Magnetic permeability
$\Phi$ Magnetic flux
$F$ Force
$E$ Energy
$V$ Volume
$m$ Mass; and slope of a line in Section III.3.1
$b$ Viscous damping; and dummy variable in Section III.3
$k$ Stiffness of spring
$f$ Frequency
$d$ Duty ratio and displacement
$A$ Dummy variable in Section III.3
$B$ Dummy variable in Section III.3
$a$ Dummy variable in Section III.3
$\Omega$ Dummy variable in Section III.3.1
$c$ Dummy variable in Section III.3.1
$\alpha$ Dummy variable in Section III.6
$\beta$ Dummy variable in Section III.6
$\chi$ Dummy variable in Section III.6

Subscripts and superscripts

$S$ Supply
$H$ High level (PWM “on”)
$L$ Low level (PWM “off”)
$A$ During PWM “on” period
$B$ During PWM “off” period
$k, n$ Indexes
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<tr>
<td>$y$</td>
<td>Longitudinal coordinate</td>
</tr>
<tr>
<td>$z$</td>
<td>Longitudinal (vertical) coordinate</td>
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<tr>
<td>$r$</td>
<td>Radial coordinate</td>
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<td>$\varphi$</td>
<td>Polar coordinate</td>
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<td>$H, L$</td>
<td>Height and length of a layer</td>
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<td>$h$</td>
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<td>$T$</td>
<td>Steady two-dimensional (2D) temperature field</td>
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<td>$\Theta$</td>
<td>Temperature solution in Section IV.4</td>
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<tr>
<td>$\Psi$</td>
<td>Temperature part-solution in Section IV.4</td>
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<tr>
<td>$\Phi$</td>
<td>Solution to the reduced 1D conduction in Section IV.4</td>
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<tr>
<td>$\Lambda$</td>
<td>Transfer matrix in Section IV.4</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>Parameter vector in Section IV.4</td>
</tr>
</tbody>
</table>
$R$ Lumped resistance
$i_L$ Lumped load current

**Subscripts and superscripts**

- $i,j,k,n,m$: indexes
- $u,b,l,r$: Top, bottom, left, right-side surface
- $hy$: Hyperbolic eigenfunction
- $tri$: Trigonometric eigenfunction
- $x$: In the $x$ direction
- $y$: In the $y$ direction
I. INTRODUCTION

1.1. Linear Electromagnetic Actuators

The linear electromagnetic actuator is a single or two phase linear motor, i.e., the moving part (spool/plunger) follows a linear, limited motion. The stroke length is usually in the range of a few millimeters. Typical fields of application of a linear electromagnetic actuator include switching operation (contactor, relay) [2-3], [17], [20], [35] and flow control (valve) [7], [12], [21], [23], [29], e.g., in an automatic transmission unit, in an internal combustion engine or in a pneumatic brake. If considering the trajectory of the spool, a linear solenoid actuator can be either push, pull or push-pull. In certain applications, the actuator may also incorporate an internal permanent magnet thus realizing a latching operation; thus, the spool remains in the actuated state without any necessary drive current. Electromagnetic devices are usually driven by the PWM (pulse width modulation) technique, the simplest configuration of which is illustrated on the right side of Fig. I.1. A detailed illustration of a single phase electromagnetic valve is also provided in the left side of Fig. I.1. In the following analyses, a solenoid valve is considered but the results can be directly applied to the switching or contacting types as well. The operation of an electromagnetic valve actuator is briefly as follows; however, a more detailed explanation about its operation and recent articles discussing the modeling and optimization of linear electromagnetic actuators can be found in e.g. [10-11], [18], [20-24], [37-38].

![Fig. I.1: Electromagnetic valve (left) and PWM drive (right): 1-valve, 2-spool, 3-electric coil, 4-return spring.](image)

In the de-energized state (no current), the actuator in the left-side of Fig. I.1 is closed. The terminal voltage that is applied to the electric coil forces electrical current through the coil, which current generates a magnetic field that exerts an attractive magnetic force to the ferromagnetic spool. The spool is displaced (valve opens) thus enabling the controlled medium (gas, fluid) to flow through the valve. The size of the orifice, i.e., the resistance to the flow is set by the position of the spool. The attractive magnetic force and the external load, (external load is caused e.g. by the pressure of the controlled medium) are counteracted by the valve return spring which pushes the spool out from the housing and keeps the valve closed. In case of contactors, there is no orifice. Based on the above description, four unique subsystems can be identified which are:

- The electrical subsystem: the input of the device, it transforms input voltage to electrical current. It is formed by the electrical coil and is usually modeled as a series resistance-inductance (LR) system,
- The magnetic (electromagnetic) subsystem: it couples the electrical subsystem to the mechanical subsystem by transforming electrical current to magnetic force. It is usually
modeled as a nonlinear magnetic circuit (reluctance), which is formed by the coil, the spool, the housing etc.,

- The mechanical subsystem: the output of the actuator, it transforms force to displacement. It is usually modeled as a second order, damped mass-spring system and is formed by the spool (mass) and the return spring. The position of the spool is set by three forces, which are the magnetic force, the spring force and the external forces,

- The hydraulic/pneumatic subsystem: it is relevant only in flow controlling applications. The outflow orifice represents a hydraulic resistance, which is set by the position of the spool and by the geometry of the orifice. The outflow area, thus the hydraulic resistance which is represented by the valve, can be obtained through the geometric constraints.

The model or functional block diagram of a linear electromagnetic actuator can be constructed as it is illustrated in Fig. I.2. The three major input parameters of a linear electromagnetic device are the input voltage, temperature and the external load, respectively. The electrical subsystem converts the input voltage to coil current; it represents an electrical impedance that consists of resistances, inductances etc. Under normal operation (flow control), the return spring counteracts the magnetic force and an additional force that comes from the pressure of the controlled medium. This external load may arbitrarily change and represents an excitation to the mechanical subsystem. Nonetheless, the temperature has an influence on the behavior of all of the subsystems. With higher temperature the coil’s resistance increases; therefore, the output current becomes smaller for the same input voltage. Additionally, the overall magnetic permeability; thus, force linkage may decrease and other magnetic losses (e.g. hysteresis) may increase with the temperature. Considering the mechanical subsystem, thermal elongation causes deformations and the force of the return spring can vary with temperature.

![Block diagram of a linear solenoid actuator](image)

**Fig. I.2: Block diagram of a linear solenoid actuator.**

The block diagram in Fig. I.2 illustrates a feedback from the mechanical subsystem (spool position) to the electrical and electromagnetic subsystems. The magnetic force is exerted through a working air gap, the length of which is in a direct relation to the position of the spool. In case the length of the air gap varies (position of the spool), the overall magnetic reluctance thus the impedance (inductance) of the electrical subsystem also changes. Considering the electromechanical subsystem, the length of the working air gap determines the flux linkage (e.g. leakage); thus, the force transfer. For example, a smaller gap produces a larger force at the same input current.
I.1.1. The electrical subsystem

The input electrical side of a solenoid actuator represents electrical impedance that converts the terminal voltage to coil current. In fact, this subsystem is formed by the coil (winding) and its corresponding magnetic reluctance (inductance). In technical literature, a series resistance-inductance (LR) approximation is commonly used for describing the electrical subsystem [1]-[3]; however it is also possible to use more sophisticated models [6], [29]. The governing equation for the electrical subsystem is the voltage equation law that is expressed in (I.1), where \( R \) represents the resistance of the coil (temperature dependent), \( u(t) \) is the input voltage, \( i(t) \) is the current, \( x(t) \) is the position of the spool, \( T \) is the temperature and \( \Psi(x,i) \) represents the flux linkage,

\[
U(t) = i(t)R(T) + \frac{d\Psi(x,i)}{dt}.
\] (I.1)

In (I.1) the flux linkage depends both on the current and on the position of the spool. The dependence on position had been already explained in Fig. I.2. The dependence on the current is caused by the nonlinear magnetization curve of the core materials, i.e., the relative magnetic permeability is a function of the current. Further rearrangement of (I.1) yields (I.2),

\[
U(t) = i(t)R(T) + \frac{\partial\Psi(x,i)}{\partial i} \frac{di}{dt} + \frac{\partial\Psi(x,i)}{\partial x} \frac{dx}{dt}.
\] (I.2)

The first term in (I.2) is the resistive voltage drop, the second one is the voltage induced by changes in the flux linkage due to changes in the permeability, and the third term refers to the motional back EMF (electromotive force), i.e., there is voltage induced by spool motion. In case the velocity of the spool (\( dx/dt \) term) is small, the EMF term becomes insignificant. In [1], measurements are presented that demonstrate that the effect of back EMF can be less than 5% for certain solenoids. With the \( \Psi i \) substitution for \( L \), (I.2) can be equivalently rewritten to (I.3) that describes the voltage equation in an alternate way which uses the inductance, i.e.,

\[
U(t) = i(t)R(T) + L(x,i) \frac{di}{dt} + i(t) \frac{dL(x,i)}{dt}.
\] (I.3)

I.1.2. The electromagnetic subsystem

The middle subsystem of a solenoid actuator is the electromagnetic one which couples the electrical, magnetic and mechanical subsystems to each other. It can be represented as a complex magnetic circuit [6], [10], [18], [20], [23], which is formed by the coil (winding), housing, core (high permeability material), air gaps etc. In fact, it is a controllable electromagnet that exerts an attractive magnetic force (flux) to the spool through a working air gap. The length of the air gap; thus, the overall magnetic behavior depends on the position of the moving part. This subsystem can be modeled by means of a magnetic reluctance network [18], finite element analyses [11], [21], [23], sectional linearization [1], and by means of polynomial approximation to an empirical
flux linkage data [6]. The electromagnetic subsystem exhibits some special features that render it to be the most difficult to model:

- Eddy current intensity: depending on the material of the core and on the geometry of the device, changes in the flux linkage induce eddy currents in the actuator that contribute to self-heating and degraded magnetic performance,
- Nonlinear magnetization curve: the permeability of the core materials is not constant, but varies with the magnetic field intensity $H$ (magnetizing current). Additionally, the overall magnetic reluctance depends on the position of the spool, i.e., on the length of the working air gap,
- Saturation: it is also a nonlinear effect, i.e., the relative magnetic permeability severely decreases thus the magnet stops behaving as a magnet,
- Hysteresis of the core materials: the magnetization curve (flux density versus field intensity) is different if the polarity of the magnetic excitation changes. This effect results in additional energy dissipation inside the core materials of the actuator.

In the following, a short explanation of the relevant magnetic quantities/definitions is provided for the sake of better understanding. The upcoming definitions can be found in [73]. According to Ampere’s law, a current carrying conductor produces a magnetic field intensity $H$ that is expressed in (I.4). The line integral of $H$ equals the sum of enclosed currents [73],

$$\oint \vec{H} \cdot d\vec{l} = \sum_i i = N_i. \quad \text{(I.4)}$$

The flux density, which is denoted by $B$, is related to the $H$ field by the property of the medium in which these exist (I.5) [73]. The term $\mu$ (permeability of medium) is the slope of the $H$-$B$ (magnetizing) curve. The term $\mu_0$ is the permeability of free space and the term $\mu_r$ is the relative permeability of the medium. An exemplary magnetization curve, i.e., the flux density as the function of the field intensity, ($H$-$B$ curve) is plotted in Fig. I.3,

$$B = \mu_0 \mu_r H = \mu H. \quad \text{(I.5)}$$

Fig. I.3: General magnetizing curves with permeability.

The surface integral of the flux density yields the amount of flux crossing that certain cross section (I.6) [73]. According to the continuity of flux, the flux lines are closed loops thus the leaving and entering “lines” are the same,
\[ \Phi = \int_A \vec{B} \cdot d\vec{A}. \]  

(I.6)

According to Faraday’s voltage induction law, the induced voltage across an inductor equals the rate of change of flux linkage (\( \Psi \) equals the number of turns \( N \) times the flux \( \Phi \)) (I.7),

\[ u_{\text{ind}}(t) = \frac{d\Psi}{dt} = N \frac{d\Phi}{dt}. \]  

(I.7)

The magnetic force that attracts the spool can be derived from the energy that is accumulated in the air gap. It can be expressed by (I.8), [23],

\[ F_{\text{magnetic}} = \frac{\partial E_{\text{gap}}(x,i)}{\partial x} = \int \frac{\partial \Psi(x,i)}{\partial x} di = \left( \frac{\partial}{\partial x} \int V H dBdV \right). \]  

(I.8)

I.1.3. The mechanical subsystem

The spool (mass) and the return spring form mechanical energy storage elements. According to Fig. I.2, the output of the mechanical subsystem is the position of the moving part and its input is the net force. In technical literature [6], a second order differential equation is usually established to describe the behavior of the mechanical subsystem. The equation of motion is provided in (I.9), and a common way to extract the necessary parameters in (I.9) is from the damped free oscillations of the mechanical subsystem [6],

\[ m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = F(t). \]  

(I.9)

The position of the spool is represented by \( x \), the mass of the spool by \( m \), the stiffness of the spring by \( k \) and the viscous damping by \( b \). In real applications, it is possible that the parameters \( k \) and \( b \) depend on the position or on the temperature. The net load force input \( F_L \) to the system can be decomposed to the magnetic force, which depends on the current and on the position, to the external force which is caused by e.g. fluid pressure and to subsidiary forces such as dry friction, mechanical hysteresis and the gravitational force of the spool (I.10),

\[ F(t) = F_{\text{magnetic}}(t) + F_{\text{external}}(t) + F_{\text{subsidiary}}(t). \]  

(I.10)

I.2. The Sensorless Principle in Linear Electromagnetic Actuators

A brief overview of the sensorless principle is provided in Fig. I.4. The concept is that a certain output quantity of the system is not measured directly with a dedicated sensor but estimated (if possible) on the basis of an elaborate model of the system and by measuring alternative input and output quantities of the system. Considering electromagnetic devices, the
position/velocity of the spool is estimated on the basis of a model of the electromagnetic subsystem and by measuring its electrical signals, e.g., current and voltage. The main advantage of the sensorless principle is that costly external sensors, e.g. a position sensor, can be saved along with its mechanical and hardware layout.

Fig. I.4: Block diagram of the sensorless principle compared to the traditional approach.

Due to the fact that linear electromagnetic actuators are most commonly used in an embedded environment the hardware and software, which are necessary for driving the device and applying the sensorless approach, have to be minimal for the sake of cost effectiveness. Additionally, the sensorless methods have to be compatible with the PWM technique and have to ensure that the actuator fulfills its original actuating roles.

The sensorless methods for estimating the mechanical parameters (position, velocity, force) exploit the dependence of the electrical and electromagnetic subsystems (electrical impedance) on the position of the spool, e.g., the change in the inductance e.g. [1-3], [29], [35] and in flux linkage, e.g. [6-7], [9], [16], [27]. For this reason, it is necessary to identify the electromagnetic subsystem and to measure and compute the necessary parameters. As the inductance is a dynamic quantity and it is related to a specific location in the magnetization curve, the inductance is computed from the system’s response that is given to a dedicated scan signal. This scan signal can be, for example, the PWM itself or a sinusoidal component in the input voltage. Contrarily, the flux is an integral quantity and its measurement may require continuous integration and the use of auxiliary windings [9], [27], which are disadvantageous from the perspective of cost effectiveness. The use of complex electromagnetic models and sophisticated control schemes [28-31] is also an alternative approach; however, these may be less compatible with the special requirements of the low complexity and computational load.

I.2.1. Estimation of the position of the spool

The estimation of the position of the spool in linear electromagnetic actuators exploits the dependence of the electromagnetic subsystem on the position of the spool; thus, information about the position is extracted from e.g. inductance or flux linkage data. Considering the physical quantity on which the estimation is based, the available methods can be grouped in the following ways:

- EMF based methods [8], [25], [33-34]: According to (I.2) voltage is induced by the movement of the spool because the magnetic reluctance of the system is also determined by the length of the air gap. The EMF based methods require that the velocity of the
spool is significant therefore they are not applicable if the spool is stationary or has a low velocity.

- **Inductance (electrical impedance) observers [1-5], [17], [29-31], [35], [39]:** These methods exploit the dependence of the inductance (overall magnetic reluctance) on the position of the spool. However, the inductance depends on the magnitude of the drive current as well (saturation, nonlinear magnetization). The position versus inductance and position (inductance) versus current relationships can be established on the basis of experimental measurements (2D look-up-table) [1-3] or on the basis of numerical models [23]. Then, the position is estimated from the experimental/numerical model by measuring the inductance. The inductance is a “local” quantity that is related to a specific point in the magnetization curve, i.e., it is related to the slope of the H/B curve at a given field intensity. Therefore, the inductance based estimation is more susceptible to measurement noise. Furthermore, it is possible that the inductance is the same for multiple spool positions, which fact can render the estimation of the position to be difficult. Since the inductance is a dynamic quantity its measurement requires a dedicated scan signal that excites the system locally. Such scan signals can be:
  - Inherent chopping of the PWM [1-3], [17], [35]: the input voltage of the electrical subsystem is not a continuous function but a sequence of rectangular waveforms with a specific frequency and duty ratio if the PWM technique is applied. If considering a single PWM cycle, the system is first excited by a high (PWM on) and then by a low (PWM off) voltage pulse. This causes the current to fluctuate during each PWM cycles (current ripple) and enables to compute the inductance if the ripple in the current is measurable. This approach requires a lower PWM frequency to obtain sufficient current ripple,
  - Sinusoidal scan signal [4-5], [8], [25], [39]: the input voltage of the electrical subsystem consists of the main drive voltage and of a sinusoidal part, which is superposed onto the main drive voltage. The sinusoidal scan signal can be generated by either adding a continuous sinusoidal waveform to the supply voltage or by modulating the duty ratio. Compared to the previous method, the generation of the sinusoidal scan signal requires a more complex hardware, e.g. additional capacitors [4] or a H-bridge (four switches) [5].

- **Flux linkage observers [6-7], [9], [16], [26-28]:** These methods exploit the dependence of the flux linkage on the position of the spool. According to (I.7), the flux linkage is an integral and “global” quantity, the measurement of which requires continuous integration (exciting voltage). Compared to the inductance based methods, this approach is less sensitive to measurement noise, enables to compute the attractive magnetic force and represents a global state of the system, which is more preferable if estimating the position of the spool. However, its computation may require the use auxiliary windings [9], [16], [26-27] and integration. Therefore, it has a more expensive hardware layout, which is disadvantageous from the perspective of cost effectiveness. Also, flux linkage based methods may be susceptible to integration error (offset) and to initial conditions.
1.2.2. Estimation of the external load

In certain applications the electromagnetic actuator is subject to a time-varying external load, e.g., to fluid pressure in flow control applications. This input force excites the system and influences the drive current that is necessary to reach the desired position of the spool. If the effect of external load is neglected, then it can cause significant error in the estimation of the position of the spool. In certain applications, it could be beneficial if the magnitude of the external load could be estimated as it could enable to save force or pressure sensors; thus, improving the cost effectiveness of the system. However, a literature review has shown that the effect of an externally applied load, its estimation and its compensation in the estimation of the position is still an open issue considering solenoid actuators. In the majority of the corresponding articles the effect of external forces is not considered; thus, the external force is not estimated; and there are no experimental results presented in the situation when the load changes during the estimation. In [1], external forces are considered to be difficult to predict and model; thus, omitted. However, [1], [2], [3] recorded the set of inductance and current ripple values at fixed spool positions and average currents. From these data, the compensation for external forces might be possible, although information about its magnitude is lost. Measurements were also carried out at fixed spool positions in [6] but with the sliding mode concept considerable parameter insensitivity is ensured. Yet robustness, tracking error, etc. are not tested in case of sudden and significant changes of the mechanical inertia for [6], [7], [9], nor the problem of external forces on position estimation is discussed in [4], [5]. However, [8] achieved and presented results on combined position and force estimation by exploiting the spool motion generated back EMF; yet, this principle has limitations and is not applicable if the spool is stationary or has low velocity. In [16], the flatness-based tracking of an electromechanical variable valve timing actuator is presented with disturbance observer feedforward compensation to account for external forces that are caused by gas pressure. However, the method is based on flux linkage reconstruction; therefore, it has a relatively costly hardware (refer to section 1.2.1) and may become computationally exhaustive for embedded controllers. Improved methods for the combined estimation of position and external force at a low hardware and computational complexity are of great interest and could be of great practical use.

1.2.3. Changes in the electrical resistance of the coil

Knowledge and tracking of the parameters of the model of electromechanical devices are important for achieving high effectiveness at sensorless control applications, because the parameters of the model may change during normal operation. From the viewpoint of possible sources of parameter sensitivity, temperature has a major role as it can cause the resistance of the coil of electromagnetic devices to increase by 40% for a 100 °C temperature rise. In numerous practical applications, the change in the temperature of the actuator is significant (even more than 100 °C), e.g., the starting temperature can be -20°C during winter at start and then the actuator warms up to 125 °C in an automatic transmission unit. Therefore, the measurement or estimation of the electrical resistance of the coil is important for reducing bias in control and sensorless schemes which rely on a model of the actuator. Due to the fact that cost effectiveness is a major principle in engineering practice, especially in embedded applications, model based estimation methods are preferred than direct measurements with dedicated external sensors.
For rotary induction machines the literature on estimating the resistance of the windings is comprehensive; some major contributions include [13-15]. However, for solenoid actuators the methods that are available for rotary motors are not applicable; because solenoid actuators have a single phase structure, they lack cyclic signals and have unique electrical drive conditions [1]. A review of literature has shown that the sensorless methods, which are available for solenoid actuators, do not satisfactorily consider the variations in the coil’s resistance caused by thermal effects [1-9], [16-17], [26-31], [35]. Therefore, efficient methods for estimating the resistance of solenoid actuators are important for improving robustness and effectiveness. Furthermore, an estimate of the resistance of the winding can also provide information about the thermal state of the actuator, i.e., about the average temperature of the coil.

In [6] the resistance is estimated from a lumped, dynamic thermal model that is continuously evaluated accordingly to the internal and external thermal boundary conditions. A possible drawback of [6] is that the estimate of the resistance is susceptible to the parameters of the thermal model, to the initial conditions and to the external thermal boundary conditions which require extra sensors or additional models to be measured, e.g., the ambient temperature. A detailed thermal model is also presented in [18] but it has the same disadvantages as [6]; and may become computationally exhaustive, which is not advantageous in embedded applications. An alternative method is presented in [19] for PWM driven solenoids that directly estimates the resistance from the electrical signals; thus, the aforementioned modeling problems are avoided although the electrical model considers a simple LR model. Since solenoid actuators are most commonly used in embedded applications, the complexity and the computational needs of the methods for the estimation of the resistance have to be as low as possible.

In engineering practice, a common way for driving solenoid actuators is by means of PWM (pulse width modulation) in a single switch battery powered configuration, as it is illustrated in Fig. I.5. According to Fig. I.5 (right side), the current of the actuator flows through different sections of the circuit (energizing paths) during the “on” and “off” periods of each PWM cycle. Considering real applications, the voltage source, the connecting cables, the connections, the switching transistor and the PCB (printed circuit board) all have some resistance which add up to each other and add to the resistance of the coil. Therefore, the current of the actuator encounters different overall resistances in the energizing paths and; by only measuring electrical signals, it is the separate overall resistances that can be estimated. In engineering applications, the difference between the resistances of the energizing paths can be comparable to the resistance of the coil, e.g. 0.5 Ohms to 4 Ohms. If this difference is neglected, then the estimate of the coil’s resistance may become biased.

Fig. I.5: An exemplary PWM drive configuration and the problem of resistance estimation.
I.3. The Experimental Setup

A considerable part of my PhD research was dedicated to the studying of linear electromagnetic actuators and to developing new sensorless methods. In order to elaborately study linear electromagnetic devices, to create and indentify the necessary models and to test the sensorless methods under real operating conditions; I have designed and built an experimental setup that was used during my research, e.g., in Section II. The experimental setup consists of four main parts. The first part is a dedicated mechanical device that clamps the actuator and applies and measures the necessary mechanical signals, i.e. external force and position. The second part is a custom made an analog front end (electrical hardware) which produces the drive PWM, supplies the sensors and conditions their signal. The third one is a data acquisition device, the NI-USB 6150 from National Instruments, which samples and stores the necessary signals. The fourth part is a high level PC (personal computer) interface at which the necessary settings of the measurement can be set and the measurement data can be evaluated. The PC interface was realized by the LabView 2011 software from National Instruments. From the viewpoint of designing and creating the measurement layout, the most difficult part was the design of the unique, dedicated mechanical device which is illustrated in Fig. I.6. A detailed description of the experimental setup can be found in the Appendix 3; here, only a brief description is provided.

The main purpose of the experimental setup is the identification and the modeling of different types of solenoid actuators (wide range of possible spool strokes); therefore, cost effectiveness and flexibility were key design principles. In solenoid actuators, the relevant mechanical quantities are the position (velocity) of the moving part (spool position) and the external load forces. In the proposed setup, the position is measured by a reflective optical sensor (photo diode and transistor) with marginal cost and hardware requirement compared to other traditional transducers e.g. inductive sensors. The optical sensor hosts a light source (diode) and photo detector (transistor) in parallel which are operated at the infrared region. The amount of light arriving to the photo transistor is determined by the distance of the reflective medium (circular disc) thus the position is computed from the emitter current of the transistor. As illustrated in Fig. I.6, the reflective disc is attached to the spool, thus the spool’s motion and its position is captured. The transfer function of the optical position sensing had been captured by a micrometer. The experimental setup has a vertical configuration and the external load forces are realized by the gravitational force of masses. Thus, the load is exactly known in the steady state and does not depend on the position of the spool; therefore, the force sensors and their hardware are not necessary. Friction that is associated with the experimental setup is also overcome by the vertical orientation and by the use of linear bearings. The linear bearings were donated by NBG Masters Ltd. to support my research. The test setup is designed in particular consideration of measuring very small spool strokes (~1 mm).
I.4. Thermal Modeling of Linear Electromagnetic Actuators

An elaborate thermal model of linear electromagnetic actuators is useful for the design, optimization and diagnosis of such devices. Furthermore, a thermal model can be used to determine the internal thermal state of the actuator during operation. With the knowledge of the internal thermal state, the parameters (e.g. the electrical resistance of the coil) of the model, which model serves as the basis of the sensorless methods, can be updated. This is important because these parameters can depend on the temperature. Alternatively to an electrical model, the resistance of the coil can be also determined from a thermal model (internal thermal state) of the actuator. In the following list, the most important “thermally” induced model variations are summarized:

- Resistivity of copper. In almost every model of electromagnetic devices the electrical side consists of some sort of coil resistance. If the device is either subject to a wide operating ambient temperature range or intensive internal heat dissipation, the resistance of the coil (resistivity of copper) significantly changes, e.g. ~40% for a temperature rise of 100 °C,
- The electromagnetic subsystem incorporates a magnetic “circuit”. Considering a temperature change of 100 °C, the core material might have a spatial temperature profile and thermal dependence, which affects its permeability thus the magnetizing curve,
- Magnetic phenomena such as hysteresis, core reluctance and saturation are also related to temperature,
- The electromagnetic subsystem, e.g. the air gap is related to the geometric dimensions of the actuator. Elongation and deformation due to temperature might not be neglected in certain situations,
- The parameters of the mechanical subsystem, e.g., the stiffness of the return spring can also depend on the temperature,
• Friction and viscous damping has to be considered as well,
• If the working temperature range of the actuator varies dynamically in a wide range, subsidiary mechanical phenomena such as fatigue and component wear may have to be accounted for. The properties of the components, e.g. spring stiffness, are also likely to alter permanently.

Considering a valve actuator, three modes of heat transfer are to be distinguished: conduction inside the solid material, convection at the surfaces and radiation at the surfaces [28]. The latter; however, is usually not significant due to the ambient and surface temperatures being relatively low. Extreme temperatures would not be permitted anyway, as it would lead to device malfunction. Therefore, surface convection and internal conduction is considered; however, I will focus on conduction because conduction is the major heat transfer process that determines the spatial thermal distribution inside the solenoid. For convection, a simplified approach is used. The general approaches for creating a thermal model are listed below:

• Lumped network model [18], [63], [74-75]: this is the simplest and computationally most efficient; yet, the least accurate. The model relies on “compressing” the studied area into a single component which describes some kind of mean or average property of the simplified domain. In case of low temperature gradients, lumped models give acceptable result. However, information about spatial distribution and parameter dependence is lost,

• Analytical solution from physical model [52], [63], [75]: a closed form expression of the system is given by solving the main underlying physical equations. Analytical solutions give the most information about and insight into real physical processes and parameter dependence. However, in many situations closed form solutions are not available or restricted to simplified geometries,

• Numerical methods (finite element) [63], [74-75]: depending on the e.g. mesh resolution; they give an accurate solution of the problem irrespective of geometrical complexity, although at the cost of extreme computational needs. Since the governing partial differential equations are approximated, only the solution at discrete locations is determined and any physical or model knowledge is lost. The accuracy of numerical (FE) solutions greatly depends on the size of the mesh and may be susceptible to numerical instability with a large element size.

