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DYNAMICS OF DIGITALLY CONTROLLED
UNSTABLE MECHANICAL SYSTEMS

by

László E. Kollár

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1 Preliminaries and scopes

Stabilization of unstable equilibria of mechanical systems is frequently needed in engineering practice. A number of applications can be found in this field, e.g. the articulated bus powered through the rear axle, the shimming wheels of trailers and airplanes, the thrust control of aircrafts, the control of cranes during weightlifting operations, the steering problem of motorcycles or the standing and walking robots. In the digital control of such systems, time delay always appears due to the sampling of the computer. This delay is short because of the great sampling frequency of the digital processor. Nowadays, it is in the range of milliseconds, but its effect on the stability of the system cannot be neglected. If the delay exceeds a certain critical value then the desired equilibrium of the controlled system becomes unstable.

A family of continuous delayed systems can be described by retarded differential equations. Systems considering also sampling delay can be obtained by discretizing these equations. The theory of retarded differential equations and discrete systems were required to develop fast in the past decades due to the widespread usage of computers but many questions are still to be answered in this field.

A typical basic example of controlling unstable equilibria by control force is the balancing. The control strategies of existing balancing models need an absolute reference line which serves as vertical direction. One of the goals of the present work is to construct a balancing model where the control strategy uses relative angular velocities instead of absolute angles and angular velocities. As a result, balancing works successfully also on a surface of unknown inclination.

The theory of linear systems is well-known and many methods have been published in order to examine weakly nonlinear systems. However, many mechanical problems lead to strongly nonlinear systems due to the piecewise linear terms in their governing equations of motion. Gear pairs with backlash, impact dampers, moving parts with dry friction and adjacent structures during earthquake are modelled by systems with piecewise linear stiffness, damping or restoring force. These phenomena lead to strongly nonlinear governing equations, so analytical solution techniques which are applicable to weakly nonlinear equations are not suitable for their analysis.

Another goal of this research is to stabilize unstable equilibria of piecewise linear systems and examine nonlinear vibrations in these systems. In the present model backlash is considered in driving by teeth belt and causes periodic motions. The sampling delay in such systems leads to more complicated motions that may be even chaotic. The specialized literature suggests several methods to examine periodic solutions. Some of them are applied in the present work but the dynamics of discrete, higher dimensional, strongly nonlinear systems is very complicated. A simplified one dimensional mathematical model is constructed which has analogous mathematical properties as the examined higher dimensional model. Since chaotic motions arise in many physical phenomena, many publications have been appeared in this field. Several methods presented in these publications are applied to analyze the one dimensional map. The orbits of this map represent a motion which can occur in piecewise linear systems with sampling.

2 The applied methods

The present work is related to nonlinear vibrations and digital control. Principally theoretical methods are applied due to the objectives of the project but some results are
verified experimentally. In the present work controlled unstable mechanical systems are examined analytically but if the complexity of the model or the computation justified, numerical methods are applied and numerical results are presented. The following software packages are used: Mathematica and Maple for symbolic computations, AUTO for constructing bifurcation diagrams as well as MATLAB and LabVIEW for simulations.

This research is also related to biomechanics. A balancing model based on the structure and functioning of the human balancing organ is constructed. This model works successfully also on a surface of unknown inclination. According to the applicability in mechanical engineering or, in particular, in robotics, digital control is applied. The methodology summarized below is followed during the study of this model. The simplest possible model of balancing, the inverted pendulum, is considered and examined analytically. Then the improved model of balancing is investigated numerically and the results are compared with those obtained by the analytical study of the simple inverted pendulum.

The other main goal of this research is to examine the effect of backlash arising at the contact of machine elements and sampling delay which always appears in digital control. Backlash itself means already a strong nonlinearity in the examined system. Several methods are applied to study its effect. I worked on this project for a couple of months at the University of Bristol where professor S. John Hogan supervised my research. Conclusions on mechanical systems with backlash are given after the evaluation of several numerical methods. Systems considering backlash and sampling delay together are also examined numerically, furthermore, a simplified one-dimensional map which has analogous mathematical properties as digitally controlled systems with backlash is studied analytically.

The scientific results are published frequently and are presented in several national and international conferences. These conferences gave me the possibility for personal discussions with well-known professors and researchers of this scientific field. Regarding to control theory, professor Éva Gyurkovics (Institute of Mathematics, Budapest University of Technology and Economics) and professor Jean Levine (Ecole des Mines de Paris) had helpful advices. I had illuminating consultations with professor János Somló (Department of Manufacture Engineering, Budapest University of Technology and Economics) on the application of the harmonic balance method for piecewise linear systems. Professor Tamás Tél (Department of Theoretical Physics, Eötvös Loránd University) had valuable advices with respect to the examination of maps. My supervisor, professor Gábor Stépán, provided comprehensive guidance as well as helpful discussions and advices which meant the greatest help during my work.

3  The new scientific results

Thesis 1

The stability conditions of the upper equilibrium of the digitally controlled inverted pendulum on inclined surface, the critical sampling delay and the oscillation frequency at the loss of stability is determined.

1.1 The inverted pendulum is stabilized by control force which points in the direction of the inclination and proportional to the angle and the angular velocity of the pendulum. The sampling delay and the inclination of the slope have great influence on the stability conditions. The size of the stability domain in the plane of the control parameters
decreases