Linear electromagnetic actuators (valves) are usually implemented in embedded applications which set special requirements towards the needs of CPU and memory. Therefore, if a thermal model of an electromagnetic actuator is to be used then special considerations are necessary for the complexity its thermal model. With the FEM programs requiring a lot of memory and computational effort, either analytical or lumped models are preferred. Due to the fact that lumped models are obtainable from analytical ones, an analytical solution is sought for the thermal processes. Furthermore, analytical solutions are ideal for the optimization and design of solenoids in the sense that they provide full information about the underlying physical processes and show how the behavior of the system depends on the parameters [40], [47-50].

Besides the conduction of heat, the actuator is subject to another important thermal phenomenon which is energy generation (dissipation) inside the actuator. The major phenomena that contribute to the temperature rise inside a solenoid actuator are summarized as follows:

• Joule heating: the winding has a finite resistance due to the conducting material (copper). The drive current which is fed through the coil generates heat. This
dissipation is the most significant one and affects the largest volume fraction of the solenoid. Its magnitude equals the square of the RMS (root mean square) current multiplied by the resistance of the coil,

- Skin and proximity effects [73]: In real applications the coil is excited by PWM (pulse width modulation) thus the current is not merely DC but has AC components as well. The skin effect states that with increasing frequency the effective cross section of the conducting material (penetration depth) decreases, resulting in a higher AC resistance. This leads to increased losses if the same current is to be maintained. Proximity effect, on the other hand, is a well known phenomenon for transformers. It is similar to skin effect but in a larger geometric scale. If there are many turns coupled tightly, the magnetic fields of these turns repel the current into the boundaries; thus, the input AC resistance further increases,

- Core losses [73]: These are mainly eddy current losses in the core material and hysteresis, which are dissipated inside the magnetic elements. Both of them are material and frequency dependent. Since valve actuators are usually operated at relatively lower PWM frequencies, these sources are not so dominant compared to conduction losses; however, at high frequency applications they might be relevant,

- Absorption of mechanical energy: Friction, bouncing of the spool and deformation of the valve return spring. Usually these terms are insignificant compared to the sources which are mentioned previously.

I.4.1. The multilayered medium approach

Heat is mainly transferred inside a solenoid actuator via conduction. From the point of thermal analyses, internal heat generation also has to be considered. With the solenoid actuator being an axisymmetric device, it is reasonable to use a cylindrical coordinate system. The underlying conduction process is multi-dimensional heat conduction in finite, solid media. Assuming a linear, homogenous, isotropic cylindrical body, the unsteady conduction problem is governed by the diffusion equation (Laplace equation), and by the Poisson equation in (I.11) in cylindrical coordinates, provided that heat is generated inside the volume [62-63], [75],

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2} = \alpha \frac{\partial T}{\partial t} - \frac{q(r, \varphi, z, t)}{k}.
\]

(I.11)

The process in (I.11) is the unsteady Poisson equation. If the time derivative vanishes, the steady state is obtained. If the heat generation term \( q \) is zero, the Laplace equation is obtained [62-63], [75]. In (I.11) \( T \) denotes the temperature field in the solid medium, \( r, z \) and \( \varphi \) denote the cylindrical coordinates, \( k \) is the thermal conductivity and \( \alpha \) is the diffusivity of the medium. Solutions of these equations are not trivial and rely on the applied boundary conditions. Regarding valve actuators, the material properties such as thermal conductivity or heat capacity depend on the spatial coordinates, e.g., in case of the winding which incorporates an insulating resin and highly conductive copper. Thus, (I.11) can not be written for the whole domain but for well defined, smaller homogenous fractions. According to the mechanical structure of a valve actuator, quasi homogenous sub domains can be located, even for the winding. With some geometric simplification, a good approximation of the real problem can be provided by considering the actuator as concentric, cylindrical shells that are layered to each other in the radial
direction as shown in the left side of Fig. I.7. For an additional refinement of the model, the previous approach can be extended along the horizontal coordinate as well (2D array of layers). The shells are chosen such that they represent a homogenous domain respectively; thus, (I.11) can be written and solved separately and the corresponding diffusion equations. The part solutions are linked to each other through the layer interface and boundary conditions [62], which impose some sort of continuity constraint (temperature, flux). Contact resistances are also likely to be present.

![Fig. I.7: Simplified model of cylindrical multilayered structure [45] (left); and an exemplary picture of a multilayered medium for slabs (right).](image)

However, it is possible that certain components show such a complex structure that the above partitioning can not be reasonably fulfilled, e.g. the winding. From a general approach, the winding consists of two different materials: a randomly coupled array of wires that is cast out with resin. Obviously, it is not possible to explicitly model such a system, although an estimation might be given by homogenization. This approach considers the composite domain made of a single material that is described by a single, homogenized material parameter. Corresponding literature dealing with this topic can be found in [24], applying winding homogenization to a thermal model of a stator.

I.4.2. Model of the multilayered medium

An exemplary illustration of a multilayered medium (slabs) is provided in the right side of Fig. I.7, and the modeling principles can be briefly described as follows [40-52], [54-56], [59-72]. The multilayered medium is represented as the group (array) of \( N \) linear layers (slabs) with equal height but with different lengths, where the slabs connect to each other at the interfaces (junctions). The layers are piecewise homogeneous, isotropic and are described by the \( k \) parameter of diffusivity. The underlying thermal process is the conduction of heat in solids. On the right side of Fig. I.7, the diffusion takes place only along the \( x \) transverse (in the direction perpendicular to the layers) and in the \( y, z \) longitudinal (in the direction parallel to the layers) coordinates. The layers incorporate the \( f(x,y,z) \) internal energy generation; and on their surfaces the layers are subjected to the homogeneous (excitation is zero) or non-homogeneous (excitation is not zero) boundary conditions. The boundary conditions can be of the following types [62-63], [75]:

- Boundary condition of the first kind: the potential (voltage, temperature) is prescribed for the respective surface,
• Boundary condition of the second kind: the flux is prescribed for the respective surface,
• Boundary condition of the third kind: impedance type boundary condition, i.e., the flux is proportional to the difference between the potential of the ambient and the potential of the surface. The ratio of proportionality is denoted by the $h$ convection coefficient.
• Periodic: it is relevant in cylindrical or spherical coordinates.

The transverse boundary conditions are the boundary conditions that are defined on the surfaces the normal vector of which points into the $x$ direction, i.e., they are in the direction parallel to the layers [42]. The longitudinal boundary conditions are the boundary conditions that are defined on the surfaces the normal vector of which points into the $y$ or $z$ direction, i.e., they in the directions parallel to the layers [42]. The solution of the thermal (potential) field in the layers is sought in the steady-state under the prescribed homogeneous, non-homogeneous boundary conditions and internal energy generation. For obtaining a detailed, in depth solution of the temperature field in the multilayered medium, the solution is preferred to be expressed in an analytical way because analytical solutions give direct insight into the physical processes and lead to a closed from, compact solution. These features of the analytical solutions are preferable in embedded systems and at the design, optimization and fault diagnosis of linear electromagnetic devices. The studied problem is general; therefore, the results are applicable not only to electromagnetic devices but to a far wider field of research and application, which can be described by the diffusion equation, e.g., electrostatics, and by the multilayered medium model, e.g., composite materials. In the related technical literature, this problem is succinctly described as the analytical solution of steady heat conduction in multilayered media.

I.4.3. Analytical solutions to the heat equation in multilayered media

To date, many studies and a variety of analytical solutions have been published that are related to the modeling of heat conduction in composites (multilayered media). Examples of these solutions include the Green’s function approach [41], orthogonal expansion [59-60], eigenfunction expansion [47-48], integral transforms [50], [52], [55-56] and two port network formalism [57-58]. A detailed collection of available techniques can be found in Özisik’s heat conduction book [62]. From the viewpoint of treating the governing partial differential equations, the majority of approaches are based on separation of variables (SOV) or integral transforms, an essential part of which is to determine the eigenvalues of the system. The application of SOV requires homogeneous boundary conditions, from which the eigenvalues are extracted (opposite homogeneous sides). For a single layer, this condition can be easily satisfied by superposition or by an appropriate temperature transform. However, in the case of multilayer laminates, the eigenproblem can become quite complicated due to the boundary conditions that the junctions represent. In such situations, general orthogonality conditions may have unique features [40], [43] and traditional SOV may not be applicable unless special considerations are made. Examples of these considerations include the “natural” orthogonality concept [43-44] to relate the transverse eigenvalues during unsteady conduction and a novel separation of variables [45-46]. Eigenvalues in Cartesian coordinates can become imaginary [40], [42], thereby making analytical solutions even more difficult.

Considering the necessary eigenconditions, another problem arises when the multilayer system is subject to longitudinal excitations. In such situations, the boundary conditions in the directions longitudinal to the layers (parallel to the junction plane) are non-homogeneous and cannot be homogenized; thus, the traditional SOV approach is not applicable. This problem has
caught the attention of many researchers and highlights the limitations of the available analytical solutions [42]. Haji-Sheikh et al. studied the steady [40] and unsteady [41] thermal behavior of a two-slab 3D body with contact resistance, taking into account longitudinal excitations. Tables were also included to assist the eigenvalue computation, although the suggested expressions (transcendental equations) were only given for two layers. De Monte presented a similar structure (two layers) [42] and gave better bracketing bounds and initial guesses for the transverse eigenvalues to aid the searching algorithms. If all of the longitudinal excitations are the same, e.g., far field temperatures, the problem can also be resolved by an appropriate temperature transform [45-46]. Kayhani et al. studied steady [54] and Nourozi et al. studied transient [56] conduction in a cylindrical laminate in the radial and angular directions. By exploiting the periodic angular boundary conditions, longitudinal (angular) excitations did not have to be dealt with. An innovative solution was presented by Kayhani et al. [55], considering general inhomogeneous linear boundary conditions on all of the surfaces for cylindrical composites. The homogenized boundary conditions in the longitudinal direction (with the longitudinal excitations set to zero) had the same form for every layer, from which a Fourier transformation was derived and applied to all the layers. A possible limitation of [55] is if the previous condition for the longitudinal boundary conditions is not satisfied because the layers have different corresponding integral transforms; thus, the solution method in [55] may not be applicable.

If an analytical solution of the heat equation is sought, another problem arises if there is dissipation in the layers. The analytical treatment of such heat diffusion problems requires special attention because the governing PDE becomes non-homogeneous, so separation of variables is not applicable. A literature survey has shown that only a few studies have considered the effect of heat generation. Singh et al. [47] and Jain et al. [48-49] used eigenfunction expansion for the inhomogeneous steady-state, considering an arbitrary distribution of volumetric heat source, and Pradhan et al. [52] applied the method of integral transforms. Mulholland and Cobble [51] gave a general treatment for a 1D conduction problem. In [63], a simple approach with a modified SOV for a single slab was presented. As heat generation is relevant to several engineering applications, e.g., the windings of electrical machines, there is a continuous need for analytical solutions that incorporate either complex or simpler forms of dissipation.

**I.4.4. Numerical difficulties at computing the analytical solution**

There is a consensus in the literature on multilayered structures that boundary conditions of the third kind in directions parallel to the layers (i.e., longitudinal directions) can produce mathematical inconsistencies [42], [44], [46]-[48], [64], [72]. Therefore, most current techniques deal only with boundary conditions of the first and second kind in longitudinal directions, see e.g., [40-41], [45]-[49], [53-54], [56]-[60], [64], [66-67], [69], [72]. Although these theoretical solutions may still be valid for problems with boundary conditions of the third kind, difficulties are encountered when evaluating the parameters of the analytical solution. This problem arises because the longitudinal eigenfunctions of the layers become layer-wise non-orthogonal (the kernel differs layer by layer); therefore, the contributions from all of the eigenfunctions of the adjacent layer must be considered to compute the parameters of a particular transverse eigenfunction. This computational difficulty may also arise if the layers have different combinations of longitudinal boundary conditions of the first and second kind. Formally, this problem is described by a system of linear equations with a coefficient matrix that contains very large and very small elements from the exponential transverse eigenfunctions; thus, the solution (by inversion) of the system of linear equations may be an ill-conditioned problem [58].
Implementation and computation have a limited arithmetic precision in practice, which can result in erroneous solutions because of the accumulation of numerical errors (e.g., from truncation and round-off) during the calculation process. This phenomenon is termed an exponential dichotomy in [58], and the DGTM (direct global transfer matrix) method is recommended for delaying this effect for unsteady conduction problems in two-dimensional semi-infinite media. For cylindrical composite laminates that are layered in the radial direction [47], [49], [54], [56], this computational problem is resolved by exploiting the periodicity of the angular eigenfunctions. A general analytical solution was presented in [55] for steady heat conduction in cylindrical multi-layer composite laminates with general linear boundary conditions in the longitudinal direction. A Fourier transform was derived for the problem, where the kernel of the integral transform was the same for each layer because all of the layers were subject to the same homogenized longitudinal boundary conditions, thus ensuring “global” orthogonality.

I.5. Highlights of The Research

The introductory sections above provided a general overview of the studied fields and highlighted some problems and possibilities for further improvement and research. Accordingly, I now provide a short summary about the main scopes, intentions and contributions of my research.

- **Part 1:** Improved sensorless methods for solenoid actuators with respect to HW&SW requirements, computational complexity and with respect to estimating external forces,
  - The simultaneous estimation of the position of the moving part and the magnitude of the external load. The estimation of the external load is important in e.g. flow controlling applications where a varying load pressure can be present. In technical literature, the estimation of the external load is still an open issue.
  - The estimation of the electrical resistance of the winding from an electrical model. The estimate of the resistance provides information about the internal thermal state of the device. Furthermore, the resistance can change during operation thus biasing the estimate of other parameters (e.g., position) because the sensorless methods rely on a model of the system. Consideration is also given to a non-ideal PWM drive configuration.
  - The methods have to be compatible with PWM drive and with the HW in Fig. I.1.

- **Part 2:** Extension and improvement of analytical solutions to the steady-state heat conduction in multilayered media,
  - The inclusion of non-homogenous longitudinal boundary conditions in the analytical solution. A literature review has shown that these boundary conditions are usually not considered except some special situations.
  - The simplification of the associated eigenvalue problem.
  - Internal energy generation (dissipation) is a relevant phenomenon when modeling e.g. solenoid actuators. The inclusion of forms of internal energy generation in the analytical solution is still an open issue.
  - The computation of the analytical solution in multilayered media can become numerically unstable under some circumstances, which limits its practical applicability. A computational process resolving these numerical difficulties is necessary which is also computationally effective and has a fast convergence even in case of numerous layers.
The body of the dissertation is divided into two main parts, *Part 1: Methods for Estimating the Parameters of Linear Electromagnetic Actuators* and *Part 2: The Multilayered Medium Approach and Analytical Solution to the Diffusion Equation*, respectively. This structuring is intentional, because the research addresses two major, completely different, standalone topics. However, the motivation behind *Part 2* originates from *Part 1*. The workflow of my research, the main problems, motivations and the connection between *Part 1* and *Part 2* are explained in the *Preface*.

At the end of each Section, a conclusion is provided which summarizes the studied problem, my corresponding results with their major properties; and the possible fields of application. This choice of structuring is also intentional. The conclusions provide a general overview; however, the *New Scientific Results* in SectionVI.1 provide only the “essence”, i.e., the core of my research result. Therefore, these contain only the details which are the most substantial yet completely define the studied problem and contribution; and are kept to be as short as possible.
PART 1

METHODS FOR ESTIMATING THE PARAMETERS OF LINEAR ELECTROMAGNETIC ACTUATORS

The first part of my research is dedicated to the development of improved sensorless methods for linear electromagnetic actuators. Due to the fact that cost effectiveness is a major principle in engineering practice, especially in embedded applications, the low computational load and complexity of the methods are key principles. Also, the sensorless methods have to be compatible with the PWM drive technique. Under certain working conditions, varying external forces might be present on the spool of the actuator, e.g., from fluid pressure. In the majority of technical literature, the effect of an externally applied load and its compensation in the sensorless principle of solenoids is still an open issue, only a few articles consider or directly study this effect [16]. Also, no experimental test results are presented on the load disturbance rejection of the proposed methods. In Section II, a method is presented for simultaneously estimating the position of the moving part and the external load. In Section III, a set of methods are presented for estimating the electrical resistance of the coil; thus, the error in the estimation that is associated to changes of the parameters can be reduced.

II. THE ESTIMATION OF THE EXTERNAL LOAD AND OF THE POSITION OF THE SPOOL

This section presents a detailed experimental analysis of a linear electromagnetic actuator (valve); and a PWM based method is developed for the simultaneous estimation of the position of the moving part, and of the external force on the valve. The estimation of the force is advantageous in flow control applications because the actuator can be subjected to a significant, time-varying external load due to the pressure and flow of the controlled medium. The developed method has low computational complexity and uses an experimental model of the device, i.e., the set of inductance and average current curves which are recorded as the function of the position and of the external force. For the identification of the inductance (it is a dynamic quantity), a scan signal is used during the PWM drive, which scan signal is first generated by the inherent chopping of the PWM (current ripple); and secondly by adding a sinusoidal component to the base of the duty ratio. The hardware requirements of the method and of generating the scan signals are simple, i.e., the minimal hardware that is illustrated in Fig. I.1. Therefore, the developed method is suitable for cost effective embedded systems. During the experimental analyses the major phenomena, e.g., the effect of scan frequency and supply voltage, are studied.
Despite temperature being an important input quantity, thermal analyses are not in the scope of this section but discussed in Section III. The related measurement results were captured at the same ambient (room) and valve operating temperatures. The remainder of this study is structured as follows. In Section II.1 the two scan signals are described. In Section II.2, a detailed experimental analysis of a valve actuator is presented; it studies the effects of e.g. supply voltage, external load on the main parameters of the actuator. In Section II.3, the method for the estimation of the position and force is presented, and in Section II.4 the developed method is applied to a solenoid valve and tested under real operating conditions. The experimental setup that was used for the experimental analyses is described in Section I.3 (Appendix).

II.1. Scan Signal Generation

The sensorless principle is based on measuring electrical signals (current, voltage) from which information about e.g. spool position is extracted. From one point, one can rely on observing and tracking global parameters such as flux linkage [6], [7], although initial conditions and integration error may yield some problem. Alternatively, a scan signal can be superposed onto the drive signal thus exciting the system “locally” [1]-[5], and measuring its local properties, e.g. inductance or impedance. However, these local states might overlap, thereby making estimation more difficult. In this study, the latter process is preferred because it has a considerably simpler hardware requirement, and a method similar to e.g. [1]-[5] is presented.

II.1.1. Scan signal by direct PWM drive

Since the overall magnetic reluctance of the solenoid is partly determined by the length of the working air gap, which is in a direct relation to the position of the spool, the inductance is expected to be position dependent. By taking a simplified LR electrical model [1]-[3], the voltage equation can be expressed as (II.1),

\[ u_s(t) = iR + \frac{d\psi(x,i)}{dt} = iR + \frac{\partial \psi(x,i)}{\partial i} \frac{di}{dt} + \frac{\partial \psi(x,i)}{\partial x} \frac{dx}{dt}. \]  

(II.1)

The first term in (II.1) refers to the resistive voltage drop and the second one to the induced voltage, \( \psi \) is the flux linkage and \( u_s \) is the supply voltage. Equation (II.1) can be further expressed to (II.1). The third term in (II.1) refers to the motional back EMF (electromotive force), a voltage induced due to reluctance changes by spool movement. In steady state or at low spool velocity it is insignificant thus will be omitted in further analyses. With an \( \psi = Li \) substitution, the inductance can be expressed from (II.1), although it is meaningful only if the \( di/dt \) value is nonzero and sufficiently large. Practically, inductance measurement will be processed using the inherent ripple caused by PWM [1]; therefore, the switching frequency was chosen low enough (500 Hz) to ensure significant ripple current without any spool oscillation. In [2], [3] the current change over the PWM on period is recorded and [1] calculates the incremental inductance. Here, an average inductance from the time integral of (II.2) on a specific PWM period (on or off) is used (II.3), rather than simply current ripple. Moreover, no auxiliary coils are needed compared to [9].
\[ L(x,i) = \frac{\Delta V_S}{\Delta i} = \frac{\int_{t_1}^{t_2} (u_s - iR)dt}{i(t_2) - i(t_1)}. \]  

(II.2)

The denominator contains current difference; therefore, (II.2) has some susceptibility to noise which can be reduced by taking averages. The main steps of implementing this approach are listed below:

- Use the (minimal) PWM drive hardware that is illustrated in Fig. I.1,
- Apply PWM drive, which has a switching frequency \( f_{PWM} \) and the duty ratio \( d \), to the actuator,
- Choose the \( f_{PWM} \) to be low enough to have a sufficient current ripple (but no spool oscillations) thus the effect of measurement noise is reduced,
- Consider the “on” and “off” periods of a PWM cycle and sample the corresponding exciting voltage \( U_s \) and current \( i \) waveforms with the \( f_s \) sampling frequency. The sampling frequency has to considerably higher (x100) than the switching frequency,
- Compute (II.2) for the “on” and “off” periods of the PWM cycle respectively, and compute the average (integral) of the current for the PWM cycle,
- If steady-state is reached, perform the above process for multiple, successive PWM cycles and average the computed parameters in order to reduce measurement noise.

Here, two parameters of a solenoid are to be estimated, namely spool position and external load. Regarding spool position, it seems reasonable to associate it with the inductance that is yielded by (II.2) due to the position dependent air gap (magnetic reluctance). However, estimation of two independent parameters from a single variable is not possible. For this reason, it is necessary to find a second quantity which is closely related to force. The force equation in steady state is given in (II.3),

\[ F_{spring}(x) = F_{magnetic}(i,x) + F_{external} \]  

(II.3)

At equilibrium position, the return spring compensates the pulling magnetic force plus all external disturbances. In fact, the magnetic force, which is exerted through the air gap, is produced by the magnetizing (coil) current. Of course, the magnetic force also depends on the position since less current is needed for a smaller gap (higher reluctance) to produce the same force. Therefore, associating the average coil current (denoted by “\( i \”) with load estimation; thus, choosing it to be the second parameter seems reasonable.
II.1.2. Scan signal by sinusoidal duty ratio

In [5], [6] the idea is to excite the system with a sine close to the resonance frequency and to measure either phase shift or attenuation, which requires auxiliary hardware for scan signal generation and signal reconstruction. I use a considerably simpler hardware layout, which is illustrated in Fig. I.1, to realize scan signal generation and signal reconstruction. The device is driven by PWM in a low-side single switch configuration. The proposed concept is the following: using the single switch (unipolar), low-side drive configuration, a sinusoidal component is added to the base duty ratio of the PWM (II.5). The necessary signals to be measured are coil current and supply voltage,

\[
Dr(t) = ZOH \left[ D_0(t) + d(t) \left( 1 + \sin \left( \frac{2\pi f_{\text{scan}}}{f_{\text{PWM}}} t \right) \right) \right]. \quad (II.4)
\]

Since the duty ratio cannot change until the end of its pulse period, the sine is fed through a ZOH (zero order hold) unit, introducing some additional phase lag. That is, at the beginning of a new PWM cycle the corresponding duty ratio is the base duty ratio plus the value of the sine wave at the beginning. This way of signal generation can be readily implemented in embedded applications. The offset by “1” is necessary to avoid negative duty ratios. Also, the term \( D_0 \) corresponds to the base duty ratio, as the valve has to be actuating as well. The sine wave can be generated using a dedicated LUT (look up table). Regarding the scan frequency, it is chosen to be significantly lower than the switching frequency. The left side of Fig. II.1 illustrates such a waveform and the corresponding PWM output, where the ZOH effect is quite clear. The right side of Fig. II.1 shows the FFT (Fast Fourier Transform) of both duty ratio and PWM output, with the sinusoidal part visible at the first bin. In fact, its amplitude is proportional to the amplitude of the duty ratio and of the supply voltage. With the frequency of the sinusoidal part exactly known; the parameters (impedance) of the valve are to be measured and determined only at the scan frequency; therefore, a full FFT computation is not necessary. It has to be noted that the criteria of coherent sampling must be met to obtain a good frequency estimate; and the sampling frequency must be high enough to minimize anti aliasing due to the rectangular switching waveform, although the electrical subsystem of the valve behaves as a low pass filter. Sampling can be internally generated by a DSP and triggered when the LUT turns over.

![Fig. II.1: Duty ratio and corresponding PWM output (left). FFT of the duty ratio and PWM output (right).](image)

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Compared to [5], this method uses a far simpler hardware configuration. Instead of a sinusoidal voltage source, excitation is created by chopping a DC one, i.e., a battery in a sinusoidal way. Additionally, the external capacitor and other dedicated phase detecting circuitry are omitted and the same coil is used for actuating and sensing purposes. In [6], a DC source is used and the valve is driven by an $H$ bridge consisting of four switches and additional drive circuits. The valve is excited at its internal resonance frequency and the sum of the squared current difference between the signal and average current is linked to position. Instead, I use a single switch unipolar layout and I compute an FFT (coherent sampling) at the scan frequency to evaluate current amplitude and phase shift. The FFT is a robust estimation. Major drawback of the proposed method is its possibly lower estimation bandwidth as the estimation speed equals the scan frequency of the sine, which is significantly less compared to the direct PWM drive method (Section II.1.1), where the position can be estimated in every PWM cycle.

In the proposed method, mainly three effects determine the uncertainty of estimation. First, jittering during the trigger period might introduce an error in the phase of the FFT vector. However, sampling can be synchronized internally by the DSP; therefore, it is almost zero or a constant offset. Secondly, if a significant ambient noise is present in the sampled current waveform then some error might be injected into the corresponding FFT bin. In case the length of the FFT increases, the “constant” noise power spreads through a wider spectrum thus enhancing SNR (signal to noise ratio). If the error is uncorrelated with the input signal, the quantization error has a uniform distribution in the [-LSB/2; LSB/2] interval, where LSB stands for least significant bit. If assuming a white noise, (II.5) gives a good approximation for its power. This uncertainty can be imagined as the signal’s FFT vector with a circle attached to its tip with variance (II.5),

$$P_{\text{quantization}} = \frac{\text{LSB}^2}{12}. \quad (\text{II.5})$$

The main steps of implementing this approach are listed below:

- Use the (minimal) PWM drive hardware that is illustrated in Fig. I.1,
- Apply PWM drive, which has a switching frequency $f_{\text{PWM}}$, to the actuator. In every PWM cycle, update the duty ratio accordingly to (II.5), i.e., a sinusoidal component is superposed onto the main drive signal (duty ratio) $D_0$.
- The amplitude of modulation $d$ has to be small enough not to cause significant oscillation of the spool,
- The frequency of the scan signal $f_{\text{scan}}$ has to be considerably smaller compared to the switching frequency for the sake of an appropriate scan signal generation,
- Use a switching frequency that is an integer multiple of the scan frequency,
- Synchronize the following measurement process to the internally generated, sinusoidal component of the duty ratio (e.g. to zero angle),
- Wait for the system (electrical subsystem) to reach steady state,
- Sample the current and supply voltages with the rate of $f_s$ sampling frequency for the duration of integer multiples of the time period of the scan signal,
- Assuming that the supply voltages remained constant, compute the average of the supply voltage,
- Perform an FFT on the recorded current waveform at the scan frequency to obtain the amplitude of the current response,
- Compute the impedance ($LR$ circuit) at the scan frequency.
II.2. Experimental Analyses

An experimental model of a valve actuator is established by applying the proposed scan signals and methods for impedance identification. Furthermore, I also study the effects of supply voltage, scan signal frequency and external load on the electrical impedance of the solenoid, which impedance (\(RL\) model) is derived by both of the methods. Regarding further measurement processes, the duty ratio of the drive PWM was swept forwardly and then reversely; thus, recording full cycles of current and inductance curves as the function of spool position. On the different directions (reverse, forward), a polynomial approximation was performed and a much densely interpolated (128 points) curve was stored in a LUT (Lookup table) as a “numerical” model of the actuator.

II.2.1. Effect of supply voltage

During the PWM “on” period, the solenoid valve is subject to the full supply voltage which can vary significantly, e.g., in battery powered applications. Since secondary magnetic phenomena, such as eddy current intensity, is partly related to the speed of flux change, the behavior of the valve is expected to depend on the chopped voltage level. Therefore, the effect of supply voltage was investigated at a constant external load condition.

Considering the direct PWM drive inductance identification, the current commutates through the freewheeling diode (almost constant voltage source) during switch off; therefore, the core’s (de)magnetization is rather current (coil resistance) than supply voltage dependent than compared to turn on. This might cause the effect of eddy current intensity being less significant at switch off. In [1]-[5], [8], [9] the possible influence of eddy current intensity caused by voltage variations is not studied. In [6], secondary magnetic phenomena are neither modeled and [7] also omits the possible problems of eddy current intensity, yet flux linkage is recorded at different input voltages. Therefore, (II.2) was calculated for the “on” and “off” periods respectively, at the different drive voltage levels. The results are presented in Fig. II.2, where the left side illustrates the “off” period and the right side illustrates the “on” period inductance. The measurement data indicate that the previous assumptions were correct. At switch off, (left side of Fig. II.2) the inductance depends only slightly on the supply voltage, although it varies significantly during turn on. Besides, the average of the current (Fig. II.2) is insensitive to the input PWM voltage, because it is related to a global, absolute state (flux) despite (II.2), which is locally calculated. In the following, the PWM “off” inductance was calculated and used for further analyses.

Considering the case of a sinusoidal scan signal, the same analysis was carried out at a 5 kHz switching and 83.33 Hz scan frequencies and an \(RL\) model was identified. The duty ratio’s sinusoidal part was adjusted such that the scan voltage amplitude remained constant; thus, the amplitude of the scan signal was the same. With average coil current being a global parameter of the system, it is not expected to vary with the different supply voltage levels compared to the inductance. Experimental analyses in Fig. II.4 indicate the previous assumptions to be correct. According to the left side of Fig. II.4, the average coil current almost remains the same but the inductance (right side of Fig. II.4) slightly increases with the supply voltage. The effect of EMF is also visible; therefore, the amplitude of the modulation has to be reduced.
II.2.2. Effect of external forces

The electromagnetic actuator can be subject to varying load excitations from e.g. fluid pressure in flow control. For a complete analysis, this effect has to be considered because the necessary magnetic force; thus, coil current to reach the same spool position changes if a different external load is applied. In the following analyses, the coil current and the inductance curves were recorded with the two scan signals at different external forces but at a constant 12 V supply voltage. Considering the sinusoidal scan signal, it used a 5 kHz switching frequency, 83.33 Hz modulation frequency and 5% peak to peak sinusoidal component in the duty ratio, thus EMF
becomes negligible. The results that were obtained by the direct PWM drive and by the sinusoidal duty ratio method are plotted in Fig. II.5 and II.6, respectively. In Fig. II.5-6, a strange phenomenon can be observed, i.e., the inductance decreases after a certain position despite being more immersed into the housing. Concerning further estimation methods, this non-monotonic behavior causes difficulties, as multiple position values are linked to the same inductance. This effect can be denoted to secondary nonlinearities (e.g. saturation, nonlinear magnetization curve [1], [10], [12]), namely that the inductance is also current dependent. The eddy currents also contribute to this effect [3], because during dynamic excitation the flux lines are more repelled with the spool being more immersed. The recorded inductance data show a similar shape and behavior to the results presented in [10], which were obtained from finite element simulation.

![Fig. II.5: Inductance (left) and average current (right) vs. position and external force, direct PWM drive.](image)

![Fig. II.6: Inductance (left) and average current (right) vs. position and external force, sinusoidal scan signal.](image)

**II.2.3. Effect of the frequency of the scan signal**

Considering the identification of the inductance by the sinusoidal scan signal, the inductance is expected to depend on the frequency and amplitude of the scan signal. Furthermore, the scan frequency limits the speed of position estimation. Contrarily, the necessary coil current is not expected to significantly vary with the frequency. The measured inductance, however, is expected to increase with a smaller frequency as secondary magnetic phenomena, such as eddy current intensity, become less significant. Experimental measurements have been conducted at a constant supply voltage of 12 V and 0 N external load, the results of which are presented in Figs.
II.7-8. The “pp” symbol in the figures refers to the peak to peak percent in duty ratio, which is twice the amplitude of the sine. Thus, 0.1 pp refers to a sinusoidal component which has the amplitude of 5%. The switching frequency was 5 kHz.

The results match the expectations; the necessary average coil current remains the same irrespective of the scan frequency and modulation amplitude because coil current is a “global” quantity of the system. However, the inductance in Figs. II.7-8 shows significant changes. It can be also observed that the drop in the inductance (valley) at small spool positions becomes more significant with increasing modulation amplitude. This effect can be denoted to the back EMF, i.e., the spool is oscillating. Apart from the effect of EMF, the inductance varies slightly with modulation amplitude in Fig. II.7, but at higher frequencies the variations of the inductance with respect to position diminishes due to secondary magnetic phenomena e.g. eddy current intensity.

![Fig. II.7: Average coil current (left) and inductance (right) at different amplitudes of the scan signal.](image)

![Fig. II.8: Average coil current (left) and inductance (right) at different frequencies of the scan signal.](image)

II.3. Estimation of the Position and External Load

For estimating the position and the external load simultaneously, at least two parameters are necessary which are related to force and position, respectively. Regarding spool position, it seems reasonable to associate it with the inductance due to the position dependent air gap (magnetic reluctance); however, it also depends on current. On the other hand, the (average) coil current determines the magnetic force that is exerted onto the spool; thus, it can provide information about its magnitude. Yet, magnetic force is also a function of position as less current is needed to produce the same force in case of a smaller gap (higher reluctance).
Compared to the coil current curves, which are strictly monotonic, the inductance takes the same values at multiple positions. This can cause problems during the estimation because the inverse of the captured transfer function has to be performed, which can be imagined as drawing a horizontal line from the measured values and taking those positions where it intersects the recorded inductance curve. With the help of the monotonic coil current; however, this position overlapping can be resolved, because different currents belong to those positions that yield the same inductance. Therefore, only that position is the “valid” one from the multiple choices which is the best related to the measured average coil current.

In the above reasoning, the external load was assumed to be constant; thus, it was a local decision problem. Now the general process is introduced. The magnitude of external load and spool position are estimated using the previously described inductance and average current \((L, i)\) pair. From the interpolated position versus current and inductance curves, which were obtained at different loads (Fig. II.4-5), a main LUT is formulated that handles the forward and reverse directions separately due to hysteresis. Let us assume that the spool is at position \(d\) and force \(F\) is applied. If all \(L, i\) pairs are taken at \(d\) from the main LUT, there will be only a single pair being mutually the closest to the calculated \(L\) and \(i\) values respectively, at \(F\). In our case, \(d\) and \(F\) are unknown but \(L, i\) available. Regarding the search algorithm, only data at specific force levels are available (main LUT); therefore, the estimated force also takes discrete values. Strictly speaking, this is not an estimation, but decision making problem. The method is the following:

- The inductance and average current curves are organized by a given set of external force in the LUT. At each force level, the spool positions, which are related to the calculated average coil current and inductance, are retrieved separately.
- At all force levels, there is a position estimated from coil current and another (or multiple) from inductance. The absolute difference of these two positions is taken and a minimum is to be found along the force, see Table II.1. However, there might be multiple position values that are related to inductance at the same force; therefore, the above process is also performed “locally”, for each force level.
- The force, at which the difference takes its minimum, will be the chosen force. Notice that at the “real” force the \((L, i)\) pair must give the same position. At all different load levels; however, these will significantly differ.
- In case the force is found, the chosen position will be the one that is derived from the average current, because it is less susceptible to measurement noise.
- Hysteresis is taken into account by monitoring the change in the average coil current; and the corresponding curves are used for the decision making.
- Data are stored at only specific force levels; therefore, it is not strictly an estimation but rather a decision making problem with a force quantizer. However, interpolation is possible but it is not in the scope of the present study.

<table>
<thead>
<tr>
<th>(d^*) from (i)</th>
<th>(F_1)</th>
<th>(F_2)</th>
<th>(F_3)</th>
<th>(\ldots)</th>
<th>(F_N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d^*) from (L)</td>
<td>(y_1)</td>
<td>(y_2)</td>
<td>(y_3)</td>
<td>(\ldots)</td>
<td>(y_N)</td>
</tr>
<tr>
<td>difference</td>
<td>(</td>
<td>x_1 - y_1</td>
<td>)</td>
<td>(</td>
<td>x_2 - y_2</td>
</tr>
</tbody>
</table>
II.4. Experimental Results

Experimental analyses were carried out to estimate the position of the spool and the external force in an open loop configuration. The scan signal (inductance computation) utilized the direct PWM drive approach in II.2.1. The supply voltage was held constant at 12 V. During the analyses such external forces were chosen that interpolation in force was not necessary, namely it coincided with one of the exciting forces at which the experimental model had been captured. For this reason, the output force is “quantized” and only such values can be returned which are defined in the main LUT. In case a different force is applied, the decision process rounds it up and gives the closest match. However, interpolation, i.e., to estimate force levels that are between the recorded ones, is not in the scope of present study. First, the position was estimated under a triangular duty ratio. The external force was zero but unknown to the system. The results of position estimation are illustrated in Fig. II.9. According to Fig. II.4, the coil current and the inductance curves are relatively flat at small displacements and at zero load; therefore, a small measurement error causes a large difference. Data concerning the “static” external force estimation is presented in II.10, i.e., the duty ratio was held constant and from a zero force condition the magnitude of the external load was altered to 0.5N, 0N, 1N, 1.5N, 0N, 2N and back to 0N. The results indicate that both the position and the force could be estimated. Analyses related to the dynamic behavior of position and load estimation are presented in Fig. II.11, where the duty ratio was following a triangular trajectory while external load was arbitrary altered. Presently, the forward and reverse curves during the estimation are switched “hardly”; however, the transition during hysteresis is usually softer thus some error is produced at turnover.

![Fig. II.9: Position estimation at a triangular duty ratio (left) and error (right), 0 N load. (direct PWM drive)](image1)

![Fig. II.10: Open loop position (left) and external load estimation (right). (direct PWM drive)](image2)
The proposed method was also tested by using the concept of sinusoidal duty ratio for generating the scan signal (Section II.1.1). The parameters of the scan signal were 5 kHz PWM frequency; 83.33 Hz scan frequency, 0.05 pp amplitude and 85.33 kHz sampling frequency. The measurements were carried out similarly to the previous investigations.

Data concerning the “static” external force estimation is presented in Fig. II.12. The duty ratio was held constant and from a zero force condition the magnitude of external load was consequently altered to 0.5N, 0N, 1N, 1.5N, 1N, and back to 0N. A few dips and spikes are observable in the estimates of the force and position. This error can be denoted to the quantization of output force and measurement noise. Given a noisy input signal, if its variance is high enough, there is a chance that the algorithm makes a false decision thus jumps to adjacent force levels. Since force is decided first and position is associated with it afterwards, position error in Fig. II.12 is large in case of false force decisions due to the rough force resolution in the main LUT. Altogether, the results demonstrate that the presented methods are suitable for the simultaneous estimation of the position of the spool and of the external load.
II.5. Conclusion

This section presented a method for simultaneously estimating the position of the moving part and the external force in a solenoid actuator. Two approaches were also presented to generate a scan signal with which the actuator was studied and its empirical model was identified. In the first case, the inherent chopping of the PWM was employed thus the estimation could be performed in every PWM cycle. In the second case, the scan signal was a sine that was generated by adding a sinusoidal part to the duty ratio. The developed methods use PWM and are applicable in a very simple hardware configuration, which property makes them suitable for cost effective embedded applications. A detailed experimental analysis of a solenoid actuator was performed as well; and the effects of supply voltage variations, external forces and scan signal on the identified model were investigated. From the experimental results, a model of the actuator was derived; and the position and the external load could be successfully estimated. Therefore, the cost effectiveness of the systems which utilize solenoid devices can further improve.
III. THE ESTIMATION OF THE ELECTRICAL RESISTANCE OF THE COIL

Section II discussed methods for the combined estimation of the external load and of the position of the spool. However, the resistance in the electrical model of the actuator was considered constant although in practical applications it can change significantly due to energy dissipation inside the coil and due to changes in the ambient temperature. In this section, a set of PWM based methods are developed for estimating the resistance of the coil of solenoid actuators from both steady and transient PWM signals. Furthermore, consideration is given to the practical case that the current flows through different circuits during the on and off periods of the PWM cycles, the overall resistance of which circuits can be different. Thus, an error which depends on the duty ratio is produced in the estimate of the resistance if the difference between the overall resistances is neglected. The special requirements of embedded systems, e.g. limited memory and CPU capacity, are also considered. The remainder of this study is structured as follows: in Section III.1 the studied problem is presented. In Section III.2 the resistance is estimated from the steady PWM waveforms and in Section III.3, a set of methods are presented that estimate the resistance from the transient PWM waveforms. In Section III.4 the methods are studied through computer simulations, and through experimental analyses in a real electromagnetic valve (solenoid actuator).

III.1. Statement of the Problem

A solenoid actuator is driven by PWM in a low-side single switch configuration as it is illustrated in Fig. III.1. By measuring only the main electrical signals, i.e., supply voltage and coil current; its resistance is to be estimated for compensating the changes in the resistance which are caused by e.g. thermal effects. In practice, the voltage supply, the switching transistor, the connecting wires and the electrical connections have resistances which add to each other and to the resistance of the coil. Therefore, the separate energizing paths, between which the coil current commutates in each PWM cycle, have different total resistances. If the switch is turned on (PWM “on”), the coil is energized and its current flows from the supply through the switching transistor and through some connecting wires, as it is illustrated in the left-hand side of Fig. III.1. If the switch is turned off (PWM “off”) the coil’s current commutates through the freewheeling diode and flows through a different path that is illustrated in the middle side of Fig. III.1. In Fig. III.1 the terms $R_C$ and $R_D$ refer to the resistances of cables and connections, $R_T$ refers to the junction resistance of the switching transistor, $R_S$ and $L_S$ refer to the resistance and inductance of the solenoid respectively and $U_H$ refers to the supply voltage. The drive PWM has a time period $T$ and duty ratio $d$, and the sampling frequency is considerably higher than the switching one.

The aforementioned problem can be generalized accordingly to the right-hand side of Fig. III.1, where $R_H$ and $R_L$ denote the sum of the corresponding subsidiary resistances and $R_S$ denotes the coil’s resistance. In each energizing path, the solenoid is subject to the $U_H$ (PWM on) and $U_L$ (PWM off) supply voltages, respectively. Considering Fig. III.1, the $U_L$ is represented as the forward voltage drop on the freewheeling diode.

The resistances which can be estimated are the overall resistances in the energizing paths which consist not only of the coil’s resistance; therefore, they are always higher than (or equal to)
the coil’s resistance. If the resistance difference between the overall resistances is not considered but a single “equivalent” resistance is estimated, then the estimate of the resistance will be between the values of the overall resistances of the energizing paths. Furthermore, it will also depend on the duty ratio thus a bias is induced which depends on the duty ratio. However, if the difference in the overall resistances is considered, then a better estimate of the coil’s resistance can be provided which will correspond to the lower one of the resistances of the energizing paths, i.e., \( R_H + R_S \) or \( R_L + R_S \) in Fig. III.1. Further on, the overall resistances of the energizing paths are denoted as in (III.5).

In the following sections, I differentiate between two situations. In Section III.2, the overall resistances of the energizing paths are estimated from the PWM steady-state signals. The electrical model of the solenoid is assumed to be a series inductance-resistance circuit with an inductance that depends on the current; however, eddy current effects are negligible. As the steady-state waveform is considered the back EMF has no significance.

In Section III.3, the overall resistances of the energizing paths are estimated from the transient electrical signals, before steady-state is reached. Thus, if the system undergoes a long transient period, i.e., the control signal changes, resistance data can be still provided. Compared to the previous modeling assumptions, the inductance is considered as a linear, constant inductance and eddy currents are negligible. From a modeling viewpoint, the effect of EMF can be represented as voltage source to the electrical system. For the methods in Section 3, the EMF is considered to be negligible or as a constant voltage input. In case of solenoid actuators (or contactors), the stroke, i.e., the total change in the working air gap can be relatively small (less than one millimeter) thus the change in the magnetic reluctance is small [1]; enabling the EMF to be omitted. For actuators which have larger stroke, this assumption can pose some limitation. However, it is possible to compensate the supply voltage input with the EMF part by using a previously defined EMF model and measuring the velocity of the moving part. Considering the fact that mechanical time constants are usually much larger than electrical time constants, it can be reasonable to assume that the changes in the velocity of the spool and in the position dependent magnetic reluctance are insignificant during a single estimation; thus, the EMF voltage can be considered to be constant. Then, the methods in Section III.3 apply.

In both situations (sections III.2 and III.3), the resistance depends only on temperature; and the thermal time constant of the system is considered to be larger than the electrical time constants; therefore, changes in the resistance are negligible during the period of a single estimation.

Fig. III.1: Non-ideal PWM drive configuration (subsidary resistances) and generalization (right).
III.2. The Steady-State Based Method

In the steady-state of the PWM the coil current fluctuates between a high and low level which are unchanged during the PWM cycles. Additionally, the overall flux change in a PWM cycle must be zero; otherwise it would lead to accumulation in magnetic energy thus increase in current (continuity of flux). An exemplary steady-state PWM current waveform, with a hysteretic inductance function, is illustrated in Fig. III.2.

![Exemplary steady state PWM current waveform with hysteresis in the inductance.](image)

Using the voltage induction law for the PWM “on” and “off” periods and exploiting flux balance; the governing equations become (III.1)-(III.5). The symbol $d$ refers to the duty ratio, and $T$ refers to the time period of a PWM cycle, and $\Psi$ refers to the magnetic flux, The subscripts $H$ and $L$ refer to the high and low periods of the PWM cycle,

$$\Delta\Psi_H = \int_{i_H}^{i_L} L_H(i) di = \int_0^{dT} (U_H(t) - R_A i(t)) dt,$$  

(III.1)

$$\Delta\Psi_L = \int_{i_H}^{i_L} L_L(i) di = \int_{dT}^{T} (U_L(t) - R_B i(t)) dt,$$  

(III.2)

$$\Delta\Psi_H + \Delta\Psi_L = 0,$$  

(III.3)

$$R_A \int_0^{dT} i(t) dt + R_B \int_{dT}^{T} i(t) dt = \int_0^{dT} U_H(t) dt + \int_{dT}^{T} U_L(t) dt,$$  

(III.4)

$$R_A = R_H + R_S, \quad R_B = R_L + R_S.$$  

(III.5)

According to (III.4), the inductance function has no effect on estimating the resistances from the steady-state waveform because only the average supply voltage and average current values are that matter. With (III.4) expressing a linear combination of the resistances at a given supply voltage and duty ratio setting, multiple steady-state current waveforms are necessary,
which are captured at different supply voltages or duty ratios. From the successive evaluation of (III.4), a system of linear equations can be formulated (III.6), the solution of which yields the overall resistances in the energizing paths,

$$
\begin{bmatrix}
\int_0^{d_1T} i_1(t) dt \\
\vdots \\
\int_0^{d_nT} i_N(t) dt \\
\end{bmatrix}
\begin{bmatrix}
U_1(t) dt \\
\vdots \\
U_N(t) dt \\
\end{bmatrix}
= 

\begin{bmatrix}
R_A \\
R_B \\
\end{bmatrix}
\Rightarrow

\begin{bmatrix}
R_A \\
R_B \\
\end{bmatrix} = \begin{bmatrix}
D_1 & \cdots & D_n \\
\end{bmatrix} \cdot u.

(III.6)

In practical PWM applications, it is more convenient to change the duty ratio than the supply voltages because the supplies are usually constant as they provided by a battery and voltage stabilizer. Having the elements of the coefficient matrix in (III.6) computed from average values (III.4), the noise suppression of the estimation is greatly improved. Drawback of the presented method is that the steady-state waveform is required which may take a long time to reach, especially if the control signals are changing. Furthermore, the duty ratio has to be always altered in order to construct (III.6).

Considering embedded applications, the computational complexity of the presented method is low. In order to construct (III.6), the current waveform has to be integrated during the “on” and “off” periods of the PWM cycle respectively, and the average exciting voltage has to be calculated for a PWM cycle. The integration can be replaced by a finite summation, which is computed rapidly even in a low-end microcontroller, although the sampling frequency (number of samples in a PWM cycle) has to be high enough to reduce truncation error at the integration. The only computational difficulty may arise at solving the system of linear equations for the overall resistances (inversion). However, present microcontrollers have an increasing computational power and some devices have hardware support for floating-point operations, e.g., in Cortex M4.

### III.3. The Transient State Based Methods

Compared to the previous section, the steady-state of the PWM is now not reached but the system undergoes a transient period due to a change in the control signal (duty ratio). However, in certain situations it might be still necessary to provide an estimate of the resistance, e.g. the transient period is long enough for the resistance to change considerably. In the following subsections III.3.1 to III.3.3, a set of low-complexity computational methods are presented for the “transient” estimation of the coil’s resistance. Unlike the method in section III.2, the following methods can not directly scope with the nonlinear inductance behavior, which may pose some limitation with respect to practical applicability. However, this limitation can be mitigated as the inductance can be considered “almost” constant if the underlying range of the coil current, from which the estimation is performed, is not very large. In subsection III.3.4, some further improvement is presented for reducing the bias if the inductance depends on the current.

With the modeling assumptions described in Section III.1, the estimation of the resistance is based on the following principle: if the supply voltages are constant and the duty ratio does not change, then the evolution of the “average current in a PWM cycle” follows the three parameter exponential function in (III.7) throughout the PWM cycles, even if the overall resistances in the
energizing paths (time constants) are different. A mathematical proof of this statement is provided in the Appendix 1. Thus, full information about the underlying process (time constant and steady-state current) can be extracted from the average current waveform by e.g. fitting a three parameter exponential curve to the samples. In this study, the resistance data are extracted by means of exponential fitting to the samples of the average current on a PWM cycle.

In the following methods, the basis of the estimation is the average current on a PWM cycle and not the current signal under a PWM cycle. The main reasons are noise suppression and computational effectiveness. By taking the average (integral) of the current on a PWM cycle, the noise which disturbs the measurement can be considerably reduced. The magnitude of the noise suppression is related to the number of samples in a PWM cycle (the higher the better), which can be considerably higher than the PWM switching frequency (e.g. 100 times), even in low-end microcontroller applications. The operation of averaging or integral, which is a summation, can be rapidly computed and require insignificant resources. Furthermore, the matrix operations, recursive and fitting methods, from which the resistance data are extracted e.g. exponential fitting in subsection III.3.1- III.3.2, are thus to be performed on far less samples by considering the average current. Since these operations can be highly resource consuming (iteration, matrix inversion etc.) and their resource needs may non-linearly increase with the sample size, their application to the average current signal (less samples) results in a significantly improved computational speed at the same or better signal to noise ratio.

From the viewpoint of practical application, the upcoming methods require that: the sampling frequency has to be high enough (e.g. 100 times the frequency of the PWM) to satisfactorily compute the average current and average supply voltage on the PWM cycles, i.e., the discrete integration (summation) results in insignificant truncation error. Next, the duty ratio has to be held steady for at least three subsequent PWM cycles and the corresponding current and voltage signals are to be captured, so the exponential fitting can be performed. However, the more cycles (e.g. 8) available the more robust the estimation of the resistance becomes (better noise suppression). Additionally, at least two transient waveforms are necessary, which belong to different duty ratios so a system of linear equations, that is similar to (III.6), can be created. Using the concept of the exponential fitting, the method in subsection III.3.1 extracts the resistances from the exponent, and the method in subsection III.3.2 uses an extrapolation to the steady state of the current. The methods in subsection III.3.3 bypass the exponential fitting and may also apply if the supply voltages change from PWM cycle to PWM cycle.

Further on, the following simplifying notations (III.8-11) are introduced; which are also better explained in the III.6. The average current function, which can be taken only at the end of the PWM cycles, is derived as (III.7); and the steady-state of the average current can be derived as (III.8),

\[
i_{AVG,n} = i_{AVG}(nT) = \left( i_{AVG,0} - i_{AVG} \right) e^{\sigma T (d_d + b(1-d_d))} + i_{AVG}. \tag{III.7}
\]

\[
i_{AVG} = i_H d + i_L (1-d) + \frac{i_H - i_L}{T} \left( 1 - e^a \right) \left( 1 - e^b \right) \left( \frac{1}{A} - \frac{1}{B} \right), \tag{III.8}
\]

\[
i_H = \frac{U_H}{R_H + R_S} = \frac{U_H}{R_A}, \quad i_L = \frac{U_L}{R_L + R_S} = \frac{U_L}{R_B}, \tag{III.9}
\]
\[
A = -\frac{R_H + R_S}{L_S} = -\frac{R_A}{L_S}, \quad B = -\frac{R_L + R_S}{L_S} = -\frac{R_B}{L_S}, \quad (\text{III.10})
\]

\[
a = AD_T, \quad b = B(1-d)T. \quad (\text{III.11})
\]

### III.3.1. Estimation of the resistance from the exponent

Under the conditions which are described in the introductory part of section III.3, the average current is captured and computed for a few, e.g. six, successive PWM cycles; thus, (III.7) applies for the sampled average current signal. On the switching frequency, there is no limitation. By performing a three parameter exponential fit to the average current data by means of e.g. least squares, the resulting time constant \( \Omega \) of the fitted exponential function can be expressed as (III.12) from (III.7) and (III.11),

\[
\Omega = (A-B)d + B, \quad \rightarrow \quad \Omega(d) = md + c. \quad (\text{III.12})
\]

According to (III.12), the fitted time constant \( \Omega \) is a linear function of the duty ratio; and the intercept and the slope express the time constants of the energizing paths. If multiple transient waveforms (at least two), which belong to different duty ratios, are captured, then the linear relationship between the duty ratios and the fitted time constant in (III.12) can be identified; thus, the time constants can be estimated as in (III.13),

\[
B^* = c, \quad A^* = m + c, \quad \rightarrow \quad \frac{A^*}{B^*} = \frac{R_A}{R_B}. \quad (\text{III.13})
\]

By substituting (III.9- III.11) to (III.8) one can rewrite (III.8) as (III.14). With the previously identified time constants and resistance ratio in (III.13), the resistances of the energizing paths can be computed from (III.14). Also, the resistances \( R_A \) and \( R_B \) of the energizing paths can be computed from multiple values of (III.14) and then averaged to reduce noise disturbances,

\[
i_{AVG} = \left(\frac{U_H}{R_A} - \frac{U_L}{R_B}\right) \left[ d + \frac{1}{T} \left(1 - e^a\right) \left(1 - e^b\right) \left(\frac{1}{A} - \frac{1}{B}\right) \right] + \frac{U_L}{R_B}. \quad (\text{III.14})
\]

Considering embedded applications, this method is computationally more exhaustive compared to the method in section III.2, because it requires an exponential function to be fitted to the average current waveform. This step may become time consuming in a microprocessor; therefore, efficient methods for performing a three parameter exponential fit may be of further research. Furthermore, the computation of (III.14) requires the computation of “exponential” functions which may also turn out to be multiple cycle instructions for a microcontroller.
III.3.2. Extrapolation to the steady-state of the current

In this subsection, the main concept is that the exponential fit is used for extrapolating to the steady-state of the average coil current, from which the overall resistances of the energizing paths can be extracted by using a simplified linear relationship (similarly to (III.6)). Compared to the previous method (subsection III.3.1), this method requires that the time period $T$ of the PWM is considerably smaller (at least 1/10 times) than the time constant of the electrical system.

Having an exponential curve fitted to a few successive samples of the transient average current, which is expressed as (III.7), the steady-state of the average current $i_{AVG}$ is also estimated by means of an e.g. exponential fitting. The analytical solution gives (III.14), which has to be compared to the steady state of the average current which resulted from the fitting. Since the resistance values are also contained in the time constant of the exponential terms, a closed form expression for the resistances from (III.14) does not exist but requires iteration, which is computationally exhaustive in embedded applications. Nonetheless, the value of the inductance also has to be approximated for evaluating (III.14). In order to bypass the previous difficulties, an approximation of (III.14) is proposed for the computation of the resistances. By taking the Taylor series of the exponential function in (III.15), the exponential part of (III.14) can be rearranged and simplified as in (III.16),

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$  \hspace{1cm} (III.15)

$$\frac{(1-e^a)(1-e^b)}{1-e^{a+b}} = -\frac{\left(\sum_{n=1}^{\infty} \frac{a^n}{n!}\right)\left(\sum_{n=1}^{\infty} \frac{b^n}{n!}\right)}{\sum_{n=1}^{\infty} \frac{(a+b)^n}{n!}} \approx \frac{ab}{a+b},$$  \hspace{1cm} (III.16)

According to (III.16), the parameters $a$ and $b$ express a ratio of the PWM time period and of the time constants in (III.11). For simplifying (III.14) and (III.16), I propose the use of the first order approximation of the exponentials, which I state that results in a very small truncation error of (III.16) in a wide range of the $a$ and $b$ parameters. With the assumption of $a=b$, the truncation error in (III.16) was computed at various values of the $a$ parameter, and the results are plotted in Fig. III.3. It can be concluded that the first order approximation of (III.16) results in negligible error at values of the $a$ parameter typical in real PWM applications. According to Fig. III.3, the error is still less than 10% at $a=1$ (time period of the PWM equals the time constant of the system); however, practical PWM applications use switching frequencies which are ten ($a=0.1$) or more times higher, in which situations the truncation error becomes insignificant (0.1%).
Fig. III.3: Truncation error of (III.16) due to the first order approximation.

Using (III.16), the analytical solution of the average current (III.14) can be written as (III.17),

$$i_{AVG} = \left( \frac{U_H}{R_A} - \frac{U_L}{R_B} \right) \left[ d - \frac{1}{T} \frac{ab}{a+b} \left( \frac{1}{A} - \frac{1}{B} \right) \right] + \frac{U_L}{R_B}. \quad (III.17)$$

The term in the round brackets in (III.17) can be factored out as (III.18) using (III.10-11),

$$d - \frac{1}{T} \frac{ab}{a+b} \left( \frac{1}{A} - \frac{1}{B} \right) = d + \frac{1}{T} \frac{R_A}{L_S} dT + \frac{R_B}{L_S} (1-d)T \left( \frac{L_A R_A - L_S R_B}{R_A R_B} \right). \quad (III.18)$$

From (III.18), the PWM time period $T$ and the inductance $L_S$ can be factored out thus the steady-state of the average current can be written as (III.19),

$$i_{AVG} = \left( \frac{U_H}{R_A} - \frac{U_L}{R_B} \right) \left[ d + \frac{d}{R_A (R_A - R_B) [d + R_B]} (R_A - R_B) \right] + \frac{U_L}{R_B}. \quad (III.19)$$

After a few algebraic manipulations (III.20-23) on (III.19), the resistances are brought to (III.24),

$$i_{AVG} = \left( \frac{U_H}{R_A} - \frac{U_L}{R_B} \right) \left[ 1 + \frac{(1-d)(R_A - R_B)}{(R_A - R_B)[d + R_B]} \right] + \frac{U_L}{R_B}, \quad (III.20)$$
\[
I_{AVG} = \left( \frac{U_H - U_L}{R_A - R_B} \right) \left( \frac{R_B}{R_A - R_B} \right) d + \frac{U_L}{R_B},
\]  
(III.21)

\[
I_{AVG} = \left( \frac{U_H - U_L}{R_A - R_B} \right) d + \frac{U_L}{R_B},
\]  
(III.22)

\[
\left[ \left( R_A - R_B \right) \cdot d + R_B \right]_{AVG} = U_H d + U_L \left( \frac{R_A d - R_B d + R_B}{R_B} - \frac{R_A}{R_B} d \right),
\]  
(III.23)

\[
R_A d + R_B (1 - d) = \frac{U_H d + U_L (1 - d)}{i_{AVG}}.
\]  
(III.24)

If the extrapolations of the average current to steady-state are available at separate duty ratios, then a system of linear equations (III.25) can be established and solved for the overall resistances. The minimum number of independent equations is two; however, the system can be also composed of more equations for better noise suppression. Note that the problem in (III.25) is in fact a line fitting problem.

Compared to the method proposed in section III.3.1, the method presented in section III.3.2 is computationally more economical. In section III.3.1, the solution requires an exponential fitting to the waveform of the average current, a line fitting to the time constant-duty ratio data and then the evaluation of (III.14) for every measurement point (to reduce noise), which can be time consuming for a microcontroller. However, the method in section III.3.2 requires an extrapolation to the steady-state by the exponential fitting or other techniques, but then the resistances are directly obtained from the solution of (III.25), which is a line fitting. Thus, the computation of (III.14) is avoided and this increased computational efficiency can be of great advantage in embedded systems. Furthermore, the necessary matrix in (III.25) is constructed readily,

\[
\begin{bmatrix}
    d_1 & 1 - d_1 \\
    d_2 & 1 - d_2
\end{bmatrix}
\begin{bmatrix}
    R_A \\
    R_B
\end{bmatrix}
= \begin{bmatrix}
    \frac{U_H d_1 + U_L (1 - d_1)}{i_{AVG,1}} \\
    \frac{U_H d_2 + U_L (1 - d_2)}{i_{AVG,2}}
\end{bmatrix},
\]  
(III.25)

**III.3.3. Estimation from the difference equation**

The previous methods relied on an exponential fit to the average current signal (or on some other methods for obtaining the exponent and steady state); and required the supply voltages to remain constant during the period of an estimation. A recursive or fitting algorithm for obtaining the exponent and the steady state may pose a significant computational burden for a low-end microcontroller; therefore, this subsection presents alternative methods which bypass e.g. the exponential fitting; and may scope with changes in the supply voltages.

If considering the average current and voltage signals at a given duty ratio, the electrical behavior can be also represented by an equivalent resistance, e.g. the time constant in (III.7) and
the “average” resistance in the right hand side of (III.25). The equivalent resistance, which is
denoted by $R_d$, is a weighted combination of the $R_A$ and $R_B$ resistances by the $d$ duty ratio,
provided the PWM frequency is high enough. According to subsection III.3, the samples of the
average current can be considered as the samples of an otherwise continuous signal in (III.7), if
the duty ratio and average supply voltage is unchanged. However, (III.7) is a closed form
analytical solution to the corresponding differential equation, which can be described as in
(III.26) in terms of the equivalent resistance $R_d$. In order to obtain (III.7) from (III.26), the
average supply voltage has to be constant,

$$L \frac{di(t)}{dt} + R_d i(t) = U(t). \tag{III.26}$$

Using (III.26), a system of linear equations (III.27) can be constructed from the
measurement data and then solved for the unknown $L$ and $R_d$ values, provided that the duty ratio
is constant. Note that by solving (III.27), it is possible to scope with changes in the average
supply voltage; however, the derivative of the average current signal is necessary, which can be
computed from the samples by using discrete differentiation as in (III.28). In the following, I
denote the approach in (III.28) as “4.3 discr”,

$$
\begin{bmatrix}
\frac{di}{dt}(t_1) \\
\vdots \\
\frac{di}{dt}(t_n)
\end{bmatrix}
= \begin{bmatrix}
i(t_1) \\
\vdots \\
i(t_n)
\end{bmatrix},
\begin{bmatrix}
L \\
R_d
\end{bmatrix}
= \begin{bmatrix}
u(t_1) \\
\vdots \\
u(t_n)
\end{bmatrix}, \tag{III.27}
$$

$$
\begin{bmatrix}
i_{AVG,1} - i_{AVG,0} \\
i_{AVG,1} + i_{AVG,0} \\
\vdots \\
i_{AVG,n} - i_{AVG,n-1} \\
i_{AVG,n} + i_{AVG,n-1}
\end{bmatrix}
\begin{bmatrix}
\frac{\Delta t}{2} \\
\vdots \\
\frac{\Delta t}{2}
\end{bmatrix}
= \begin{bmatrix}
u_{AVG,0} \\
\vdots \\
u_{AVG,n}
\end{bmatrix}. \tag{III.28}
$$

Despite being computationally economical, the solution of (III.28) is susceptible to noise
and may also suffer from discretization error due to the discrete differentiation. If the average
supply voltage does not change, then the average current signal is described by (III.7), for which
an equivalent polynomial representation is possible. As an alternative approach, it is suggested
that an e.g. 3rd or 4th order polynomial function (which reconstructs (III.7) appropriately) is fitted
to the samples of the average current signal, and then the elements of (III.28) are computed from
the polynomial representation. This way, the truncation error caused by the differentiation and
sampling can be greatly reduced, and noise effects may be also decreased due to the polynomial
fitting, although at the expense of the computational load of the polynomial fit and its evaluation.
Further on, this approach is denoted as “4.3 poly”.

Nevertheless, it is also possible to recast (III.26) into an integral form which expresses the
flux changes (III.29), thereby the discrete differentiation is bypassed. Note that this approach can
also scope with changes in the supply voltage, but the computation of the coefficient matrix
(III.29) becomes more resource consuming compared to (III.28). The parameters can be derived
by solving the resulting system of linear equations (the elements of (III.27) to be replaced accordingly to (III.29)). This approach is denoted as “4.3 int”,

\[
\int L \frac{di(t)}{dt} dt + R_d \int i(t) dt = \int u(t) dt,
\]

\[
\left( L \frac{i_n}{nT} - i_0 \right) + R_d \left( \sum_{k=0}^{n} i_{AVG,k} \frac{i_{AVG,n} + i_{AVG,0}}{2} \right) = \sum_{k=0}^{n} u_{AVG,k} - \frac{u_{AVG,n} + u_{AVG,0}}{2}.
\] (III.29)

After having the equivalent resistance \( R_d \) estimated at different duty ratios, the overall resistances of the energizing paths can be estimated accordingly to e.g. (III.25). Compared to the methods in subsections III.3.1-3.2, the (exponential) fitting procedure is avoided, which may result in a considerable improvement at the computational efficiency. However, the solution requires the inversion of two matrices: one for the equivalent resistance (e.g. (III.28)) and the other for the overall resistances (III.25).

The statistical properties, i.e., mean and variance of the estimates of the equivalent resistance from (III.14), (III.25), (III.27-29) are expected to somewhat depend on the coefficient matrix; therefore, an experimental investigation is conducted in Section III.4.

### III.3.4. Reducing bias in the estimate of the resistance

In the introductory part of section III.3, it is pointed out that the nonlinear core behavior may cause bias in the estimation because the underlying model considers a linear, constant inductance. In this subsection, an approach is presented for the reduction of this bias.

The methods in subsections III.3.1 to III.3.2 are based on estimating the steady-state current from an exponential fit to the transient step response. In real applications, the inductance of the solenoid valve might somewhat depend on the current; therefore, the exponential extrapolation that considers a constant inductance becomes biased. The exponential function in (III.7), which is expressed in terms of amplitude and offset, can be rewritten as (III.30) in terms of initial current and steady-state,

\[
i(t) = i_0 + (i_{SS} - i_0)(1 - \exp(A t)) \tag{III.30}
\]

When performing an exponential fit to the measurement data, it can be also considered as finding an appropriate initial current \( i_0 \) and current change \( i_{SS}-i_0 \). The term \( i_{SS} \) refers to the steady-state current. Since the initial current is known (measured), it is expected to change far less compared to \( i_{SS}-i_0 \) and \( A \) in case of bias due to a current dependent inductance. Assume that the inductance function is such that the \( i_{SS}-i_0 \) term in (III.30) is overestimated. In this situation, if the current (duty ratio) increases, then the aforementioned bias causes that the extrapolated steady-state current becomes larger than the original one; which corresponds to an estimated resistance being smaller than the real one. On the contrary, if the current (duty ratio) decreases, then the larger (biased) \( i_{SS}-i_0 \) term causes the extrapolation to the steady current to become smaller; thus, the corresponding resistance will be overestimated. This is opposite to the previous situation. That is how, if adding the biased resistances (previous two situations) to each other, the over and under estimations will somewhat cancel out each other and will thus result in a considerably smaller overall bias.
For reducing the overall bias in the steady-state (resistance) estimate due to a current dependent inductance, I propose that (III.25) be constructed from “averaged” terms as in (III.31). Considering (III.24), the right-hand side represents an average or equivalent resistance, which is either over or under estimated in case the inductance is not constant. Mathematically, the separate (independent) rows of (III.25) can be added to each other and averaged as in (III.31),

\[
R_A \frac{1}{N} \sum_{n=1}^{N} d_n + R_B \frac{1}{N} \sum_{n=1}^{N} (1 - d_n) = \frac{1}{N} \sum_{n=1}^{N} R_n. \tag{III.31}
\]

If the equivalent resistances, which are computed from falling and rising current signals at different duty ratios, are added to each and averaged as in (III.31), then the overall bias in the “averaged” equivalent resistance can be considerably reduced. Eq. (III.31) can be rewritten as (III.32),

\[
(R_A - R_B) \bar{d}_N + R_B = \bar{R}_N. \tag{III.32}
\]

According to (III.32), the average resistance is described as a linear function of the average duty ratio, where the slope and the intercept are simple functions of the resistances in the energizing paths. Therefore, a matrix equation similar to (III.24) can be constructed and solved, or the linear relationship identified from multiple average duty ratio and average resistance pairs.

### III.4. Computer Simulations and Experimental Analyses

#### III.4.1. Computer simulations for comparing some statistical properties

In Section III.3 a collection of methods was presented for estimating the overall resistances in the energizing paths considering the transient current waveform. Common in them all, an equivalent resistance is estimated at a certain duty ratio; then a system of linear equations is created from the equivalent resistances and solved for the resistances in the energizing paths. In a computer simulation, I studied the mean error and deviation of each of the methods in section III.3 when estimating a certain equivalent resistance. The parameters of the simulation were as follows: \( L=1, R_A=0.7, R_B=0.55, U_H=1, U_L=0, T=0.2, T_s=0.002, d=0.4, i_0=0 \) and 10 PWM cycles were recorded. The average current signal was ranging from 0 to 0.45. The samples of the current signal were loaded with a uniformly distributed white noise with expected value of 0. The simulation was carried out at different noise levels. At each of the noise amplitudes, 2048 independent simulations were performed from which an experimental mean error and deviation were computed for the separate methods. The performance of the proposed methods versus the noise disturbances is plotted in Fig. III.4. The equivalent resistance in this particular simulation was 0.61.
From left to right, the labels in the legends are the following: “4.1” refers to the method that is proposed in section III.3.1 (exponential fit to get the time constant), “4.2” refers to the method that is proposed in section III.3.2 (exponential fit to get the steady-state), “4.3.discr” refers to the method that is proposed in section III.3.3 and uses the discrete form of (III.28); “4.3.int” refers to the method which uses the integral form of (III.27) that is (III.29) and “4.3.poly” refers to the method which uses a polynomial (4th order) fit to the average current waveform and thus computes (III.27). According to Fig. III.4, the “discrete” version of the method that is proposed in section III.3.3 (from (III.28)) has a significant bias with increasing noise levels. Compared to this, the integral form (III.29) in Section III.3.3 is “unbiased”; however, it has the largest deviation. Therefore, if a method that is computationally more effective than those in Section III.3.1-III.3.2 are necessary, I suggest that (III.27) is implemented by computing the coefficients from an adequate polynomial fit to the average current signal, i.e., use “4.3.poly”.

Nonetheless, the best statistical performance (bias and deviance) is shown by the method in subsection III.3.2; however, it is computationally more demanding as it requires an exponential fitting compared to the methods in III.3.3.

**III.4.2. Experimental analyses**

The resistance of a real solenoid actuator was estimated with the methods that are proposed in Sections III.2-3. The experimental setup that was used for the analyses is described in the Appendix. The measurement settings were the following: 100 kHz sampling frequency, 2 kHz switching frequency, $10 \, V$ $U_H$ high level power supply and the $U_L$ low level or negative supply corresponded to the forward voltage drop of the freewheeling diode, the model of which had been previously captured. The solenoid actuator was driven in a low-side single switch configuration similarly to Fig. I.1. The overall resistances had been previously measured in a DC static measurement (the switch was continuously turned on); the results are listed in Table III.1.

First, the overall resistances in the energizing paths were estimated from the steady-state PWM current waveform accordingly to Section III.2. The duty ratio was swept from 0.3 to 0.4 with a spacing of 0.02 and the current waveform during a PWM cycle was recorded after 100 ms the duty ratio had changed, thus steady-state was reached. The resulting system of linear equations (III.6) was solved that yielded the resistances in Table III.1.

Secondly, the overall resistances were estimated from the transient average current signal accordingly to subsections III.3.2-3.3. Two sequences of duty ratios, each consisting of six...
different duty ratios, were used for generating the exciting PWM signal and a duty ratio lasted for approximately 18 PWM cycles. The corresponding transient waveforms are plotted in Fig. III.5. For the separate transient sections that corresponded to a certain duty ratio, the steady-state current was approximated by means of an exponential fit to the last six measurement points or using (III.27)-(III.29); and then the average duty ratios and resistances (III.32) were calculated for the two sequences. Thus, bias due to a possibly current dependent inductance could be considerably reduced. The resistance estimates were obtained by solving the 2 by 2 system of equations (III.25).

According to Table III.1, the difference in the overall resistances could be estimated within a 1% error with the proposed methods. Thus, a better estimate of the resistance of the solenoid actuator could be provided compared to computing a single “equivalent” resistance, which would also change with the duty ratio; thus, result in a bias that depends on the duty ratio.

![Table III.1: Results of resistance estimations.](image)

<table>
<thead>
<tr>
<th></th>
<th>DC Meas.</th>
<th>Sec. III.3 St. state (III.4)</th>
<th>Sec. III.4.2 Exp. fit (III.32)</th>
<th>Sec. III.4.3 discr. (III.28)</th>
<th>Sec. III.4.3 int. (III.29)</th>
<th>Sec. III.4.3 poly. (III.27)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_B$ [Ω]</td>
<td>5.755</td>
<td>5.765</td>
<td>5.736</td>
<td>5.734</td>
<td>5.740</td>
<td>5.754</td>
</tr>
<tr>
<td>$\Delta R$ [Ω]</td>
<td>0.362</td>
<td>0.372</td>
<td>0.383</td>
<td>0.396</td>
<td>0.371</td>
<td>0.353</td>
</tr>
</tbody>
</table>

Fig. III.5: Transient current waveforms of the solenoid actuator used for resistance estimation.

III.5. Conclusion

In this section, methods were developed for estimating the resistance of the coil of a solenoid actuator by considering the fact that in real applications, the overall resistance that the coil’s current encounters during the “on” and “off” periods of a PWM cycle can be different. In some situations, the difference in the overall resistances can be significant; thus, I provided better estimates of the coil’s resistance and resolved a possible source of bias, compared to estimating a single equivalent resistance. Also, I differentiated between estimating the resistance from the steady and transient waveforms of the PWM driven current signal. In the latter case, I proved that under some modeling assumptions the transient waveform of the average current on a PWM cycle could be expressed by a three parameter exponential function irrespectively of the different overall resistances (time constants). Computer simulations had been carried out to compare the statistical properties, e.g. expected error and deviance, of the developed methods. I found that in case the exponential assumption is valid, the estimation of the resistance by the exponential fitting methods had better statistical properties. However, computation of the exponential fitting may
become computationally exhaustive for certain microcontroller based applications; therefore, the methods in subsection II.3.3, which are computationally more economical, can be more suitable provided that noise effects are at an acceptable level. Experimental analyses that considered a real actuator also confirmed that the proposed methods (steady and transient) could estimate the resistance of the solenoid actuator appropriately; and that some difference in the overall resistances of the energizing paths was observable. Thus, a better estimate of the true winding resistance could be provided and bias in the estimate of the resistance, which would otherwise change with the duty ratio, could be reduced. In general, the developed methods have low computational complexity which makes them suitable for embedded applications. However, it has to be noted that the actual needs for computational resources greatly depend on the implementation; e.g., on the implementation of the exponential fitting algorithm.
PART 2

THE MULTILAYERED MEDIUM APPROACH AND ANALYTICAL SOLUTION TO THE DIFFUSION EQUATION

The sensorless methods are based on a model of the electromagnetic subsystem; therefore, changes in the electrical parameters during operation can cause error in the estimation. In Section III, it is highlighted that the electrical resistance of the coil changes due to thermal effects; and a set of PWM based methods are presented for the estimation of the resistance. Compared to estimating the resistance from an electrical model, the resistance of the coil can be determined in an alternate way, i.e., by creating a thermal model of the system and obtaining the complete temperature distribution inside the actuator. Besides, a detailed thermal model is not only useful for determining the resistance, but it is also useful for the design, optimization and fault diagnosis (hot-spots) of electromagnetic actuators. Therefore, the second part of the researched aimed at creating an in depth thermal model of linear electromagnetic actuators.

If considering the internal structure of a solenoid valve (Fig. 1.1) then the actuator, especially the coil, can be represented as a multilayered medium, i.e., separate, piecewise homogeneous solid layers that are layered along the radial direction. Therefore, the actuator is now modeled as a multilayered medium in which the complete temperature field is to be obtained. For obtaining a detailed, in depth solution of the temperature field in the multilayered medium, the solution is preferred to be expressed in an analytical way because analytical solutions give direct insight into the physical processes and lead to a closed from, compact solution. Note that the conduction of heat in solids is governed by the diffusion (Laplace) equation; thus, the second part of my research was dedicated to the analytical solution of the diffusion equation in multilayered media. The studied problem is general; therefore, the results are applicable not only to electromagnetic devices but to a far wider field of research and application, which can be described by the diffusion equation, e.g., electrostatics, and by the multilayered medium model, e.g., composite materials.

In the following Sections IV-V, I study the conduction of heat (diffusion) in multilayered media in the steady-state. Though the study considers slabs, the methods and results can be analogously written for cylindrical coordinates (cylinders, solenoid). In section IV, an extension and simplification of the analytical solution is provided, and in section V, the numerical difficulties, which are encountered at computing the analytical solution, are discussed and an iterative method is developed for computing the analytical solution.
IV. STEADY DIFFUSION IN MULTILAYER BODIES AND THE SIMPLIFICATION OF THE EIGENVALUE PROBLEM

It was found that the majority of the presented analytical solutions could only address homogeneous boundary conditions in the longitudinal directions (parallel to the layers) [43]-[45], [53], [57]-[61]. Therefore, methods to treat the effect of longitudinal excitations and to simplify the associated eigenvalue problem for an arbitrary number of layers are of interest. In this work, an attempt to simplify the eigenvalue problem for steady-state heat conduction that is associated with longitudinal excitations is presented by transforming the problem into a combination of conduction problems with only transverse excitations. As the transcendental equations and eigenvalue calculation are simpler for transverse cases, the eigenproblem becomes simplified. I will also consider forms of internal heat generation which is relevant to the modeling of the coil of electrical machines, i.e., it depends only on the transverse coordinate; and I show that with the help of the proposed method, it is possible to derive an analytical solution. For a description about the available analytical solutions for multilayered heat conduction, refer to Section I.4.3.

In this study, I do not consider how to exactly compute the analytical solution, but a general method is presented with which the effects of e.g. non-homogeneous longitudinal boundary conditions can be considered in the solution, and the associated eigenvalue problem can be simplified.

IV.1. Statement of the Problem

The underlying model, illustrated in Fig. IV.1, is a linear 2D physical model. Let the structure be composed of \#N linear, homogeneous, isotropic slabs with equal heights \( H \) but with different lengths \( L_j \) and thermal conductivities \( k_j \). Additionally, the layers incorporate the internal heat generation \( q_j(x) \), which only depends on the transverse \( x \) coordinate. The model assumes perfect thermal contact and additional flux excitations \( Q_j(y) \) between adjacent slabs. All of the exposed surfaces are subject to non-homogeneous boundary conditions of the third kind with far field temperatures \( T_j \) and constant convection coefficients \( h_j \). The subscripts \( u, b, l, r \) denote the upper, bottom, left and right surfaces. The upper and bottom ambient temperatures may depend on the transverse \( x \) coordinate, and the other temperatures may depend on the longitudinal \( y \) coordinate, i.e., \( T_{2u}(x) \) and \( T_{1l}(y) \). To simplify the mathematical treatment of the problem, separate coordinate systems are assigned to each slab that express the thermal distribution in the corresponding \([0, L_j]\) domain. The solution is derived in Cartesian coordinates.

![Fig. IV.1: Model of a multilayered structure (slabs).](image-url)
Furthermore, the term “excitation” represents a non-homogeneous boundary condition on a certain surface. Excitations on the upper and bottom surfaces, which depend on the transverse \( x \) coordinate, are referred to as longitudinal because they are in the direction parallel to the junction planes of the layers. Other excitations on the left surface, right surface and \( Q_j(y) \), are referred to as transverse because they are in the direction perpendicular to the layers.

I will solve the steady-state diffusion problem that is illustrated in Fig. IV.1 using separation of variables and superposition. The thermal equilibrium is governed by the Laplace equation or the Poisson equation, provided that there is heat generation. By using the superposition principle, the problem can be divided into three parts that can be solved separately:

- Transverse case: Heat conduction without internal heat generation but with applied transverse excitations (left, right, junction flux),
- Longitudinal case: Heat conduction without internal heat generation but with applied longitudinal excitations (upper, bottom),
- Internal heat generation with homogeneous boundary conditions.

### IV.1.1. Eigenproblem in the case of only transverse excitations

In this situation, the steady heat diffusion is governed by the Laplace equation that is expressed by (IV.1) in two-dimensional Cartesian coordinates, provided that the material is isotropic. The problem is illustrated in Fig. IV.2.

\[
\frac{\partial^2 T(x, y)}{\partial x^2} + \frac{\partial^2 T(x, y)}{\partial y^2} = 0, \tag{IV.1}
\]

\[
T_j(x, y) = \sum_{n}^{\infty} X_{j,n}(x)Y_{j,n}(y) = \sum_{n}^{\infty} \left( a_{j,n} e^{i j_n x} + b_{j,n} e^{-i j_n x} \right) \left( c_{j,n} e^{i j_n y} + d_{j,n} e^{-i j_n y} \right) \tag{IV.2}
\]
In (IV.2), the complex argument is contained in the $Y(y)$ function but could also be assigned to the $X(x)$ function. The actual values of the λ eigenvalues depend on the boundary conditions and will become imaginary if the assumption in (IV.2) turns out to be incorrect. The complex exponentials can be expressed in a trigonometric form. By grouping the coefficients and introducing new notation, (IV.2) can be written as (IV.3). The term $\nu$ belongs to the sinusoidal functions and the term $\eta$ belongs to the exponential functions,

$$T_j(x, y) = \sum_n w_{j,n} \left( \eta_{j,n} e^{\lambda_{j,n} x} + e^{-\lambda_{j,n} x}\right) v_{j,n} \sin(\lambda_{j,n} y) + \cos(\lambda_{j,n} y) \right).$$

(IV.3)

The boundary and interface conditions can be expressed as (IV.4.a - IV.4.f),

$$k_j \frac{\partial T_j(x, y)}{\partial y} \bigg|_{y=0} = h_{j,b} \left[ T_j(x, 0) - T_{j,b}(x) \right],$$

(IV.4.a)

$$-k_j \frac{\partial T_j(x, y)}{\partial y} \bigg|_{y=H} = h_{j,u} \left[ T_j(x, H) - T_{j,u}(x) \right],$$

(IV.4.b)

$$k_1 \frac{\partial T_{j1}(x, y)}{\partial x} \bigg|_{x=0} = h_{j1} \left[ T_{j1}(0, y) - T_{1,j1}(y) \right],$$

(IV.4.c)

$$-k_N \frac{\partial T_N(x, y)}{\partial x} \bigg|_{x=L_N} = h_{N,j} \left[ T_N(L_N, y) - T_{N,j}(y) \right],$$

(IV.4.d)

$$T_j(L_j, y) = T_{j,1}(0, y),$$

(IV.4.e)

$$k_j \frac{\partial T_j(x, y)}{\partial x} \bigg|_{x=L_j} - k_{j,1} \frac{\partial T_{j,1}(x, y)}{\partial x} \bigg|_{x=0} = Q_j(y).$$

(IV.4.f)

To obtain a temperature solution, the four parameters in (IV.3) and the eigencondition for each slab must be determined, and the latter should yield the infinite number of eigenvalues. In the present situation, internal heat generation and boundary conditions in the longitudinal directions are homogeneous, as illustrated in Fig. IV.2. Thus, separation of variables can be applied, and the functions that are defined in (IV.3) can be assigned to dedicated space variables; the sinusoidal function to the $y$ axis, and the exponential function to the $x$ axis. The eigenconditions (eigenvalues) and $\nu$ parameters can be calculated by evaluating (IV.4.a) and (IV.4.b) with $T_{j,b}$ and $T_{j,u}$ set to zero. From the bottom surface boundary condition, I obtain (IV.5), and, by substituting this equation into the upper boundary condition (IV.4.a), I obtain a transcendental equation with a relatively simple form, (IV.6), which holds for each layer,

$$\nu_{j,n} = \frac{h_{j,b}}{k_j \lambda_{j,n}},$$

(IV.5)
\[
\tan(\lambda_j H) = \frac{(h_{j,b} + h_{j,a}) (k_j \lambda_j)}{(k_j \lambda_j)^2 - h_{j,b} h_{j,a}}. 
\]  

(IV.6)

With the knowledge of the eigenvalues, the next step would be to completely solve the transverse problem. Such solution methods can be found in, for example, [40], [43]-[49] and [57]-[60]; however, this solution procedure is not in the scope of the present study. In the following analyses, only (IV.6) will be used to determine the eigenvalues (for slabs) because the cases with only longitudinal excitations will be traced back to transverse ones. Efficient root finding schemes [42] are available for solving equations that are similar to (IV.6). Thus, eigenvalue computation has become “algorithmic” because the same routine has to be called repeatedly.

IV.2. Proposed Method for Treating Longitudinal Excitations

In the previous Section, I showed that the eigenvalues (eigenconditions) of the slabs are readily obtainable when only transverse excitations are present. Next, consider the situation in which the transverse excitations, e.g., left (IV.4.c) and right (IV.4.d), are homogeneous and the \( Q_j(y) \) -s are disconnected (zero) (IV.4.f) but one intermediate slab \( #J \) has nonzero upper and bottom excitations applied to it. Unfortunately, the method in Section IV.1.1 cannot be used with SOV to derive the eigencondition for slab \( #J \) due to the non-homogeneous longitudinal and junction boundary conditions. I propose a novel approach illustrated in Fig. IV.3-Fig. IV.5, to solve the problem.

In the present configuration, slab \( #J \) can be represented as a heat source from the viewpoint of the remaining left side and right side arrays of slabs because heat enters the system only through slab \( #J \). Conversely, slab \( #J \) experiences certain heat sinks or impedances at its left and right sides. The solution procedure is as follows:

- Let us remove slab \( #J \) from the multilayer structure. As the perfect thermal connection with the adjacent layers is severed, the \( y \) boundary conditions at the left and right surfaces change to homogeneous ones of the second kind (the convection coefficient is zero), as illustrated in the left side of Fig. IV.3. The Laplace equation, (IV.1), can be easily solved for slab \( #J \) by SOV to give the temperature distribution \( U_{J}(y) \). (Here, the exponential term in (IV.3) belongs to the \( y \) coordinate and the sinusoidal term belongs to the \( x \) coordinate, the opposite of the case in Section IV.1.1),

- I now treat slab \( #J \) as an ideal temperature source, neglecting its internal conduction properties (impedance), with the \( U_{J} \) temperature distribution. When reconnected to the left subset, layer \( #J-1 \) experiences the temperature input \( U_{J}(0,y) \) at its right side, as illustrated in the right side of Fig. IV.3. This problem is a type of transverse excitation (Fig. IV.2), for which the eigenvalues can be readily obtained in accordance with Section IV.1.1. The right side boundary condition for the left subset changes to the first kind. I will denote the temperature solutions \( U_{J,1}, U_{J,2}, \) etc.

- From the surface temperature excitation \( U_{J}(0,y) \), the heat flux \( f_{Rx}(y) \) flows to the left subset. The source was considered ideal; however, slab \( #J \) has finite conductance, so the thermal distribution has to be corrected. Let us denote by \( f_{Rx}(y) \) the \( x \) direction flux that is entering the left subset at the right side surface of slab \( #J-1 \), as shown in the right side of Fig. IV.3. This flux excitation, which is similar to \( Q_{J-1}(y) \) in the transverse
case, is applied to the original system that is composed of slabs #I to #J with all other excitations removed, see Fig. IV.4. This problem is similar to the transverse case and can therefore be treated in the same manner to yield the temperature solutions \( V_1 \) to \( V_J \). Regarding the right subset, \( V_{J+1} \) to \( V_N \) are 0.

- The temperature solution for subset #I to #J is the difference among the sub-solutions \( U \) and \( V \) (U-V). Next, the effect of the right subset has to be considered, which can be treated similarly to the previous subset. As illustrated in the left side of Fig. IV.5, the right subset is excited by the floating (open loop), right-end surface temperature of the combined subset #I to #J. This problem is also a transverse one with \( U_{J+1} \), \( U_{J+2} \), etc. solutions for the right side slabs and the \( f_L(y) \) input flux.
- Similar to in the previous case, the finite conductance of the source term (#I to #J) has to be taken into account. The \( f_L(y) \) flux excitation is applied to the original structure at the junction of #J and #J+1, as presented in the right side of Fig. IV.5. The system is then solved to obtain the \( W_i \) to \( W_N \) slab temperatures, again incorporating only transverse inputs.

The final solution for each slab is the difference among their corresponding \( U \), \( V \) and \( W \) sub-solutions (IV.7). In the case of multiple slabs with nonzero upper and bottom excitations, the overall temperature distribution can be obtained by using the superposition principle and by separately performing the previously described procedure,

\[
T_{\text{slab}}(x, y) = U_{\text{slab}}(x, y) - V_{\text{slab}}(x, y) - W_{\text{slab}}(x, y)
\]  

(IV.7)

Fig. IV.3: Removing slab “#J” and solving the heat equation (left), then solving a “transverse” problem for the left subset (right).

Fig. IV.4: Correcting for the finite conductance of #J and solving a “transverse” problem.
IV.3. Proof of the Proposed Method

I now prove the above method using superposition and a lumped approach. An electrical analogy is exploited, in which the temperature corresponds to voltage, the heat flux to current and the thermal resistance (convection) to resistance. A more detailed explanation on the lumped model approach, on its use and on its analogy to electrical systems can be found in [74], [75]. First, let us consider a lumped two port network, as illustrated in Fig. IV.8. At point B, the potential $U_B$ and current $i_B$ can be easily calculated (IV.8),

\[
U_B = \frac{R_M}{R_M + R_J} U_J, \quad i_B = \frac{U_J}{R_M + R_J}.
\]  

This problem is now treated with the proposed method. In accordance with Section IV.2, the network is first severed at B and an open loop voltage $U_{OL}$ (IV.9) is calculated, which equals $U_J$. This “ideal” source could supply the load current $i_L$ (IV.9) to the left side of the network, i.e., to $R_M$. Next, the source $U_J$ is shorted (set to zero), and the current source $i_L$ is applied to the original network at B. The resulting voltage $U_{SH}$ at B can be readily obtained (IV.10), and $R_M$ and $R_J$ form parallel resistances. Finally, the node voltage $U_B^*$ at B is the difference between $U_{OL}$ and $U_{SH}$ (IV.11), which equals $U_B$ (IV.8). For such a lumped system, the proposed method applies,

\[
U_{OL} = U_J, \quad i_L = \frac{U_{OL}}{R_M} = \frac{U_J}{R_M},
\]  

\[
U_{SH} = i_L \frac{R_M R_J}{R_M + R_J} = \frac{R_J}{R_M + R_J} U_J,
\]  

\[
U_B = U_J = \frac{R_M}{R_M + R_J} U_J = \frac{U_J}{R_M + R_J}.
\]
Next, let us consider the partitioned, lumped representation of slabs #1 to #J, in which the arrays are split into cells and each cell is modeled by a four terminal star resistance network, as illustrated in Fig. IV.7. The left array of slabs, which is denoted \( M (\#1…\#J-1) \) is connected to \#J at terminals \( A-C \). For simplicity, terminal \( C \) and its associated quantities represent the “rest” of the structure. The rightmost impedances of \#J are floating (denoted “X”) because the right subset of the slabs is removed. The voltages \( U_{J1N} \) and \( U_{J2N} \) correspond to lumped upper and bottom excitations, respectively. The effect of convection (convection coefficient) is built into the resistances. Let us assume that the steady-state voltages (temperatures) at the junctions are \( U_A \) to \( U_C \) and that the currents (fluxes) \( i_A \) to \( i_C \) flow, provided that \( M \) is connected to \#J.

Subset \( M \) sees a voltage input from \#J, but \#J experiences an impedance, i.e., a current load, due to \( M \) through the connections. From the viewpoint of a steady state, the effect of the connection can be equivalently replaced by a voltage source for \( M \) and by a current source for \#J without altering the original solution. However, the link is severed. An example is illustrated for terminal \( A \) (\( U_A \) and \( i_A \), “X” represents a severed connection) in Fig. IV.7.

![Diagram](image_url)

**Fig. IV.7: Partitioning and lumped representation of slabs #1 to #J.**

In the following, all connections are replaced by their equivalent source terms except for one, which remains “connected”, and the effect of each link is considered separately on the basis of superposition. First, the contribution of only connection \( B \) is modeled, so the voltage sources \( U_A \) to \( U_C \) are shorted and the current sources \( i_A \) to \( i_C \) are floating except at \( B \), where the slabs are connected. From Fig. IV.7, it can be seen that \( M \) and \#J now form two poles through the ground and terminal \( B \). I recall Thévenin’s theorem, which states the following:

- Any linear, two-pole electrical network with voltage and current sources and resistances can be replaced at the terminals by an equivalent voltage source in series with an equivalent resistance.
The equivalent voltage source is the open loop voltage between the poles, and the equivalent resistance is the overall resistance of the network between the terminals with all of the internal voltage sources shorted and the current sources disconnected.

Accordingly, both \( M \) and \( \#J \) can be represented by their corresponding Thévenin equivalent circuit. Since all voltage sources are shorted in \( M \), \( M \) can be replaced by a single resistance \( (R_M) \), and \( \#J \) can be replaced by a voltage source \( (U_J) \) with an internal resistance \( (R_J) \). This reduced, equivalent network is the same as that illustrated in Fig. IV.6, for which the longitudinal approach has already been proven. Therefore, in determining the sub-solution for \( M \) that is caused by \( B \), \( R_M \) is first subject to \( U_J \), and the load current is then calculated. According to Fig. IV.7, \( U_J \) is the corresponding steady-state surface temperature of \( \#J \) with both sides (left, right) floating (cut at \( B \)). The same reasoning holds for the other connections. As a first step, \( M \) is excited at the corresponding terminal with the “open loop” voltage of \( \#J \), which causes a particular \( i_L \) flow into \( M \). Instead of performing this step for each connection separately, these steps can be conducted in a single one for all of the terminals \( A-C \). Notice that the terminals of \( M \) are connected to either the open loop voltage of \( \#J \) or to ground (Fig. IV.7), and the latter represents a shorted voltage source according to superposition. This step is the same as that in the “analytical” longitudinal method but is applied to a lumped network. (Surface temperature excitation and surface load flux.)

The “ideal” fluxes that enter \( M \) through the terminals are calculated by applying the left side surface voltages of floating \( \#J \) to the terminals of \( M \). According to the lumped model in Fig. IV.6, the solution has to be corrected to account for the internal resistance of the excitation. Analogous to the previous step, this correction can also be conducted in a single step, instead of solving for each connection separately. All upper and bottom voltage excitations of \( \#J \) are shorted, and every terminal is reconnected, but the previously calculated load currents are applied at the junctions, and the system is solved. This step is the same as that in the proposed longitudinal method but is illustrated here for a lumped network. Finally, the right subset (\( \#J+1 \) to \( \#N \)) has to be dealt with. Since the steady solution of subset \( M \) plus \( \#J \) is known, this problem is similar to that already discussed for \( M \) and \( \#J \). Thus, the same procedure can be applied, and the final steady-state solution derived.

In the above reasoning, a lumped network was considered. Since the number of cells was arbitrary, with terminal \( C \) representing the rest, this proof holds for any number of cells and internal arrangements. If the partitioning is very large (infinite), the solution will converge to the analytical one.

**IV.4. Effect of Internal Heat Generation**

In Section IV.1, the overall problem was divided into three parts on the basis of superposition. In this Section, I show that some forms of heat generation can also be traced back to a combination of only transverse effects using the method proposed for longitudinal excitations. If internal heat generation is present, the governing PDE (partial differential equation) becomes the Poisson equation. In this study, I consider the heat generation that is expressed by (IV.12), which only depends on the \( x \) transverse coordinate. With this restriction, the temperature solution \( \Theta_j \) for a slab is sought in the form of (IV.13) [63, pp. 96-98],
\[ q_i = \frac{q_j(x)}{k_j}, \quad (\text{IV.12}) \]

\[ \Theta_j(x,y) = \Psi_j(x,y) + \Phi_j(x). \quad (\text{IV.13}) \]

The governing PDE thus becomes (IV.14).

\[ \frac{\partial^2 \Psi_j(x,y)}{\partial x^2} + \frac{\partial^2 \Psi_j(x,y)}{\partial y^2} + \left( \frac{\partial^2 \Phi_j(x)}{\partial x^2} + \frac{q_j(x)}{k_j} \right) = 0. \quad (\text{IV.14}) \]

The term in parenthesis in (IV.14) is set to zero (IV.15) and solved first [63]. The interface conditions (IV.4.a)-(IV.4.f) still hold, except the problem is now reduced to 1D conduction. By direct integration, solution (IV.16) can be readily obtained for \( \Phi_j \) with two integrating constants due to the second order nature of (IV.15). The constants \( a_j \)-s and \( b_j \)-s can be determined using the boundary and continuity conditions,

\[ \frac{\partial^2 \Phi_j(x)}{\partial x^2} + \frac{q_j(x)}{k_j} = 0, \quad (\text{IV.15}) \]

\[ \Phi_j(x) = -\int \frac{q_j(x)}{k_j} dx^2 = S_j(x) + a_jx + b_j. \quad (\text{IV.16}) \]

At the junctions, the matrix relation (IV.17) can be defined in accordance with the junction boundary conditions (IV.4.e)-(IV.4.f). The double integral term in (IV.16) is represented by \( S_j \), and the upper dot symbol represents differentiation by \( x \),

\[
\begin{bmatrix}
\frac{k_j}{k_{j+1}} & 0 \\
L_j & 1
\end{bmatrix}
\begin{bmatrix}
a_j \\
b_j
\end{bmatrix}
+ \begin{bmatrix}
k_j \dot{S}_j(L_j) - \dot{S}_{j+1}(0) \\
L_j \dot{S}_j(L_j) - S_{j+1}(0)
\end{bmatrix}
= \begin{bmatrix}
a_{j+1} \\
b_{j+1}
\end{bmatrix},
\quad (\text{IV.17})
\]

The layer coefficients \( (a, b) \) can be recursively expressed in terms of each other by direct multiplication of the matrices (IV.17), in which only the integrating constants are unknown. Thus, the parameters of the last layer can be expressed by those of the first layer (IV.18),

\[ A \cdot \begin{bmatrix}
a_1 \\
b_1
\end{bmatrix} + \Gamma \begin{bmatrix}
a_N \\
b_N
\end{bmatrix} = \begin{bmatrix}
a_n \\
b_n
\end{bmatrix}, \quad (\text{IV.18}) \]

Now, the boundary conditions (IV.4.c) and (IV.4.d) are exploited to relate \( a_1 \) to \( b_1 \) and \( a_N \) to \( b_N \), respectively. By using four independent equations, (IV.4.c) to (IV.4.d) and (IV.18), the integrating constants of the first and last layers can be readily determined. The other constants can be obtained by direct substitution; thus, a specific \( \Phi_j \) solution is established for each slab.
Next, the Laplace equation that incorporates $\Psi_j$ is studied and treated by separation of variables [63]. In accordance with Section IV.1, the external excitations are removed; therefore, the boundary conditions are homogeneous for $\Theta_j$. From the substitution of (IV.13) into the corresponding boundary conditions, I obtain the upper-side and bottom-side boundary conditions (IV.19) and (IV.20) for $\Psi_j$,

$$k_j \frac{\partial \Psi_j(x,y) + \Phi_j(x)}{\partial y} \bigg|_{y=0} = h_{j,b} \left[ \Psi_j(x,0) + \Phi_j(x) \right]$$  \hspace{1cm} (IV.19)

$$-k_j \frac{\partial \Psi_j(x,y) + \Phi_j(x)}{\partial y} \bigg|_{y=H} = h_{j,u} \left[ \Psi_j(x,H) + \Phi_j(x) \right]$$  \hspace{1cm} (IV.20)

Due to the differentiation by the $y$ coordinate, the term $\Phi_j(x)$ on the left sides of (IV.19-20) vanishes. Now, (IV.19-20) have the same form as the boundary conditions (IV.4.a)-(IV.4.b), which express non-homogeneous longitudinal excitations applied to the slabs. This problem has become similar to the one that incorporates only longitudinal excitations (Section IV.2) and can therefore be treated and solved with the proposed method. (The left side boundary condition of #1 and the right side boundary condition of #N will produce a zero solution.)

**IV.5. Conclusions**

The present investigation developed a method to account for excitations in the directions parallel to the layers (longitudinal) and to simplify the eigenvalue problem in multilayer bodies that consist of an arbitrary number of layers. Considering steady-state heat conduction, longitudinal excitations could be traced back to a combination of excitations only in the direction perpendicular to the layers (transverse). In the cases of only transverse problems, the eigenconditions (transcendental equations) for the slabs could be explicitly written and had a simple form. Thus, the associated eigenvalue problem became less complicated, and the eigenvalues were easier to calculate. Separation of variables could be used to derive an analytical solution, and internal heat generation could also be considered with the presented method if the dissipation only depended on the transverse coordinate. An analytical treatment of such forms of heat generation may prove useful for modeling electrical machines, e.g., solenoid actuators. Since the overall problem(s) were decomposed into only transverse situations, the solution methodology became highly algorithmic, calling the same routine repeatedly. This property might be interesting from an implementation point of view. Furthermore, the proposed method can be extended to 3D conduction, cylindrical coordinates or orthotropic materials and may involve contact resistance.
V. A Method for Computing the Analytical Solution of the Steady-State Heat Equation in Multilayered Media

In Section IV, a general multilayered conduction problem was studied and a general method was developed for including the effects of non-homogeneous longitudinal boundary conditions and internal heat generation in an analytical solution of the temperature field. It has been also highlighted that non-homogeneous longitudinal boundary conditions and forms of internal heat generation could be traced back to a combination of multilayered conduction problems with homogeneous longitudinal boundary conditions; thus, the process of solving the thermal field becomes algorithmic. However, Section IV does not consider the exact computation of the analytical solution, i.e., how to evaluate every parameter of the eigenfunctions. In practice, numerical difficulties can be encountered at the computational process, especially if the layers have longitudinal boundary conditions of the third kind.

In this study, I develop a method to compute the analytical solution of the steady-state temperature field in multilayered laminates for an arbitrary number of layers and an arbitrary combination of homogeneous longitudinal boundary conditions of the first, second, and third kind for each layer. First, I consider a two-layer medium and use a meshless iterative method to approximate the temperature at the junction step-by-step, where the heat equation is solved analytically and separately for each layer. The developed method resolves the aforementioned computational difficulties because the computation of the analytical solution for a homogeneous single layer is numerically stable. I later show that the multi-layered media can be represented as a hierarchy of two-layered media, enabling the computational method to be generalized to an arbitrary number of layers. To enhance the computational speed, I use varying weighting coefficients during the iterations, and a method is developed to partition the multi-layered media into two-layered media. In this study, 2D slabs are considered, and the analytical temperature field is obtained using separation of variables; however, the applied analytical method can be arbitrary. A numerical example involving a four-layer structure is also presented, and the results are compared to a FEM (finite element method) solution. With the developed method, the solution can be rapidly and comfortably computed in case of problem described in Section IV.

The remainder of this study is structured as follows. In Section V.1, the underlying multi-layer model and the problem are discussed. In Section V.2, a method to resolve these computational difficulties is introduced for a two-layer structure. In Section V.3, the developed method is generalized to an arbitrary multi-layer structure. In Subsection V.4.1, a process is presented to decompose the multi-layer structure into two-layer structures to improve the computational efficiency. In Section V.4-4.1, a mathematical proof of the developed method is provided. In Subsection V.4.2, the convergence of the developed method is reviewed and improved. In Subsection V.4.3, the required weighting coefficients are estimated. In Subsection V.4.4, some practical considerations are made. In Section V.5, a numerical example is presented and the developed methodology is succinctly overviewed. In Section V.7, a detailed mathematical proof is provided for the improvement in the convergence speed.
V.1. Statement of the Problem

I consider a linear, 2D physical model, as shown in Fig. V.1. The multi-layered medium is composed of \#N linear, homogeneous, isotropic slabs with equal heights \( H \) but different lengths \( L_j \) and different thermal conductivities \( k_j \). The thermal contacts between the adjacent slabs are assumed to be ideal (from the continuity of temperature), but additional flux excitations \( Q_j(y) \) are assumed to exist at the junctions in the transverse directions. The exposed surfaces have boundary conditions of the third kind with constant convection coefficients \( h_j \) but can also be of either the first or the second kind with \( h_j \to \infty \) or \( h_j = 0 \), respectively. The subscripts \( l, r, b, \) and \( u \) refer to the left-hand side, right-hand side, bottom side, and top side, respectively, of the slabs. The transverse boundary conditions, which are in the direction perpendicular to the layers, i.e., \( T_{il}(y) \) and \( Q_{il}(y) \), are non-homogeneous; however, the longitudinal boundary conditions, which are in the directions parallel to the layers, are homogeneous. To simplify the mathematical treatment of the problem, separate coordinate systems are assigned to each slab; thus, the thermal distribution in the transverse direction is expressed in the corresponding \([0, L_j]\) domain. The solution is derived in Cartesian coordinates.

![Fig. V.1: Model of the multi-layered medium (slabs).](image)

The underlying partial differential equation for steady heat diffusion, which is governed by the Laplace equation (V.1) for isotropic materials in two-dimensional Cartesian coordinates, is written separately for the layers. The boundary and interface conditions are expressed by (V.2.a)-(V.2.f). The index \( j \) refers to a specific layer and ranges from 1 to \( N \).

\[
\frac{\partial^2 T_j(x_j, y)}{\partial x_j^2} + \frac{\partial^2 T_j(x_j, y)}{\partial y_j^2} = 0, \tag{V.1}
\]

\[
k_j \frac{\partial T_j(x_j, 0)}{\partial y_j} = h_{j,b} T_j(x_j, 0), \tag{V.2.a}
\]

\[-k_j \frac{\partial T_j(x_j, H_j)}{\partial y_j} = h_{j,b} T_j(x_j, H_j), \tag{V.2.b}\]

\[
k_i \frac{\partial T_i(0, y)}{\partial x_i} = h_{i,b} (T_i(0, y) - T_{i-1}(y)) \tag{V.2.c}
\]
\[-k_N \frac{\partial T_N(L_N, y)}{\partial x_N} = h_{N,r} \left(T_N(L_N, y) - T_{N,r}(y)\right), \quad (V.2.d)\]

\[T_j(L_j, y) = T_{j+1}(0, y), \quad (V.2.e)\]

\[k_j \frac{\partial T_j(L_j, y)}{\partial x_j} - k_{j+1} \frac{\partial T_{j+1}(0, y)}{\partial x_{j+1}} = Q_j(y), \quad (V.2.f)\]

The analytical solution for the temperature in the slabs is determined using separation of variables (SOV) as in [40], [47-49], [54], [59-60], [64], for example. The homogeneous longitudinal boundary conditions (see Fig. V.1) motivates the form of the thermal field in each slab (e.g., slab \#j) given in (V.3). In (V.3), the term \(T_j\) denotes the 2D thermal field in slab \#j, and the coefficients \(a_{j,n}, b_{j,n}, c_{j,n},\) and \(d_{j,n}\) are the constants associated with the \(n^{th}\) eigenfunction of the \(T_j\) thermal field. The term \(\lambda_{j,n}\) is the eigenvalue associated with the \(n^{th}\) eigenfunction. Since all of the longitudinal boundary conditions are homogeneous, the eigenvalues are real. The longitudinal eigenfunctions satisfy the orthogonality condition in (V.4),

\[\int_{-H}^{H} Y_{j,n}(y, \lambda_{j,n}) Y_{j,m}(y, \lambda_{j,n}) dy = \left\{ \begin{array}{ll} 0 & \text{if} \ n \neq m, \\ N_{j,n}^2 & \text{if} \ n = m. \end{array} \right. \quad (V.4)\]

The eigenvalues \(\lambda_{j,n}\) are calculated from the transcendental equations, which are derived from the homogeneous longitudinal boundary conditions of the respective layer. If the longitudinal boundary conditions are only of the first kind or only of the third kind, then a transcendental equation takes the form of (V.5) or (V.6), respectively,

\[\sin(\hat{\lambda}_j H) = 0, \quad (V.5)\]

\[\tan(\hat{\lambda}_j H) = \frac{(h_{j,b} + h_{j,a})(k_j \hat{\lambda}_j)}{(k_j \hat{\lambda}_j)^2 - h_{j,b} h_{j,a}}, \quad (V.6)\]

To determine the analytical solution of the steady-state temperature \(T_j\) in slab \(j\), the four parameters in (V.3) for each eigenfunction must be found. The constants \(c_{j,n}\) and \(d_{j,n}\) of the trigonometric eigenfunctions can be simplified or rearranged using the longitudinal boundary conditions (e.g., \(c_{j,n}\) or \(d_{j,n}\) are zero) [59]; and the constants in the hyperbolic eigenfunctions (i.e., \(a_{j,n}\) and \(b_{j,n}\)) can be computed using the interface/transverse boundary conditions in (V.2.e)-(V.2.f) and the orthogonality of the longitudinal eigenfunctions in (V.4).
If the layers have only longitudinal boundary conditions of the 1\textsuperscript{st} (or 2\textsuperscript{nd}) kind, then they have the same set of eigenvalues and eigenfunctions according to (V.5), thus the “global” orthogonality condition is satisfied. The computational process becomes straightforward (see, e.g., 47-49 and 59]) because only the eigenfunctions with same index are involved in the layers when computing the coefficients of a particular hyperbolic eigenfunction. However, if the layers have only longitudinal boundary conditions of the 3\textsuperscript{rd} kind, then they may have different sets of eigenvalues and eigenfunctions according to (V.6); thus, the “global” orthogonality condition may not be satisfied but changes to (V.7). Therefore, the computation of the parameters of a particular transverse eigenfunction involves all the other transverse eigenfunctions in the other (adjacent) layers. Thus, a system of linear equations has to be solved which contains hyperbolic terms off the main diagonal in the coefficient matrix, resulting an ill-conditioned solution process. In practice, all computational methods suffer from arithmetic errors, such as round-off and truncation error, which may cause the calculation process (matrix construction, inversion) to become numerically unstable [58],

\[ \int_{H} Y_{j,n}(y, \lambda_{j,n}) Y_{k,m}(y, \lambda_{k,m}) dy = N_{j,n,k,m} \neq 0. \]  

(V.7)

If a homogeneous single layer is considered, the computation of (V.3) is numerically stable because the transverse parameters can be computed from the transverse boundary conditions straightforwardly by using the orthogonality condition in (V.4).

V.2. Method to Compute the Analytical Solution

In the underlying multi-layered medium problem, the transverse eigenfunctions of the slabs (i.e., the parameters \( a_{j,n} \) and \( b_{j,n} \)) are linked to each other through the interface conditions in (V.2.e-V.2.f). However, this linking is absent from the solution to a single layer structure, enabling the coefficients associated with the hyperbolic transverse eigenfunctions to be readily computed from the transverse boundary conditions.

From the viewpoint of the steady-state solution in the multi-layered medium, the effects of the junctions can be equivalently represented as non-homogenous transverse boundary conditions of the 1\textsuperscript{st} or 2\textsuperscript{nd} kind for the layers. Therefore, if the steady-state temperatures or fluxes at the layers’ junctions are known, then the problem can be decomposed into single-layer problems where the transverse boundary conditions of the single layers become the corresponding junction temperatures or fluxes. The thermal fields in the layers can thus be readily calculated free of the aforementioned numerical difficulties.

Next, I develop an iterative method to approximate the junction quantities (temperature and flux) step-by-step. The method is first described for a two-layered medium, and then it is generalized to an arbitrary number of layers in Section V.3. The proof is presented in Section V.4.4.1, and its convergence properties are discussed in subsections V.4.2-4.3. The developed method is illustrated in Fig. 2 and is discussed below.

- **Step 1:** Assume the two-layered medium that is similar to Fig. V.1 and is defined in Section V.1. The layers \#1 and \#2 are in perfect thermal contact and are subject to boundary conditions of the 1\textsuperscript{st}, 2\textsuperscript{nd}, or 3\textsuperscript{rd} kind on the exposed surfaces. The flux \( Q(y) \) is also applied at the junction. Let \( t_{SP}(y) \) denote the real solution of the steady-state
temperature at the junction and let \( q_S(y) \) denote the flux entering layer \#2 from the interface, respectively.

- **Step 2:** Let the layers \#1 and \#2 be separated at the junction thus the two-layer problem is split into two independent single layer problems. The effect of the junction is replaced by a non-homogeneous b.c. of the 2\(^{\text{nd}}\) kind on the right-hand side surface \((x_1=L_1)\) for layer \#1 and is replaced by a non-homogeneous b.c. of the 1\(^{\text{st}}\) kind on the left-hand side surface \((x_2=0)\) for layer \#2. The other boundary conditions of the layers remain unchanged. The eigenvalues and eigenfunctions of the layers in the \( y \) direction can be uniquely obtained and will not be modified during the iterative process. Take an initial guess \( t_{S,0}(y) \) of the junction temperature and use it in the first cycle as the starting guess. Considering the \( k^{\text{th}} \) cycle:

- **Step 3:** Let \( t_{S,k}(y) \) be a starting guess for the steady-state junction temperature, which was resulted from the previous cycle \( k-1 \). Having the \( t_{S,k}(y) \) guess, set the boundary condition on the left-hand side surface of layer \#2 to the non-homogeneous b.c. of the 1\(^{\text{st}}\) kind with \( t_{S,k}(y) \). Next, solve the thermal field of layer \#2 (analytically) by also considering all the other of its boundary conditions. Then, compute the flux \( q_{S,k}(y) \) that enters \#2 at its left-hand side surface \((x_2=0)\) from \( t_{S,k}(y) \). The flux \( q_{S,k}(y) \) is an estimate of the interface flux of \#2 which corresponds to the \( t_{S,k}(y) \) junction temperature.

- **Step 4:** Set the boundary condition on the right-hand side surface \((x_1=L_1)\) of \#1 to the non-homogeneous b.c. of the 2\(^{\text{nd}}\) kind that equals to \( Q(y)-q_{S,k}(y) \) according to (V.2.f), i.e., the flux that enters \#2 exits \#1. Then, solve the temperature field of layer \#1 (analytically) by also considering all of its other boundary conditions. Let the term \( t_{S,k+1}(y) \) denote the resulting right-hand side surface temperature of \#1.

- **Step 5:** The guesses for the junction temperature are the present \( t_{S,k+1}(y) \) and the \( t_{S,k}(y) \) from the previous step. However, as shown in (V.8), replace \( t_{S,k+1}(y) \) by taking a weighted sum of the current guess \( t_{S,k+1}(y) \) and the previous guess \( t_{S,k}(y) \). This modified guess (V.8) becomes the updated guess for the junction temperature, which is to be used in the next iteration, i.e., when computing the junction flux at layer \#2 in Step V.3. The scalar \( \alpha_k \) parameter in (V.8) ranges from 0 to 1 and it can remain unchanged during the subsequent iterations or may vary from iteration to iteration. The significance and proper choices of this parameter are discussed in subsections V.4.1-4.3.

- **Step 6:** Repeat the iterative process from Step 3 to Step 6 until a specific convergence criterion is reached for the junction temperature, e.g., the least squared error between the new and the previous guesses is less than 1E-6.

- **Step 7:** Having the temperature distribution at the junction converged, the analytical thermal field of the layers; thus, the parameters \( a_{j,n} \) and \( b_{j,n} \) in Eq. V.3 can be readily computed by solving the single layer problems with the converged junction temperature.

\[
t_{S,k+1}(y) \leftarrow \alpha_k t_{S,k+1}(y) + (1-\alpha_k) t_{S,k}(y) \tag{V.8}
\]
Fig. V.2: The developed calculation method

The accuracy of the solution depends on the convergence criterion and on the number of eigenfunctions used in the solution to the layers. In order to satisfy the interface conditions, the temperature/flux pairs are being matched to each other during the iterations by reflecting their effect to the adjacent layer in Steps 3-6. Under the corresponding guess of the interface condition, an analytical solution (V.3) is derived for the layers separately in every cycle; thus, an approximation of the $a_{j,n}$ and $b_{j,n}$ parameters is obtained as well. The junction quantities are straightforwardly expressed from the corresponding version of (V.3) of a layer and reflected to the adjacent layer; enabling the analytical solution to be updated. According to Step 5, a linear combination of the previous and current guesses of the analytical solution has to be taken with the scalar $\alpha_k$ parameter, as in (V.8). This weighted sum of the solutions is important to ensure that the iterative process converges, as it may diverge. Furthermore, I propose the use of different $\alpha_k$ weights during the iterations because it can considerably improve the speed of convergence. These issues and how to choose the $\alpha_k$ weights are explained in sections V.4.2-4.3.

The aim of the developed method is to obtain the junction temperature because this value is the same for the two layers compared to the flux (see (V.2.f)). Furthermore, the representation of the junction is identical for the two layers within the proposed iterative scheme i.e., the same solution is obtained if the effect of the junction is represented as a b.c. of the 1st kind for layer #1 and a b.c. of the 2nd kind is applied to layer #2. However, according to subsection V.4.4, these are two different approaches and used alternately. Henceforth, the first method described above is referred to as the temperature method, and the second one is referred to as the flux method.

V.3. Generalization to Multilayered Media

In Section V.2, the developed method is presented to a two-layered structure. In this section, a generalization is provided for an arbitrary number of layers. According to Section V.2, the aim is to obtain the analytical solution of the thermal field in the layers by approximating the temperature distribution at the interfaces iteratively ($#N-1$ junctions if $#N$ layers). During the iterations, the temperature field in the layers is to be solved analytically and the analytical temperature solutions; thus, junction temperatures are to be updated step by step.

According to Section V.1, the multi-layered medium is defined as the group of piecewise homogeneous single layers, for which the analytical approaches e.g. SOV separately apply. Therefore, the analytical solution to the multi-layered medium is obtained as the “sum” of the solutions in the separate layers. Nevertheless, a group or block of single layers can also be
considered as a single, but “heterogeneous” layer from a modeling viewpoint. Considering the analytical solution to the heterogeneous layer, it requires its decomposition to the homogeneous layers, where the analytical approaches apply and the solution can be partially derived.

Consider a multi-layered medium similar to that shown in Fig. V.1 with $\#N$ layers. In the two-layer example in Section V.2, the structure is split into two homogeneous single layers at the junction. Although the multi-layered medium now has more junctions ($\#N-1$), it is still possible to split the system into two multi-layered media (blocks) at an arbitrary junction. Thus, a two-layer problem, which is similar to that in Section V.2, can be defined and the proposed method formally applies, except that a block can be a homogeneous single layer or a heterogeneous layer. According to Section V.2, the iterative process requires solving the temperature fields in the layers (analytically), which can be carried out in a single step for blocks that represent homogeneous single layers, but becomes another multi-layered medium problem for blocks that represent heterogeneous layers. Similarly as it was for the original structure, the method in Section V.2 can be applied again for the heterogeneous block(s), thus resulting in additional, “subordinate” two-layer problems. This two-layer partitioning is to be applied successively until the blocks can not be further decomposed because they have become the single layers of the original multi-layered medium; thus, the analytical solutions are to be established only in the homogeneous single layers.

Note that this way, the multi-layered medium is represented as a hierarchy of two-layered media and every two-layer problem consists of subordinate two-layer problems which have to be solved first to proceed with the next iteration in the “top/parent” two layer problem. When performing the two-layer decomposition or an iteration, the boundary and interface conditions of a particular, subordinate two-layer problem are inherited from the current state of its parent problem. Thus, the entire computation process becomes a multi-level, two-layer problem (see Fig. V.3). The developed computational process in Section V.2 can be applied separately, and independently to the two-layered media (also using separate $\alpha_k$ weights), although the required number of iterations to reach global convergence greatly depends on the two-layer partitioning of the original multi-layered medium. If using the so called straightforward approach which is illustrated in the left side of Fig. V.3, for creating the hierarchical two-layer representation, the computational process becomes extremely slow with many, e.g. 10 layers. In Fig. V.3, the capital letters define a specific two-layer problem, and the horizontal lines define the blocks (layers) between which the two-layer problem is defined. Assuming that every two-layer problem requires $x$ iterations to converge, this approach yields global convergence in approximately $x^{N-1}$ iterations. In order to greatly improve the computational effectiveness of the temperature solution, even in case of numerous layers, a method is presented in Section V.3.1 for creating the two-layer partitioning.

Fig. V.3: The straightforward two-layer partitioning (left) and the proposed (right) two-layer partitioning to decompose the multi-layered medium into two-layered media.
V.3.1. Method for decomposing the multilayered media into two-layered media

A higher computational efficiency and speed can be obtained even for large structures consisting of numerous layers by performing the two-layer decomposition using the “bisection” method that is illustrated on the right side of Fig. V.3. This method requires each multi-layered structure to be always divided at its middle junction. The detailed explanation is the following.

By applying the method in Section V.2, the multi-layered medium is split at its middle junction into the heterogeneous layers (blocks) [\(\#1;\#N/2\)] and [\(\#N/2+1;\#N\)]. The method proposed in Section V.2 formally applies and the two-layer problem \(A\) is defined. According to Section V.2, a single iteration in \(A\) requires a solution in both [\(\#1;\#N/2\)] and in [\(\#N/2+1;\#N\)]; nevertheless, these solutions can be established separately, i.e., one after the other. However, both blocks of \(A\) are heterogeneous so they represent additional multi-layered medium problems, for which the analytical solution cannot be derived directly. Therefore, the method in Section V.2 is applied again but separately to the heterogeneous blocks of \(A\), which are also divided at their middle junction. Thus, [\(\#1;\#N/2\)] is solved in the two-layer problem \(B_1\) which is defined between [\(\#1;\#N/4\)] and [\(\#N/4+1;\#N/2\)]; and [\(\#N/2+1;\#N\)] is solved in the two-layer problem \(B_2\) which is defined between [\(\#N/2+1;\#3N/4\)] and [\(\#3N/4+1;\#N\)]. Again, the parent problem \(A\) requires its subordinate problems \(B_1\) and \(B_2\) to fully converge. The boundary and interface conditions of the subordinate problems are inherited from the current state of their parent problem; and the method in Section V.2 applies separately. This two-layer decomposition is to be performed successively until the solutions are to be computed in the homogeneous single layers of the original medium.

In this case, the two-layer hierarchy consists of fewer levels \((\log_2 N)\) and each level contains many two-layer problems, related to powers of two. Compared to the straightforward approach, a new junction value from a higher-level problem does not have to propagate through the entire structure but only through its corresponding branch. This branch division shows that the required number of iterations for a particular two-layer problem is \(x\) times the sum of the number of iterations required for the subordinate two-layer problems (one level down), e.g., a single iteration in the problem \(A\) requires both \(B_1\) and \(B_2\) to converge with the inherited boundary conditions. Successively using this expansion from the topmost two-layer problem reduces the total number of cycles to reach global convergence to approximately the value given in (V.9). The developed decomposition method is useful for large structures, e.g., the required number of cycles reduces to \(8x^4\) in a 16-layer structure compared to the \(x^{15}\) cycles for the straightforward approach,

\[
S \approx 2^{\log_2 N - 1} N \log_2 N. \quad (V.9)
\]

V.4. Proof of the Developed Computational Method

For proving the proposed temperature and flux methods in Section V.2, the theory of lumped element models, networks and superposition is exploited, similarly to Section III.3. To avoid confusion, the lumped approach is used only for the proof: to show that the method is applicable and under what circumstances it converges and how. Then, it is also discussed how the lumped network approach is generalized thus the proposed method is validated (applies) to analytical solutions. A more detailed explanation on the lumped model approach, on its use and on its analogy to electrical systems can be found in [74], [75]. In subsection V.4.2, the speed of
the convergence is improved, and subsection V.4.3 discusses how to obtain an estimate of the optimal weighting coefficients in (V.8).

The following fundamental lumped element models are defined: the temperature/flux source is a discrete two-port component that produces/forces a given temperature difference/flux between/through its two ports, respectively. The resistance satisfies Ohm’s law i.e., \( q = T/R \). The corresponding symbols are provided in Fig. V.4. These lumped components can be connected to each other at their ports; thus, an arbitrary “lumped” network can be created. Using the above definitions, the boundary conditions in the analytical version can be equivalently represented in the lumped domain: the values of the lumped sources become the average of the boundary condition on the corresponding surface (volume) section that is to be modeled. Thus, the boundary condition of the \( 1^{\text{st}}/2^{\text{nd}}/3^{\text{rd}} \) kind corresponds to the temperature source/flux source/temperature source in series with a resistance, respectively. The lumped temperature is measured compared to a reference or zero point.

![Fig. V.4: Lumped model definitions and symbols.](image)

Consider the multi-layered structure in Fig. V.1, which has only two layers. Using the previously defined lumped components, an equivalent representation of the layers is created in the steady-state by a network of lumped resistances, as it is illustrated in Fig. V.6. First, the layers are divided (meshed) into \( n \) by \( m \) (rows and columns) rectangular cells. For each cell of the layers, an equivalent lumped model is derived, which is a four-terminal “star” resistance network, considering 2D conduction. In more details, the cell model consists of four resistances that are connected to each other at a central node, at which the temperature represents the average temperature in the volume of the cell. The free ends (ports) of the resistances represent the surfaces of the cell and can be connected to adjacent cells or surface boundary conditions.

By connecting the adjacent cells to each other, networks of lumped resistances are created which represent the layers. The resulting resistance network of a layer has \( 2n+2m \) free ports, which are labeled as in Fig. V.5. For layer \#1, the ports \( c_1 \) to \( c_n \) contribute to the junction, the ports \( a_1 \) to \( a_m \) and \( a_{n+m+1} \) to \( a_{n+2m} \) contribute to the longitudinal boundary conditions and the ports \( a_{m+1} \) to \( a_{n+m} \) contribute to the transverse boundary condition. The labeling is similar for layer \#2, but the ports are labeled as \( b \). The lumped equivalents of the surface boundary conditions can also be established, and the boundary cells of the layers are connected to their corresponding lumped boundary conditions through their free terminals. For example, if the boundary condition on the left-hand side of layer \#1 is of the first kind, then the ports \( a_{m+1} \) to \( a_{n+m} \) are connected to separate lumped temperature sources, the magnitude of which equals the average temperature input on the corresponding surface sections of the cells. Similarly, the lumped equivalents of the interface flux \( Q(y) \) is derived and applied between the junction ports of the two layers.

Note that the analytical temperature field is now approximated by a discrete temperature field, i.e., temperatures only at the center and surfaces of the cells are considered, which represent
the average temperatures. Let the vector $t_S$ denote the real solution of the steady temperature at the junction nodes $c_1$ to $c_n$, which represent the junction temperature; and let the vector $q_S$ denote the real steady flux which enters layer #2 through the junction nodes $c_1$ to $c_n$. Next, the junction temperature is derived by the method proposed in Section V.2.

**Fig. V.5: Lumped network representation of the layers.**

### V.4.1. Mathematical formulations

According to Section V.2, the layers are separated at the junction and the boundary (interface) conditions are set to the first kind on the left-hand side of layer #2 and to the second kind on the right-hand side of layer #1, as it is illustrated in Fig. V.5. Therefore, the $c_1$ to $c_n$ terminals of layer #1 are connected to lumped flux sources (each terminal to a separate flux source), and the $c_1$ to $c_n$ terminals of layer #2 are connected to lumped temperature sources (each terminal to a separate temperature source). In order to compute the lumped temperature distribution, the following transfer matrices of the lumped networks are derived.

I construct the $n$ by $n+2m$ matrix $A$ to express the contribution of the lumped boundary conditions, which are applied to the inputs $a_1$ to $a_{n+2m}$ of layer #1, to the temperature of the right-hand side $c_1$ to $c_n$ terminals of layer #1 if a homogeneous boundary condition of the 2nd kind (lumped flux sources with zero flux) is prescribed for layer #1 at $c_1$ to $c_n$.

The matrix $A$ of layer #1 is constructed as the following. First, all of the lumped boundary sources of #1 are set to zero. Then, an arbitrary input terminal $a_k$ from $a_1$ to $a_{n+2m}$ is selected, and the lumped boundary source at this $a_k$ is set to unity. Next, the resulting network of lumped components, representing layer #1 (resistances and boundary sources), is solved for the temperatures at its $c_1$ to $c_n$ terminals. These temperatures will form the $k^{th}$ column of the $A$ matrix. Using superposition, the previous process can be carried out for each of the input terminals separately from $a_1$ to $a_{n+2m}$. Note that in their “disabled” state (set to zero, homogeneous), the lumped temperature source becomes a zero resistance, the flux source becomes an infinite resistance and the 3rd kind becomes a resistance that is determined by the corresponding convection coefficient $h$ and surface area of the cell.

I similarly construct the $n$ by $n+2m$ matrix $D$ of layer #2 to express the contribution of the lumped boundary conditions, which are applied to the terminals $b_1$ to $b_{n+2m}$ to the flux which flows into layer #2 through its terminals $c_1$ to $c_n$, if the boundary condition is homogeneous for layer #2 at its $c_1$ to $c_n$ terminals.

I similarly construct the $n$ by $n$ matrix $B$ of layer #1 to express the contributions from the lumped flux sources, which are applied to layer #1 at $c_1$ to $c_n$, to the temperature at terminals $c_1$ to $c_n$ of layer #1 if the boundary condition is homogeneous from $a_1$ to $a_{n+2m}$. 


I similarly construct the $n \times n$ matrix $C$ of layer #2 to express the contributions from the lumped temperature sources, which are applied to layer #2 at $c_1$ to $c_n$, to the junction flux (at the terminals $c_1$ to $c_n$) of layer #2 if the boundary condition is homogeneous from $b_1$ to $b_{n+2m}$.

The vectors $a$, $b$ and $Q$ represent the magnitude of the lumped sources (b.c.-s) that are applied to terminals $a_1$ to $a_{n+2m}$ and $b_1$ to $b_{n+2m}$ and the additional fluxes at $c_1$ to $c_n$, respectively. The vectors $t_S$ and $q_S$ represent the temperature at the terminals from $c_1$ to $c_n$ and the flux that enters layer #2 through the terminals $c_1$ to $c_n$, respectively. The index $k$ refers to these values in a given iteration. These notations are used to express the variables at the junction during the iterations in (V.10)-(V.13), see Section V.2. The matrix $I$ denotes the unit matrix in (V.13),

\[ q_{S,k} = Ct_{S,k} + Db, \]  \hspace{1cm} (V.10)

\[ t_{S,k+1} = Aa - Bq_{S,k} + BQ, \]  \hspace{1cm} (V.11)

\[ t_{S,k+1} \leftarrow \alpha_k t_{S,k+1} + (1 - \alpha_k) t_{S,k}, \]  \hspace{1cm} (V.12)

\[ t_{S,k+1} = \alpha_k (Aa - BDb + BQ) + [(1 - \alpha_k)I - \alpha_k BC] t_{S,k}. \]  \hspace{1cm} (V.13)

Eq. (V.13) shows that if the parameter $\alpha_k$ does not change during the iterations (i.e., $\alpha$), the evolution of the junction temperature $t_{S,k}$ can be expressed as a geometric series in matrix form, as given in (V.14)-(V.15). Further on, the notation in (V.14) is used,

\[ E = (1 - \alpha)I - \alpha BC, \]  \hspace{1cm} (V.14)

\[ t_{S,k} = \alpha \sum_{j=0}^{k-1} E^j (Aa - BDb + BQ) + E^k t_{S,0}. \]  \hspace{1cm} (V.15)

The matrices $B$ and $C$ are full rank and positive semi-definite; therefore, their product $BC$ is also full rank and positive semi-definite and can be written in the diagonal form given in (V.16), where $\Lambda$ is diagonal and its entries contain the $\mu$ eigenvalues of $BC$. The $\alpha$ parameter and the $I$ identity matrix are merged into $\Lambda$ to yield the modified diagonal matrix $\Omega$ in (V.16), whose diagonal entries $\omega_j$ are given in (V.17),

\[ E = (1 - \alpha)I - \alpha V \Lambda V^{-1} = V \Omega V^{-1}, \]  \hspace{1cm} (V.16)

\[ \omega_{j,j} = \omega_j = 1 - \alpha (1 + \mu_j), \quad \alpha < \frac{1}{|\mu|_{\text{max}}}. \]  \hspace{1cm} (V.17)

By diagonalizing the matrix $E$, the matrix power in (V.15) simplifies to the power of the diagonal $\Omega$ matrix in (V.19). The sum of the powers of the $\Omega$ matrix also reduces to the sum of a geometric series of the corresponding $\omega_j$ parameters on the main diagonal. Further on, I use the simplifying notation in (V.18) for the load vector $x$,
\[ \vec{x} = A\vec{a} - \frac{B}{Db} + BO, \quad (V.18) \]

\[ t_{s,k} = aV\left(\sum_{j=0}^{k-1} \Omega^j\right)V^{-1}\vec{x} + V\Omega^k V^{-1}t_{s,0}. \quad (V.19) \]

Eq. (V.19) shows that the \( t_{s,k} \) solution only converges if the absolute value of every eigenvalue \( \omega_j \) of the matrix \( \Omega \) is less than unity, which is satisfied if the \( a \) parameter is chosen to satisfy (V.17). Thus, an appropriate choice of \( a \) can result in a convergent iteration process even if the original system \( (a=1) \) is divergent. If (V.17) is satisfied, then using the formula for the sum of the geometric series results in the iterative solution converging to (V.20). This equals the straightforward solution in (V.21),

\[ t_{s,k=\infty} = V < \frac{a}{1-\omega_j} > V^{-1}\vec{x} = V < \frac{1}{1+\mu_j} > V^{-1}\vec{x}, \quad (V.20) \]

\[ t_s = (I + BC)^{-1}\vec{x} = V < \frac{1}{1+\mu_j} > V^{-1}\vec{x}. \quad (V.21) \]

It can be shown that the same proof and properties hold for the alternative flux method except that the matrix power is based on the inverse of the \( BC \) matrix; therefore, the corresponding eigenvalues become the reciprocals of the eigenvalues of the \( BC \) matrix.

According to Fig. V.5, the layers \#1 and \#2 are represented by a lumped resistance network. If the lumped partitioning of the layers increases, the error due to discretization decreases thus the temperature solution in the lumped network converges to the analytical one. In theory, the meshing can be chosen to be very large until the “same” result as the analytical one is obtained. Therefore, the developed method applies to analytical solutions and there exists an \( a \) parameter that ensures the convergence of the junction quantities if using the analytical solutions. Also, the results that are obtained by the lumped network representation can be generalized.

**V.4.2. Improving the convergence**

Eq. (V.19) shows that the \( t_{s,k} \) solution is a linear combination of the sum of independent geometric series with \( \omega_j \) parameters. However, the individual geometric series can have different convergence speeds, and the smaller the specific \( \omega_j \) ratio, the less number of cycles is needed for the corresponding geometric series (the \( Q \) section) to converge. As a worst case scenario, it seems reasonable to assume that the \( \omega_j \) with the highest absolute value limits the global convergence speed of the iterative solution. The actual values of the \( \omega_j \)'s in (V.17) are set by the \( a \) scalar parameter, which is bounded from above by (V.17) such that all of the eigenvalues of the original \( BC \) matrix are suppressed to absolute values below unity to prevent divergence. If the \( BC \) matrix has eigenvalues that span a wide range, e.g., 1 to 100, then the scalar \( a \) is restricted to a small value (0.01). This limitation may result in very slow overall convergence, because the \( \omega_j \) that corresponds to a small eigenvalue will be close to unity from (V.17).

To prevent this potential slow convergence, I suggest using different \( a \) parameters \( (a_k) \) in the successive iterations rather than using a single \( a \) weight throughout the whole iteration process. Eq. (V.17) also shows that the weight \( a \) can be chosen such that a specific \( \omega_j \) can be set
to 0; thus, the corresponding geometric series in (V.19) will converge in a single step. If multiple $\alpha_k$ weights are used, (V.19) can be partially written for the different $\alpha_i$-s as a finite sum of the geometric series. However, the corresponding initial guess $t_{S,0}$ for a specific iteration process results from the previous iterations as in (V.22). This process is mathematically formulated in (V.22)-(V.24), where the index $i$ stands for the $i^{th}$ process that uses the $\alpha_i$ weight for a number of $k_i$ iterations. In (V.22)-(V.23), the subscript $i$ of the $\Omega_i$ matrix denotes the $\Omega_i$ matrix associated with the corresponding $\alpha_i$ weight,

$$t_{S,i,k} = \alpha_i V \left( \sum_{j=0}^{k_i-1} \Omega_{ij} \right) V^{-1} x + V \Omega_{ij} V^{-1} t_{S,i-1,k_i}, \quad (V.22)$$

$$t_{S,n,k} = V \sum_{i=n}^{n} \left[ \alpha_i \left( \prod_{j=i}^{n} \Omega_{ij} \right) \sum_{m=0}^{n-1} \Omega_{im} \right] V^{-1} x + V \left( \prod_{j=i}^{n} \Omega_{ij} \right) V^{-1} t_{S,0}, \quad (V.23)$$

$$\alpha_i = \frac{1}{1 + \mu_i}. \quad (V.24)$$

If the $\alpha_i$ parameters are chosen as in (V.24), then each eigenvalue of $\Omega$ is zeroed out in the corresponding $i^{th}$ sub process. Thus, the exact solution in (V.21) is reached in a finite number of steps, i.e., a minimum of $#n$ steps (the number of distinct eigenvalues of $BC$), irrespective of the $x$ load vector. The mathematical proof is provided in the Appendix 2.

V.4.3. Estimation of the “$\alpha$” weights for the analytical solution

At the end of subsection V.4.1, it is discussed that the developed computational method applies if using analytical solutions during the iterations; and an $\alpha$ parameter exists which ensures that the iterative process remains convergent even if being originally divergent. In subsection V.4.2, it is shown that the convergence speed can be significantly improved by using different $\alpha$ weighting coefficients during the subsequent iterations (Eq. V.8); and these weighting coefficients are related to the eigenvalues of the $BC$ matrix of the lumped network model. Since the lumped network approach is actually an approximation of the analytical solution (the finer the meshing the less the error), the same conclusion can be drawn for the convergence properties and $\alpha$ weighting coefficients if using the analytical solutions.

Considering the analytical solutions, they correspond to the solution of an infinitely fine lumped resistance network of the layers (Fig. V.5); thus, the corresponding matrices $B$ and $C$ become infinitely large. However, only the main spectral properties of the $BC$ matrix are of interest from the viewpoint of the iterative process, i.e., the largest, the smallest eigenvalues and their distribution. Since the lumped network model approximates the analytical solution, the “lumped” transfer matrix $BC$ approximates the spectral properties of the analytical one. Therefore, for the analytical solutions, it is suggested that the necessary $\mu$ eigenvalues thus $\alpha_i$ weighting coefficients to be extracted from a $BC$ matrix, which is constructed from a “rough” lumped network representation of the layers as in subsection V.4.1. If using a finer partitioning the analytical system is approximated more accurately, yet at the cost of increasing computational load. As a general rule, I would say that an 8 by 8 lumped partitioning produces an acceptable

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estimate of the $BC$ matrix. Altogether, the $BC$ matrix can also be iterated by increasing the partitioning until the smallest and largest eigenvalues converge or can be extrapolated.

**V.4.4. Practical considerations**

In some situations, the eigenvalues of the approximate $BC$ matrix can be closely spaced. Based on the distribution of the eigenvalues, these can be sorted into groups; and an “equivalent” eigenvalue can be determined for each group. By using these equivalent eigenvalues during the iterative processes, the number of iterations required to reach the desired convergence may reduce compared to using every eigenvalue of the $BC$ matrix.

In practical applications, the eigenvalues are only approximate; therefore, the $\omega_{i,i}$, which is the $i^{th}$ element in the main diagonal of the $Ω$ matrix, is not zero but close to zero; thus, it results in non-zero residual terms in (V.22)-(V.23). This residual error can be overcome if the sub-iterations, especially those which are associated with the larger eigenvalues, are run for more cycles, because the corresponding element on the main diagonal of the powered $Ω$ matrix in (V.23) will become exponentially close to zero. Thereby, it suppresses all the other effects from the iterations which use other weights. Equation (V.17) shows that the $α_i$ parameter, which corresponds to a larger eigenvalue, has a converging effect on all of the other processes that correspond to the smaller eigenvalues but may have a diverging effect on the processes that correspond to larger eigenvalues. Therefore, I recommend that the computation process start using the $α_i$ that belongs to the highest eigenvalue for 1 or 2 cycles, followed by using the other $α_i$ weights for 1 or 2 (or more) cycles and that the algorithm is then terminated using the $α_i$ that corresponds to the highest eigenvalue again, until the process converges.

Two equivalent methods are developed to approximate the junction temperature in Section V.2, where the temperature method was related to the $BC$ matrix and the flux method was related to the inverse of the $BC$ matrix. However, the eigenvalues of the $BC$ matrix are only approximate. The $ω_{i,i}$ terms are given by (V.25), where the error between the real ($μ^*$) and estimated ($μ$) eigenvalues are expressed multiplicatively ($μ=(1+β) μ^*$),

$$ω_{i,i} = 1 - \frac{1 + μ_i}{1 + μ} = -β \frac{μ^*}{1 + μ_i}.$$  

Eq. (V.25) shows that at the same relative error, the absolute value of $ω_{i,i}$ increases for larger eigenvalues. Since the $ω_{i,i}$ parameters directly determine the convergence speed and the suppression/divergence of the residual terms in (V.23); the $ω_{i,i}$ must be kept as small as possible. Therefore, if the reciprocal of the smallest eigenvalue in the $BC$ matrix is larger than its largest eigenvalue; the first method i.e., the temperature method, is preferred. Otherwise, the second method i.e., the flux method, is preferred because the $μ$ eigenvalues generally decrease, and smaller $ω_{i,i}$s are obtained at the same estimation error. This result is obtained because the eigenvalues in the flux method are the reciprocals of the eigenvalues of the matrix $BC$.

**V.5. Numerical Example**

The method developed in Section V.2 is demonstrated on a four-layer structure that is similar to Fig. V.1 and illustrated in Fig. V.6. This is not a real physical problem, but a dimensionless one that represents a worst case scenario, i.e., the layers greatly differ in both
transverse extension and thermal conductivity. The steady-state temperature field in each of the four layers is sought by separation of variables as in (V.3).

First, the multi-layered medium is decomposed into the hierarchy of two-layered medium problems. According to subsection V.3.1, the problem is split into three separate two-layer problems that are: between [#1;#2] and [#3;#4], between [#1] and [#2], between [#3] and [#4], respectively. The process is the same that is depicted in Fig. V.4, except that there are only two levels that include the two-layer problems A, B1 and B2 because the number of the layers is four.

Next, the proper iterative method (temperature/flux in Section V.2) and the corresponding $\alpha$ weights have to be selected for each of the two-layer problems; starting from the topmost problem A and proceeding to the subordinate ones B1 and B2. Using the initial guess of temperature method for A, the corresponding $B$, $C$ and $BC$ matrices for the heterogeneous layers (blocks) [#1;#2] and [#3;#4] are established on the basis of a 8 by 8 lumped representation of the layers, see Fig. V.5 and subsection V.4.1. Since the reciprocal of the smallest eigenvalue in this $BC$ matrix is larger then its largest eigenvalue, the temperature method is preferred for the problem A (subsection V.4.4); and the corresponding weights $\alpha_A$ are computed from the eigenvalues (V.24) as in Table V.1. Considering B1 and B2, they are independent of each other thus the appropriate iterative methods can be chosen separately; however, the boundary conditions are inherited from the parent problem A which uses the temperature method. Therefore, the boundary condition on the right-hand side of #2 changes to the second kind and it changes to the first kind on the left-hand side of #3. The corresponding $B$, $C$ and $BC$ matrices are also derived for B1 and B2 similarly as for A; but the flux method is preferred for both of them. The weights $\alpha_{B1}$ and $\alpha_{B2}$ to be used in these iterative processes are listed in Table V.1.

The computational process is the following. Having an initial guess, e.g. zero, for the junction temperature in A, the analytical thermal field is first sought in the block [#3;#4] that is subject to the “guessed” temperature input (b.c. of the 1st kind) at its left-hand side surface. The temperature field in [#3;#4] is solved in B2 that applies the flux method that uses the $\alpha_{B2}$ weights to obtain a converged analytical result in the homogeneous layers #3 and #4. The flux at the left-hand side surface is now known for [#3;#4], and the iteration in problem A continues to the next step, which is to apply the opposite of this surface flux plus $Q_2(y)$ to the right-hand side surface of the block [#1;#2], and search for its thermal field analytically. Here, B1 applies and a converged analytical result is obtained in the layers #1 and #2 by using the flux method with the $\alpha_{B1}$ weights. The corresponding temperature at the right-hand side surface of layer [#1;#2] is now known, and the next guess for the junction temperature in problem A is taken as the weighted sum of the previous guess and the current solution (V.8) using the appropriate $\alpha_A$ weight. Then, a new cycle begins and the process continues until every junction temperature satisfies the convergence criteria in (V.26). Finally, the analytical temperature solution can be established by solving the
single layer problems with the converged interface conditions, or it is also possible that the solution \((a_{j,n} \text{ and } b_{j,n})\) is already in memory because it is established in every step with the updated boundary/interface conditions.

In each layer, the analytical solution by SOV considered the first 64 eigenfunctions, and the solution required a total of 51 two-layer iterations (451 ms runtime) to fully converge. The indicated runtime data are related to the iterative part of the algorithm; therefore, it did not consider the time that was spent computing the eigenvalues from the transcendental equations. The program was implemented in LabView 2009 and was run on a PC with AMD Athlon II X2 250 (2 CPU-s @ 3 GHz) with 2GB RAM and with a 32-bit Windows 7 operating system. The runtime data in the parentheses is only informative because the code could have been further optimized thus the speed of the computation could have been improved. The underlying four-layer problem was also simulated by a commercial FEM program (SolidWorks 2011). The FEM simulation used a planar 2D standard mesh with element size of 0.01 (relative unit) thus it consisted of 134871 nodes (total number of elements of 67000) and it required ~3s to compute the solution (omitting the time spent with meshing), which is significantly more than the 0.45s in the case of the proposed method. The analytical temperatures at the left-hand side, at the junctions, and at the right-hand side are compared with the FEM solution in Fig. V.7-9. A very good agreement can be found between the two solutions; except the small discrepancy between the analytical and FEM temperatures at the interface of #1-2 and #2-3 (Fig. V.7-8). Note that layer #2 has very high conductivity thus its interface temperature is almost a straight line. I conducted further analyses and found that by increasing the number of eigenfunctions in the layers (e.g. 32, 64, 96, 128) the accuracy of the analytical solution improved; altogether, Fig. V.7-8 are “misleading”. The large offset in the temperature field is not properly illustrated as the y axis starts from 0.82; thus, compared to the tiny variations in the thermal field any difference between the solutions is disproportionately magnified. Therefore, it is more reasonable to consider the full signal and the relative error between the analytical and FEM solutions; which turns out to be less than 0.09%, 0.028%, 0.017% and 0.013% if the first 32, 64, 96 and 128 eigenfunctions are used, respectively. Furthermore, the FEM solution may also have some discretization error. The errors reported above are negligible. The parameters of the first 4 eigenfunctions, expressed as in (V.3), are listed in Table V.2 (the \(d_{j,n}\) terms are unity),

\[
\frac{\int_0^H [t_{S,k+1}(y) - t_{S,k}(y)]^2 \, dy}{\int_0^H [t_{S,k}(y)]^2 \, dy} \leq 0.0001. \tag{V.26}
\]

| Table 1: Weighting constants (\(\alpha\)) used during the iterations |
|-----------------|-----------------|-----------------|-----------------|
| \(\alpha_A\)    | 4.720E-01       | 9.978E-01       | 4.720E-01       |
| \(\alpha_B1\)   | 1.538E-02       | 9.985E-01       | 1.538E-02       |
| \(\alpha_B2\)   | 8.953E-01       | 9.684E-01       | 8.953E-01       |

<p>| Table 2: Parameters of the first 4 SOV eigenfunctions of the layers (An.64) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Layer #1        | Layer #2        | Layer #2        | Layer #4        | Layer #1        | Layer #2        | Layer #2        | Layer #4        |
| (\lambda_{j,n}) | (a_{j,n})    | (\lambda_{j,n}) | (a_{j,n})    | (\lambda_{j,n}) | (a_{j,n})    | (\lambda_{j,n}) | (a_{j,n})    |
| 1.859E+00       | 1.601E-02       | 1.214E+00       | 5.914E-01       | 1.724E-01       | 4.297E-01       | -2.653E-02      | 1.224E-01     |</p>
<table>
<thead>
<tr>
<th>$b_{j,n}$</th>
<th>$c_{j,n}$</th>
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<tr>
<td>7.052E-01</td>
<td>3.919E-01</td>
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<td>9.001E-02</td>
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<tr>
<td>5.475E+00</td>
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<tr>
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<td>-7.309E-05</td>
</tr>
<tr>
<td>1.033E+01</td>
<td>-4.144E-05</td>
</tr>
</tbody>
</table>

Fig. V.7: Temperature at the left-hand side of layer #1 (left) and at the interface of layers #1-#2 (right).

Fig. V.8: Temperature at the interface of layers #2-#3 (left) and at the interface of layers #3-#4 (right).

Fig. V.9: Temperature at the right-hand side of layer #4.
V.6. Conclusion

I developed a method to resolve the computational difficulties typically encountered when computing the analytical solution of the steady-state temperature distribution in multi-layered media. I derived a mesh-free iterative method that uses analytical basis function in the single layers for approximating the temperatures and fluxes at the junctions. The accuracy of the proposed solution depends on the iterative convergence criterion and on the number of eigenfunctions used in the solution. I showed that using varying weighting coefficients during successive iterations greatly improved the speed of convergence, irrespective of the loads of the multi-layered medium. The required weighting coefficients for the analytical version were estimated from a “rough” lumped representation of the layers. The developed method was initially derived for a two-layered medium and then extended to an arbitrary number of layers using a hierarchical two-layer representation of the multi-layered medium. Thus, the computation was a multi-leveled two-layer solution process. A method for decomposing the multi-layered medium into two-layered media was also presented that greatly improved the computational efficiency even for cases with numerous layers. The solution method for the two-layered media was proven for an arbitrary structure; therefore, the developed method is also applicable to a multi-dimensional array of layers (e.g., 2D-stacked) and can be applied to model internal heat generation problems in the layers or anisotropic material properties, e.g., orthotropic properties. The developed method can also be applied to 3D conduction, and to cylindrical or spherical coordinates. A numerical example was presented consisting of four significantly different layers and the results were compared to a FEM solution. Good agreement was found between the results and the FEM solution.
VI. SUMMARY

The research presented here has contributions to two fields. In the first part of the dissertation, sensorless methods that target embedded applications were developed for estimating the parameters of linear electromagnetic devices. Such devices are commonly used in switching and flow controlling purposes and their typical applications include hydraulic and pneumatic systems e.g. an automatic transmission unit, pneumatic brake, internal combustion engines and relays, contactors. The results have relevance to engineering practice and to every application where linear actuation is necessary. Compared to already existing solutions, the major contributions of this work are the sensorless methods which have lower model (computational) and hardware complexity, and the capability of the combined estimation of the position and external force. Thus, the cost effectiveness and reliability of the systems that utilize solenoid actuators can further improve and it is possible to save additional sensors, e.g. pressure sensors, from the systems. The change in the electrical resistance of the coil during normal operation was also considered and methods were developed for its identification, enabling to reduce error in the estimations which would be caused by the variation of the resistance with the temperature. Additionally, the estimate of the resistance can be also used to save external temperature sensors from the system because the resistance provides information about the internal thermal state of the actuator.

In the second part of the dissertation, the analytical solutions to the conduction of heat in solids (diffusion equation) were studied in multilayered media. The original motivation behind this research was to create an in depth thermal model of solenoid actuators, from which model the resistance of the coil could be determined. However, such elaborate thermal models can be also used for the design and optimization of electromagnetic actuators. Besides, the multilayered medium approach had lead to a more general problem; therefore, the corresponding results are applicable to every field where the diffusion equation and the multilayered medium modeling principle apply. Examples are the conduction of heat in solids, electrostatics and composite materials, which have high relevance to engineering practice. The major contributions of the second part of the research are the extension of analytical series solutions to the steady-state heat equation in multilayered media and the improvement of their computational effectiveness. A general method has been developed with which non-homogeneous longitudinal boundary conditions and internal energy generation can be comfortably included in the analytical solution; and an iterative method has been also developed and optimized with which the numerical difficulties at computing the analytical solution are resolved. These results of the research enable a more detailed analysis of multilayered media and extend the applicability of analytical solutions to the diffusion equation in multilayered media to a wider field of application.

Considering further research and improvements of the presented results, a few possibilities are listed below:

- Sensitivity analysis of the sensorless methods in Section II-III, i.e., how the accuracy and variance of the estimations vary with noise disturbances and variations in the parameters of the model,
- The analysis of the sensorless methods on more and on different types of linear electromagnetic actuators,
- Interpolation in the estimation of the external load in Section II,
- Research of alternate sensorless methods for linear electromagnetic devices,
- The estimation of the resistance (Section III.) should consider a more realistic inductance function, i.e., that it depends on the current,
- In case of numerous layers, the computational effectiveness of the method in Section IV.2, which considers non-homogeneous longitudinal boundary conditions, becomes very low (many part solutions),
- An alternate way to estimate the optimal weighting coefficients in Section V.4.3, as it currently uses a network of lumped resistances,
- An extension of the methods in Sections IV and V to unsteady heat conduction and to multilayered media that are layered along more directions,
- Application and evaluation of the methods in Sections IV and V in a real embedded platform.

VI.1. New Scientific Results

I. I have developed methods, which are applicable under PWM drive conditions and target embedded applications, for estimating the parameters of linear electromagnetic actuators.

I.1 I have developed a method to simultaneously estimate the position of the moving part and the external load in linear electromagnetic actuators. The estimation is based on an empirically acquired set of position to inductance and position to average current curves, which curves are recorded at different external loads. The inductance is measured by dedicated scan signals (chopping of the PWM and sinusoidal duty ratio component), which are generated with the simplest PWM drive hardware. [di4-5]

I.2 I have developed a set of methods, which have low computational complexity, to estimate the electrical resistance of the coil of linear electromagnetic devices. The difference between the overall resistances during the on and off periods of the PWM cycles is also considered thus the error (bias) at the estimation is reduced. [di1]

I.3 I have proven that under PWM drive conditions with constant supply voltages and with a constant duty ratio, the time evolution of the average of the current, which is computed for the PWM cycle, can be expressed by the general exponential function for a linear L/R circuit, even if the system changes its electrical time constant during the on and off periods of the PWM cycles. [di1]

II. I have shown that the non-homogeneous longitudinal boundary condition can be included in the solution of the steady-state diffusion equation in multilayered media by solving a single layer problem in the corresponding layer and by solving a set of multilayered medium problems which have homogeneous longitudinal boundary conditions. [di3]

II.1 The eigenvalue problem that is associated with the analytical solution of steady diffusion can be simplified in multilayered media having non-homogeneous longitudinal boundary conditions by reducing the transcendental equation. Thus, the computational resources which are necessary for computing the eigenvalues decrease. [di3]

II.2 I have shown that the internal energy generation, which depends only on the transverse coordinate, can be represented as a one dimensional diffusion problem and as
non-homogeneous longitudinal boundary conditions if considering the solution to the steady-state diffusion in multilayered media. [di3]

III. I have developed a mesh-free iterative algorithm to resolve the numerical difficulties typically encountered when computing the analytical solution of the steady diffusion equation in multilayered media. [di2]

III.1 The convergence of the iterative solution improves if varying weighting coefficients are used during the iterations. [di2]

III.2 I have provided an estimate of the optimal weighting coefficients, which are to be used during the iterations, from a lumped-impedance-network representation of the layers. [di2]

III.3 I have generalized the iterative solution to multilayered media on the basis of a hierarchical two-layered medium representation. [di2]

III.4 The number of iterations required to reach convergence decreases if the bisection method is used to create the hierarchical two-layered medium representation of the multilayered medium. [di2]
VII. REFERENCES


VII.1. Publications of the Author

VII.1.1. Peer-reviewed journal articles

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Resistance Estimation in Solenoid Actuators by Considering Different Resistances in the PWM Paths.
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DOI: 10.1115/1.4027838

[di3] Ivor Dülk, Tamás Kovácsházy
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DOI: 10.1016/j.ijheatmasstransfer.2013.08.070

[di4] I. Dülk, T. Kovácsházy
*CARPATHIAN JOURNAL OF ELECTRONIC AND COMPUTER ENGINEERING* 6:(1) pp. 36-43. (2013)

VII.1.2. Conference proceedings

[di5] Dülk Ivor, Kovácsházy Tamás
Sensorless position estimation in solenoid actuators with load compensation.

[di6] I. Dülk, T. Kovácsházy
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In: Petras I, Podlubny I, Kacur J, Nawrocka A, Sapinski B

Dülk Ivor, Kovácsházy Tamás
Modelling of a linear proportional electromagnetic actuator and possibilities of sensorless plunger position estimation.

Dülk Ivor, Kovácsházy Tamás
A Computationally Effective Method for Calculating the Exponential Fit
DOI: 10.1109/CarpathianCC.2014.6843577
APPENDIX 1: PROOF OF THE TIME EVOLUTION OF THE AVERAGE CURRENT

I prove the principle in section III.3: if the inductance is constant, then the step response of the average current on a PWM cycle can be expressed as (III.7) if the time constants in the PWM “on” and “off” periods are different. During a PWM “on” period (switch is turned on) and during a PWM “off” period (switch is turned off) the coil current can be expressed as (III.a.1)-(III.a.2) respectively according to Fig. III.2. The terms $i_{A,0}$ and $i_{B,0}$ in (III.a.1)-(III.a.2) refer to the initial conditions of the coil current. The following notations (III.9)-(III.11) are also made use of,

$$i_{ON}(t) = (i_{A,0} - i_H) e^{at} + i_H,$$  \hspace{1cm} (III.a.1)

$$i_{OFF}(t) = (i_{B,0} - i_L) e^{bt} + i_L.$$  \hspace{1cm} (III.a.2)

For a particular $k^{th}$ PWM cycle the following notations (III.a.3)-(III.a.5) are introduced for the current signal. At the beginning of the PWM cycle (“on” period) the initial or start current is denoted as $i_{A,k}$. Next, the symbol $i_{B,k}$ refers to the current that is reached by the end of the $k^{th}$ PWM “on” period. The coil current at the end of the $k^{th}$ PWM cycle, which is the start current of the next $k+1^{th}$ PWM cycle, is denoted by $i_{A,k+1}$. Using (III.a.1)-(III.a.2) the specific current values in an arbitrary PWM cycle can be expressed as (III.a.3)-(III.a.5),

$$i_{B,k} = (i_{A,k} - i_H) e^{at} + i_H,$$  \hspace{1cm} (III.a.3)

$$i_{A,k+1} = (i_{B,k} - i_L) e^{bt} + i_L,$$  \hspace{1cm} (III.a.4)

$$i_{A,k+1} = i_{A,k} e^{a+b} - i_H e^{a+b} + (i_H - i_L) e^b + i_L,$$  \hspace{1cm} (III.a.5)

$$\alpha = -i_H e^{a+b} + (i_H - i_L) e^b + i_L.$$  \hspace{1cm} (III.a.6)

Using the formulas (III.a.5)-(III.a.6) successively, the initial current of the PWM cycles can be expressed as (III.a.7) from an arbitrary start current $i_{A,k}$,

$$i_{A,k+1} = i_{A,k} e^{a+b} + \alpha,$$

$$i_{A,k+2} = i_{A,k} e^{2(a+b)} + e^{a+b} \alpha + \alpha,$$

$$\ldots$$

$$i_{A,k+n} = i_{A,k} e^{n(a+b)} + \alpha \sum_{j=0}^{n-1} e^{j(a+b)}.$$  \hspace{1cm} (III.a.7)
According to (III.a.7), the evolution of the current, which is taken at the start of the PWM cycles, follows a geometric series. Using the formulae for the sum of geometric series, the summation can be transformed to (III.a.8),

\[ i_{A,k+n} = i_{A,k}e^{n(a+b)} + \alpha \frac{1-e^{n(a+b)}}{1-e^{a+b}}. \]  

(III.a.8)

Through the substitution and rearrangement of the coefficients, (III.a.8) can be brought to (III.a.9) which possesses the form of (III.7); that is, the initial current of the PWM cycles can be expressed by the exponential relationship irrespective of the different time constants. Though the initial current is sampled by the PWM switching frequency, it still fits an otherwise continuous exponential signal. The overall time constant is a weighted sum of the time constants,

\[ i_{A,k+n} = \left( i_{A,k} - \frac{\alpha}{1-e^{a+b}} \right)e^{n(a+b)} + \frac{\alpha}{1-e^{a+b}}. \]  

(III.a.9)

In Section III.3, the resistances are estimated from the average current in a PWM cycle. In terms of \( i_{A,k}, i_{B,k} \) and \( i_{A,k+1} \) the average current in the \( k^{th} \) PWM cycle can be written as (III.a.10)-(III.a.11),

\[ i_{AVG,k} = \frac{\int_0^T (i_{A,k} - i_H) e^{aT} + i_H dt + \int_0^T (i_{B,k} - i_L) e^{bT} + i_L dt}{T}, \]  

(III.a.10)

\[ i_{AVG,k} = \frac{i_{A,k} - i_H}{AT} (e^a - 1) + i_H d + \frac{i_{B,k} - i_L}{BT} (e^b - 1) + i_L (1-d). \]  

(III.a.11)

By substituting (III.a.2) to (III.a.11), the average current can be rearranged as in (III.a.12),

\[ i_{AVG,k} = \frac{i_{A,k} - i_H}{AT} (e^a - 1) + \frac{(i_{A,k} - i_H)}{BT} e^a + i_H (e^b - 1) + i_L (1-d) = \]

\[ = i_H \left[ d + \left(1-e^a \left( \frac{1}{AT} - \frac{1-e^b}{BT} \right) \right) + i_L \left[ 1-d + \frac{1-e^b}{BT} \right] + i_{A,k} \left( \frac{e^a-1}{AT} + \frac{e^b-1}{BT} \cdot e^a \right) \]. \]  

(III.a.12)

Introducing the following notations (III.a.13-14), (III.a.12) can be rewritten as (III.a.15),

\[ \beta = i_H \left[ d + \left(1-e^a \left( \frac{1}{AT} - \frac{1-e^b}{BT} \right) \right) + i_L \left[ 1-d + \frac{1-e^b}{BT} \right] \], \]  

(III.a.13)

\[ \chi = \frac{e^a-1}{AT} + \frac{e^b-1}{BT} \cdot e^a, \]  

(III.a.14)
According to (III.a.15), the average current in the \( k^{th} \) PWM cycle is a linear combination of its corresponding start current. Using the previously established exponential expression (III.a.9) for the evolution of the start current values, the average current can be expressed as (III.a.16),

\[
i_{AVG,k} = \chi i_{s,k} + \beta. \tag{III.a.15}
\]

\[
i_{AVG,k+n} = \chi \left( i_{s,k} - \frac{\alpha}{1-e^{a+b}} e^{n(a+b)} \right) + \frac{\chi \alpha}{1-e^{a+b}} + \beta. \tag{III.a.16}
\]

From (III.a.16), it can be seen that the evolution of the average coil current, similarly to the start current in (III.a.9), can be also represented as the samples of a continuous exponential function. Substituting (III.a.6), (III.a.13) and (III.a.14); the steady state of the average current can be expressed as (III.8) with \( n \) set to infinity, provided that \( a \) and \( b \) are negative.
APPENDIX 2: PROOF OF THE IMPROVED CONVERGENCE

A proof is given for the iterative process in (V.23) yielding the solution in (V.21) in finite steps, if the varying weighting constants \( \alpha_i \)s in (V.24) are used for \( k_i \) periods. For the ease of mathematical treatment, I restrict the calculations only to the diagonal \( \Omega \) matrices as all the other terms, e.g., the matrix \( V \), can be factored out. By reversing the sequence of the \( \alpha_i \)s to \( \alpha_N \), \( \alpha_{N-1} \) etc. to \( \alpha_1 \), the expansion of (V.23) leads to (V.27),

\[
t_s = \alpha_1 \sum_{j=0}^{k_i-1} \Omega_1^j + \frac{\alpha_2 \Omega_1^{k_i}}{1 - \omega_{1,1}} \sum_{j=0}^{k_i-1} \Omega_2^j + \alpha_3 \Omega_1^{k_i} \Omega_2 \sum_{j=0}^{k_i-1} \Omega_3^j + \ldots + \alpha_n \left( \prod_{j=1}^{n-1} \Omega_j^j \right) \sum_{j=0}^{k_i-1} \Omega_n^j + \prod_{j=1}^{n} \Omega_j^j t_{s,0}. \tag{V.27}
\]

Since \( \alpha_1 \) is chosen accordingly to (V.24), the first diagonal element of \( \Omega_1 \) is 0 (\( \omega_{1,1}=0 \)). Therefore, the remaining terms, in which \( \Omega_1 \) stands as a multiplying “coefficient” raised at least to the first power, have no contribution to the first element of the \( t_s \) vector. Thus, the first element of \( t_s \) \((t_{s,1})\) can be easily calculated according to (V.28) and requires only one iteration \((k_i=1)\) since \( \omega_{1,1} \) is zero,

\[
t_{s,1} = \alpha_1 \frac{1 - \omega_{1,1}^{k_i}}{1 - \omega_{1,1}} = \frac{1}{1 + \mu_1}. \tag{V.28}
\]

For the second element of the \( t_s \) vector \((t_{s,2})\), only the first two terms in (V.27) have to be considered, because the second element in the main diagonal of \( \Omega_2 \) is 0 (\( \omega_{2,2}=0 \)). Thus, \( t_{s,2} \) can be written as (V.29),

\[
t_{s,2} = \alpha_{1,2} \sum_{j=0}^{k_i-1} \omega_{1,2}^j + \alpha_2 \omega_{1,2} \frac{1 - \omega_{2,1}^{k_i}}{1 - \omega_{2,1}} = \frac{1}{1 + \mu_1} \sum_{j=0}^{k_i-1} \left( 1 - \frac{1 + \mu_2}{1 + \mu_1} \right)^j + \left( 1 - \frac{1 + \mu_2}{1 + \mu_1} \right)^{k_i} \frac{1}{1 + \mu_2}. \tag{V.29}
\]

The powered terms can be expanded to (V.30)-(V.31) using the binomial identities,

\[
\left( 1 - \frac{1 + \mu_2}{1 + \mu_1} \right)^{k_i} = \sum_{j=0}^{k_i} \binom{k_i}{j} \left( \frac{1 + \mu_2}{1 + \mu_1} \right)^j, \tag{V.30}
\]

\[
\sum_{j=0}^{k_i} \left( 1 - \frac{1 + \mu_2}{1 + \mu_1} \right)^j = \sum_{j=0}^{k_i} \binom{k_i}{j+1} \left( \frac{1 + \mu_2}{1 + \mu_1} \right)^j. \tag{V.31}
\]

Equation (V.30) can be obtained as the following. Due to the summation of the power series, the coefficient \( p_m \), which stands for the term which is on the \( m^{th} \) power, becomes (V.32),

\[
p_m = \sum_{j=m}^{k_i-1} \binom{j}{m}. \tag{V.32}
\]
Using the binomial identity in the left-hand side of (V.33), it can be successively factored out as in the right hand side of (V.33) that equals (V.32),

\[
\binom{k_i}{m+1} = \binom{k_i-1}{m} + \binom{k_i-1}{m+1} = \binom{k_i-2}{m} + \binom{k_i-3}{m} + \ldots + \binom{m-1}{m} = \sum_{j=m}^{k_i-1} \binom{j}{m}.
\] (V.33)

By substituting (V.30-31) into (V.29), (V.34) is obtained. Notice that in (V.34), the terms corresponding to the indexes \(i=j=1\) factor out each other. Since the index \(i\) goes only to \(k_i-1\), (34) reduces to the element which belongs to \(j=0\),

\[
t_{S,2} = \sum_{i=0}^{k_i-1} \binom{k_i}{i+1}(-1)^i \frac{(1+\mu_2)^i}{(1+\mu_1)^{i+1}} + \sum_{j=0}^{k_i-1} \binom{k_i}{j}(-1)^i \frac{(1+\mu_2)^j}{(1+\mu_1)^{j+1}} = \frac{1}{1+\mu_2}.
\] (V.34)

Next, the third element of the \(t_S\) solution vector, that is \(t_{S,3}\), is considered. Similarly to (V.29), it can be expressed as (V.35),

\[
t_{S,3} = \alpha_1 \sum_{j=0}^{k_i-1} \omega_{13}^j + \alpha_2 \sum_{j=0}^{k_i-1} \omega_{23}^j + \omega_{33} \sum_{j=0}^{k_i-1} \alpha_3 \frac{1-\omega_{33}^j}{1-\omega_{33}} = \frac{1}{1+\mu_1} \sum_{j=0}^{k_i-1} \left(1-\frac{1+\mu_1}{1+\mu_2}\right)^j + \left(1-\frac{1+\mu_3}{1+\mu_1}\right) \left(1-\frac{1+\mu_2}{1+\mu_3}\right) \left(1-\frac{1+\mu_1}{1+\mu_2}\right)^{k_i-1}
\] (V.35)

According to (V.29)-(V.34), the term in the square brackets in (V.35) can be simplified; thus, (V.35) can be factored out to (V.36), which is similar to (V.29),

\[
t_{S,3} = \frac{1}{1+\mu_1} \sum_{j=0}^{k_i-1} \left(1-\frac{1+\mu_3}{1+\mu_1}\right)^j + \left(1-\frac{1+\mu_1}{1+\mu_3}\right) \left(1-\frac{1+\mu_3}{1+\mu_1}\right) = \frac{1}{1+\mu_3}.
\] (V.36)

For the other terms of the \(t_S\) vector, the solution becomes straightforward by using the successive factorization which was applied to (V.29) and to (V.36). Thus, if every \(k_i\) is unity, the exact solution in (V.37) is reached in \(\#n\) steps,

\[
t_{S,j} = \frac{1}{1+\mu_j}.\] (V.37)
APPENDIX 3: A NOVEL EXPERIMENTAL SETUP FOR SOLENOID ACTUATORS

A.1. Introduction

Solenoids are limited travel electromechanical converters that are most commonly used as relays, on-off contactors or for flow control purposes in hydraulic and in pneumatic systems. Their ruggedness and low prime cost make them preferred in applications where cost effectiveness, reliability and high force density are required e.g. an automatic transmission unit. However, the need for enhancing the present designs and models of the valve actuators is ever increasing to reduce system costs, to improve robustness and to enhance controllers. Therefore, a proper experimental setup is of great importance if carrying out in depth solenoid analyses such as modeling, identification and control loop verification.

The picture of a valve actuator and a better explanation on its main operating principles are given in Fig.A.1 and in section A.1.1. Regarding solenoid actuators the corresponding literature agrees that their related mechanical quantities are the position and velocity of the moving part and the force it exerts (or external load force). Velocity is relevant in on-off, e.g. intake and exhaust applications [6], [16], [26], [36-37] to control the landing speed, thus bouncing and hard hitting of the spool can be avoided. However, for flow or pressure regulating purposes [1], [3], [6], [38] position is more meaningful to consider. Depending on the field of its application the force excitation has to be accounted as well [16], as it may be present from e.g. fluid pressure. Therefore, if attempting an elaborate analysis of solenoids the experimental setup has to satisfy basically three requirements. First, the accurate and reliable measurement of spool position (or velocity) has to be ensured. Secondly, there has to be a way to provide external load to the actuator and to measure the load forces. Last, the subsidiary friction and hysteresis that are associated with the test setup have to be minimal not to bias the measurement results and the identification of the tested solenoid.

In the corresponding literature the available solenoid models and the experimental setups are vast. In Table A.1 an overview is given over the stroke length of the studied solenoid actuators in the corresponding articles. It can be seen that the stroke length may vary in a wide range, but in general it is in the range of one centimeter except for extreme cases where it can be even less than one millimeter. If the detectable position range is very small then it becomes challenging to ensure the reliability of the measurements and to apply the external load, because special effects have to be considered that would not be relevant for longer strokes e.g. deformation. For a simple example even a 80 micrometer difference which might be caused by deformation would result in an unacceptable 10% error.

<table>
<thead>
<tr>
<th>Reference</th>
<th>Stroke Length [mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Studied valve</td>
<td>0.8</td>
</tr>
<tr>
<td></td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>16</td>
</tr>
<tr>
<td>[23],[37]</td>
<td>[2]</td>
</tr>
<tr>
<td>[39]</td>
<td>[6],[16]</td>
</tr>
<tr>
<td>[26],[36]</td>
<td>[1],[4]</td>
</tr>
<tr>
<td>[5]</td>
<td>[3]</td>
</tr>
</tbody>
</table>

Table A.1: Stroke Length of Valve Actuators in References
The experimental setups found in the technical literature implement well known “industrial” position transducers to measure the position of the spool. In [1], [3], [36] LVDTs (linear variable differential transformer) are used for sensing purposes by attaching one end of the LVDT to the spool. In other studies the concepts of using laser transducers [6], [16], [39], eddy current sensors [26] and inductive transducers [2] are presented. Common in all, such sensing devices are fairly expensive and usually require a dedicated signal conditioning and driving circuitry (LVDT), which further increases the cost of the test setup. Scalability in case if a different solenoid actuator is to be studied is also questionable. For inductive transducers, bandwidth issues have to be considered as well.

The other key feature in the solenoid testing environments is the way to provide and measure the external load forces. In many test setups the option of applying and measuring force excitations is not realized [5], [26], [36], [38], [39], or the solenoid is clamped by a linear stage and its characteristics are captured at fixed spool positions [1], [2], [3], [5], [6], [23], [37]. This approach is adequate for solenoid modeling and identification, but the possibility to apply different, prescribed external forces is lost. Also, the experimental setups may not be suitable for control loop verification and for dynamic analyses with varying external forces. In corresponding experimental setups solenoid forces are usually measured by load cells [6], [23], [37]. In [4] a double coil actuator is studied and the necessary force is produced by actuating the release coil, a model of which had been previously defined. However, this method greatly relies on the underlying model and requires a complex control apparatus. By building a complete test-bench engine emulator incorporating a pressure transducer [16] studies the solenoid valve under real pneumatic working conditions. Load cells and pressure transducers are quite expensive and require an additional hardware overhead. Nonetheless, the proper design of force transfer elements and the guiding of the moving parts are of paramount importance, because friction and hysteresis might be induced associated with the load forces and guiding components. Especially at very small spool strokes this friction deteriorates the reliability of the measurements, thus solenoid modeling, therefore it has to be minimized.

In this study, I propose a prototype of a novel experimental setup for studying different types of linear electromagnetic actuators. The setup is applicable for a wide range of spool strokes but designed with particular respect to solenoids which have a very small spool stroke (less than one millimeter). The setup is intended for modeling and identification purposes; therefore, cost effectiveness and flexibility are major design principles. Existing solutions use expensive position (LVDT, laser, eddy current) and force transducers that often require dedicated signal conditioning and driving hardware. Conversely, I use inexpensive reflective optical sensors. With these sensors and with the novel mechanical design of the presented layout the possible measurement range of spool position becomes highly flexible. Force excitations are also applicable to the solenoid. I exploit the gravitational force of reference masses, thus force transducers become unnecessary in steady state. Possible sources that would reduce the reliability of the measurements, especially in the case of very small spool strokes, are also highlighted and reviewed. The solenoid studied in this study has a spool stroke of 0.8 mm and a load of 10 N.

A.1.1. Operation of a solenoid actuator

The literature dealing with the modeling of solenoid actuators is vast [6], [23], [37-38]. A picture of a solenoid, used for flow control purposes, is given in Fig. A.1. From an operational viewpoint current builds up in the winding due to the terminal voltage (usually from PWM). Through the air gap a magnetic force, which is related to the coil current, is exerted. This
magnetic force and external forces (e.g. fluid pressure) immerse the spool into the housing, thus, the outflow orifice can be altered. These forces are counteracted by the valve return spring and cancel each other in steady state.

![Diagram of a solenoid actuator](image)

**Fig. A.1: Picture of a solenoid actuator (flow control): 1-orifice, 2-spool, 3-coil, 4-return spring.**

A solenoid actuator’s major input parameters are the voltage, external load and temperature. Strictly speaking its output quantity is hydraulic resistance (outflow orifice); however, in case of relays or force actuators (free spool, no orifice) the hydraulic side does not exist. Besides, it is the position of the spool which determines the intake and outlet areas through only geometric constraints. Therefore; the corresponding literature agrees on spool position being the characteristic output quantity of a solenoid actuator.

### A.2. The Proposed Experimental Setup

The complete experimental setup should basically consist of three main parts, which are illustrated in Fig. A.2. The first one is a mechanical test rig that is “built around” the actuator and realizes the proper clamping of the tested device, the mounting of the position and force sensors and also provides a way to apply the force excitations to the actuator. The second part is a signal conditioning, drive and data acquisition hardware which generates the necessary PWM signal to drive the valve actuator, scales the output signals of the sensors (position, force, coil current, supply voltage), samples the necessary waveforms and transmits the recorded data for further post processing. The third part is a “top level” PC interface onto which the sampled data is transmitted and evaluated. Through this interface the measurement parameters such as PWM frequency, duty ratio etc. are set, thus the full test is bench controlled by the PC. As the measurement of electrical signals (current, voltage) is far less difficult than the measurement of mechanical signals the major focus is on the proper design of the mechanical rig, further on referred as the experimental setup. Accordingly to section A.1, the main requirements are the following:

- Sensing of spool position considering very small spool displacements. The solenoid to be studied in this study has a stroke of 0.8 mm,
- The possibility to apply external force excitation to the solenoid valve (here the maximum is 10 N),
- Sensing of load excitation and proper guiding. Friction associated with the setup must be minimal as it would influence the modeling of the solenoid,
- Flexibility of the experimental setup (measurement), so a wide range of possible stroke lengths and different solenoid types can be studied,
- Simple design and cost effectiveness, because the setup is intended for identification and “proof of concept” purposes not for large scale production.
In the following chapters I will discuss the main features of the proposed experimental setup.

![Main blocks of the experimental setup.](image)

**Fig. A.2: Main blocks of the experimental setup.**

### A.2.1. The concept of position measurement

The position of the spool is measured by a reflective optical sensor (CNY 70). This device consists of an infrared photo diode (light source) and a photo transistor parallel to each other. The sensor is also suitable to detect very small displacements (1mm). If a reflective medium is placed in front of the device some light is reflected into the transistor, thus current flows in its emitter. This current depends on the distance, reflectance of the medium and on the light emitted from the source. In the final layout the photodiode is fed by a constant current source and the position signal is associated with the voltage drop of the emitter current on a shunt. The shunt resistor is placed between the emitter and ground.

![The concept of position measurement: 1-CNY70 sensor, 2-reflective disc, 3-“lower” shaft, 4-slit.](image)

**Fig. A.3: The concept of position measurement: 1-CNY70 sensor, 2-reflective disc, 3-“lower” shaft, 4-slit.**

The mechanical design of the position sensing module is illustrated in Fig. A.3. The spool is contacted by a so called lower shaft (refer to section A.2.2), therefore this shaft and the reflective circular disc, which is attached to the shaft, will move on the same trajectory as the spool does if the contact is always ensured. The CNY 70 position sensor is fixed onto a main board by a figure “L” element and faces the reflective disc in the perpendicular direction. The initial position (position offset) can be set by locating the reflective optical sensor. According to Fig. A.3 the reflective disc has a narrow slit on its opposite side. This is necessary to prevent any rotation of the reflective disc as it may cause significant measurement uncertainty if the total spool displacement is small, refer to section A.3.4. The proposed position sensing layout is applicable for solenoids for which the spool is accessible from the outside, as depicted in Fig.
A.3. It is also to be mentioned that the size and weight of the components (shaft, disc) has to be minimal not to alter the mechanical behavior of the valve.

The CNY 70 reflective optical sensor, being a semiconductor based device, is far less inexpensive compared to inductive, eddy current or laser sensors. Furthermore, the position sensing module does not require special driving conditions such as AC drive (LVDT), bipolar supply or high supply voltages (12 V) but runs from a single DC supply of e.g. 3.3 V or 5 V because of the photodiode-photo transistor operating principle. Since the output signal is the voltage drop of the emitter current on a shunt resistor the position signal can be directly connected to the sampling A/D module and easily scaled to any level. There is no need for any dedicated signal conditioning hardware, thus, the hardware requirements from the point of position measurement are kept very simple. Bandwidth issues, which may concern inductive transducers, are not relevant as the speed of measurement depends on the speed of the optical semiconductor devices, which have a few orders larger bandwidth compared to mechanical systems in general. By setting the drive current of the photodiode and the value of the shunt (gain) the measurement range can be extended without any difficulty.

The position sensing system was calibrated by a micrometer, i.e., the spool was continuously displaced by the micrometer, with which the position was also measured, while the output voltage of the reflective optical transducer was recorded. Then, a polynomial function was fitted to the position-output voltage data and used as the transfer function of the position measuring system in the later analyses. According to the measurement data, the transfer function of the position measuring system could be well approximated by a straight line if the total displacement was only a few millimeters. The measuring setup was then also set to an initial configuration which had the “best” properties for measuring the position.

A.2.2. The concept of applying and measuring external forces

In certain applications, e.g. flow control, the solenoid actuator is subject to varying force excitations from fluid pressure. These effects have to be considered if an in depth solenoid model is to be established; therefore, the measurement and application of the load forces and the proper guiding of the moving parts are key requirements. The latter has a paramount importance in order to reduce friction and hysteresis in solenoid analyses as these would influence the reliability of the measurements in an intolerable way.

In the experimental setup I propose that the solenoid actuator and the experimental setup be arranged in the vertical direction. The load excitations for the solenoid are realized by exploiting the gravitational force of reference masses, the concept is illustrated in Fig. A.4. Therefore, the load force is constant regardless of the position of the spool in steady state and its magnitude is exactly known, the mass multiplied by the gravitational constant. Thus, any load cell or pressure transducer and the associated hardware are not necessary, greatly reducing the cost of the experimental setup and simplifying the layout. The gravitational force of the masses is transferred to the spool by the upper shaft, damping spring and the lower shaft. Due to the vertical configuration some residual force due to gravity is continuously exerted onto the spool, thus the contact between the spool and lower shaft is ensured. According to section A.2.1 this residual force is necessary for position measurement. The vertical arrangement is also a proper choice to reduce possible friction and hysteresis, which are associated with the normal (pressing) forces on the guiding elements. The guiding elements have a key role from this point as well; therefore, two linear ball bearings are used (donated by NBG Masters ltd.), one for each shaft. According to Fig. A.4 the two shafts are separated by a damping spring. This spring is intent to enhance to
dynamics of the test rig and to compensate any misalignment or deviation from the desired collinearity of the two shafts which would otherwise induce additional pressure onto the bearings, resulting in friction. I would like to mention that the size and mass of the moving elements (shafts) should be minimal because it reduces the measurement bandwidth.

![Fig. A.4: The concept of applying external forces. The layout is in a vertical configuration: 1-lower shaft, 2-linear bearings, 3-spring, 4-upper shaft, 5-mass holding plate, 6-mass, 7-solenoid valve.](image)

### A.2.3. The assembled experimental setup

The sketch of the assembled experimental setup is illustrated in Fig. A.5. The studied solenoid, shafts and linear bearings are oriented in the vertical direction, thus friction is minimized. The “L” shaped mount of the optical sensor and the spacers of the bearings along with the other accessorial elements are mounted on a strong base plate and are fixed by screws and bolts (omitted from the picture). The reflective disc and the shafts are deprived of their rotational freedom by a “stabilizer” rod which passes through the slit of the reflective disc. The valve is clamped onto the base plate by two clamps at its “neck”. The position of the spacers (upper and bottom bearing) and of the optical sensor can be adjusted through slots that are cut into the main board; thus, the allowable stroke of the shafts and the measurement range can be adjusted accordingly to the tested solenoid.

![Fig. A.5: The mechanical test rig: 1-main board, 2-linear bearings, 3-CNY70 mount, 4-upper valve mount, 5-solenoid valve, 6-board foot, 7-lower valve mount, 8-spacers, 9-disc stabilizer.](image)
In section A.2.2 the importance of reducing friction has been already highlighted from the viewpoint of solenoid identification and modeling. In fact, friction depends on the magnitude of applied external forces, because if the load is higher the bearings experience a higher pressure. In the final design (Fig. A.5 friction is overcome by the use of dedicated linear bearings and by introducing an intermediate damping spring and by the vertical arrangement of the test setup.

Experimental analyses have been carried out in the final test bench concerning subsidiary friction and hysteresis it might induce. In Fig. A.6 the position of the moving part (full cycle there and back) of a solenoid is plotted as a function of average coil current and external forces. Position is taken such that it increases with the spool being more immersed to the housing. Since hysteresis, which is associated with the guiding of the shafts, depends on the external force (greater normal force in the bearings) the recorded curves should increasingly widen (larger hysteresis area) with respect to the increasing external force if the induced friction is significant. However, the gaps between the corresponding forward and reverse curves and the hysteresis area in Fig. A.6 are almost the same irrespective of the external force; therefore, it can be concluded that the observed hysteresis is associated with the valve itself and not with the experimental setup.

![Fig. A.6: Results concerning friction induced by the experimental setup.](image)

According to section A.2, the complete experimental setup consists of three major parts. For the purpose of signal conditioning and solenoid driving a dedicated hardware has been built. Coil current is measured on a high side shunt by a common mode differential amplifier and the CNY 70 is fed by a current source realized by operational amplifiers. The solenoid valve is driven in a low side single switch configuration. The PWM signal (frequency, duty ratio, special waveforms) is generated by a dsPIC33FJ128GP 16-bit DSC (digital signal controller) which is accessible through dedicated interfaces from PC. Data acquisition is performed by an NI (National Instruments) USB DAQ device and is transmitted to PC where the measurement data are evaluated by LabView. LabView is also used to set the necessary measurement parameters e.g. sampling frequency and PWM properties and to control the entire test setup, including the custom HW and the NI DAQ device.

### A.3. Sources of Measurement Uncertainties

The proposed experimental setup is designed in a way that it has to be appropriate for studying solenoid actuators which have a very small spool stroke, e.g. 0.8 mm. At this displacement range reliable position measurement becomes challenging because I have to account
for special effects that are typically not so relevant if the spool stroke is large. For a simple example consider that 80 um uncertainty, which may be simply resulted from mechanical deformation due to different load forces, causes a 10% error in solenoid modeling. In this section I discuss the error sources I have encountered during the design of the test bench.

### A.3.1 Static deformation (by load forces)

According to Fig. A.5 the applied external force propagates through the upper shaft, damping spring, lower shaft and through the valve mounts. The main bolts fixing the bottom valve clamp (Fig. A.5) encounter the full load force. It is well known that mechanical components suffer from some elastic deformation, compression or deflection, which results in a “load dependent” error term in position measurement. Since I am interested only in the relative displacement of the reflective disc from the optical sensor due to deformation; the upper shaft and damping spring are not of concern. However, the lower shaft is subject to compression and the main bolts to bending. The deformation of the former can be estimated using Hooke’s law and of the latter by Castigliano’s theorem. In order to minimize error from mechanical deformation, shortening the shafts and bolts besides increasing their cross section is a reasonable design principle. However, this might be contradictory from the viewpoint of reducing component weight (increasing bandwidth) and achieving a scalable setup.

In our case the diameter of the cylindrical lower shaft is 5 mm and 2.5 mm for lengths of 40 mm and 11.5 mm. At a maximum load force of 10 N and Young modulus of 200 GPa (steel), the maximum deformation of the lower shaft is found to be 0.2 micrometer due to compression. Regarding deflection, M10 bolts are used with an effective thickness of about 7.5 mm and relevant length of 32 mm, which yields 3.5 micrometer at 10 N for one bolt. This has to be halved as two bolts are used. These errors are insignificant in our case.

### A.3.2 Dynamic deformation

Since the lower shaft contacts the reflective disc not on the entire, but on a small surface part of the reflective disc, the disc is subject to a non uniform dynamic force excitation in case of transient measurements when the spool is not stationary. This results in the bending and oscillation of the reflective disc. A similar situation is if the lower shaft stops and the kinetic energy of the reflective disc is unleashed in the form of vibration. This oscillation of the reflective disc may cause considerable error in position measurement during transient analyses. So as to reduce the amplitude of bending vibrations, the stiffness of the reflective disc has to be increased either by reducing its diameter or by increasing its thickness. However, it is contradictory to the requirements of decreasing the mass of the components and extending the measurable position range.

### A.3.3 Manufacturing inaccuracy and misalignment

Due to ever present machining and assembling imperfections, neither the collinearity nor the perfect verticality of the upper and lower shafts can be guaranteed. Since the applied load forces propagate through the entire test setup these “mismatching”s inevitably cause a higher stress in the bearings, which results in additional friction and in the possible jamming of the moving elements. Therefore, two separate shafts are used (upper and lower) with a damping
spring between them which compensates the effect of misalignment from the point of force transfer. The spring also enhances the dynamic behavior of the measurement setup.

A.3.4. Surface roughness of the reflective disc

According to section A.2.1 the measured position signal (reflected light) is a function of the reflectance of the reflective medium (circular disc). Since manufacturing imperfections are always present the surface roughness of the reflective disc is not uniform but spatially different. Since reflectance is associated with geometric and surface properties the rotation of the lower shaft and thus of the reflective disc induces measurement error, as the amount of reflected light will change despite the spool being stationary. This problem is resolved by preventing the lower shaft from rotating. According to Fig. 4-5 the reflective disc has a narrow slit and a rod which is fixed onto the main board, passes through it. This rod ensures that the disc cannot rotate and the optical sensor faces the same spot. However, lubrication is also necessary between the slit and the rod to reduce any possible friction between them.

A.3.5. Stray ambient light

Despite the CNY 70 optical sensors being operated at the far infrared region and equipped with optical filters, experiments have shown that the stray ambient light can significantly influence the measurements. For this reason the mechanical test bench is put into a dark box in order to shield it from ambient light disturbances.

A.3.6. Thermal elongation

Deformation cannot only be associated with mechanical loads but it can be associated with temperature as well, thus the thermal elongation of the shafts may significantly disturb the measurement of spool position. From this point, only the lower shaft has to be concerned because the reflective disc is attached to the lower shaft. An estimation of the thermally induced error can be given using the linear elongation relation. For steel the thermal expansion coefficient, which expresses the relative elongation for a temperature change of one degrees of Celsius, is 1.17E-5 [1/°C]. The corresponding length of the lower shaft is 51.5 mm, thus the error is 0.6 [um/°C]. This term is usually insignificant even if the ambient temperature changes a few degrees of Celsius. However, the solenoid actuator might be subject to thermal analyses therefore the setup could be put into a heat chamber (-40°C to +120 °C). In such situations the thermally induced error becomes considerable but can be considered as an offset, thus it can be simply zeroed out.

A.4. Design of the DSP Based Custom Hardware

The mechanical test bench is driven and connected to the NI DAQ device/PC interface by a dedicated hardware. Though the NI USB DAQ is capable of PWM signal generation and multi-channeled sampling, the custom gateway card also hosts a DSP (digital signal controller) with which PWM generation and sampling can be alternately realized, e.g., if a “true” embedded environment is to be simulated. However, sampling and PWM generation is preferred to be handled by the NI USB DAQ device. The main tasks of the gateway card are: producing and adjusting the drive PWM, conditioning and acquiring the necessary signals and communication
with the host PC, i.e., receiving commands and transmitting the recorded data. The gateway card can be separated to an analog front end (towards the electromagnetic actuator and sensors) and to a digital processing unit, practically a microcontroller and other accessories. Its block diagram is illustrated in Fig. A.7.

**A.4.1. Analog front end**

The block diagram of the custom HW, including the analog front end and its main roles is presented in Fig. A.2. For proper analyses, the following signals have to be measured: supply (drive) voltage, coil current, spool position (reflective optical sensor) and temperature. Considering the external load, it is known and given “manually”. A major task of the analog front end is to scale and adjust the above signals so those can be converted by an A/D converter. The supply voltages are sensed via a simple resistor divider up to 25 V. The concept for monitoring the coil current is shown in Fig. A.8 along with the switching (PWM) part. Current is measured on a high side shunt resistor by a dedicated high side current shunt monitor IC (INA139 since a significant, varying common mode voltage is present. In the final layout, the low side switch configuration is used with a dedicated FET driver (TC1413 from Microchip) gated by the microcontroller’s or NI USB DAQ’s pulse width modulated pin. The freewheeling diode is very important so the inductor’s current can commutate when the main FET turns off.

![Fig. A.7: Block (functional) diagram of the custom HW.](image)

![Fig. A.8: Schematic of driving the actuator and the current sensor, INA139 (right side).](image)
The supply network consists of four supply rails. The first one is the drive voltage of the actuator that is chopped by the PWM. The other part of the board, however, is fed from 7.5-9 VDC called “Logic”, which is stepped down to 5V and then to 3.3V but also serves as the switching voltage for the FET driver. The latter 3.3V is needed by the microcontroller. The other ICs, such as the current monitor and the operational amplifiers, operate from 5V.

A.4.2. Embedded controller and digital interfaces

The gateway incorporates a DSP microcontroller that interfaces the host PC and can perform the measurement of the necessary signals, some low level signal processing and an arbitrary PWM waveform generation. The DSP is the dsPIC33FJ128GP802 from Microchip which is a high performance, 16 bit DSC (digital signal controller) dedicated to signal processing and general microcontroller purposes. It also features an internal DSP engine, high speed 12 bit A/D module (of 470 ksp at 12 bit resolution and 1.1 Msps at 10 bit resolution) and a rich set of remappable peripherals, making it quite attractive in low cost, high performance applications. The controller’s maximum computational throughput is 40 MIPS at 80 MHz clock frequency. For more information about dsPIC-s, visit Microchip’s official website (www.microchip.com). The communication with the host PC is realized by serial RS232 (UART) and USB channels. For the former, the dsPIC’s UART1 is used with a MAX3232 line driver IC allowing 1 Mbps bandwidth. On the contrary, USB interface uses the microcontroller’s UART2 peripheral and an MCP2200 USB 2.0 to UART protocol converter from Microchip supporting full speed USB operation.

A.4.3. Code structure of the embedded microcontroller

A basic block diagram of the code running in the microcontroller is shown in Fig. A.9. After performing all necessary initializations, such as interrupts, peripheral remapping, PWM, UART channels (baud rate) etc., the processor goes to an infinite loop. In case a valid command package arrives from the host PC (NI LabView interface) (UART interrupts), the microcontroller decodes the package and jumps to the task specified in it. For example it can be setting the frequency and duty ratio of the PWM output, measuring different signals etc. In case a signal acquisition is initiated, the given sampled frequencies, the enabled channels (current, voltage etc.) and record lengths are also described in the command package. After collecting the necessary data it is sent back to the PC terminal for further processing. Code was written in “C” using MPLABX IDE from Microchip.

![Fig. A.9: Simplified block diagram of the main routine running in the DSP.](attachment:image-url)
A.5. Conclusion

In this study, I have presented a novel experimental setup for studying solenoid actuators. Since the setup was intended for modeling and identification purposes, cost effectiveness was a key requirement. Usability considering a wide range of spool strokes, with respect to spool displacements that could be as small as 0.8 mm, was also a main design principle in order to study different solenoids. The measurement of the position of the spool was realized by semiconductor based reflective optical sensors; therefore, the sensor and hardware costs associated with position measurement became marginal. With the experimental setup load excitation could be also applied to the studied solenoid actuator. For this purpose the entire setup was arranged vertically and the gravitational force of masses was used for excitation. Thus, load forces were exactly known and were constant irrespective of the spool position in steady state; therefore, load cells and their hardware overhead could be saved. With the vertical arrangement and with the novel mechanical design the friction associated with the experimental setup (guiding) was minimized, which would have otherwise deteriorated the identification of the studied solenoids. Since the experimental setup had to be applicable for solenoids which had a very small spool stroke many special effects, which would not have been significant in case of “long” strokes, had to be considered. These effects were also reviewed and resolved in the design. A picture of the prototype of the experimental setup is given in Fig. A.5.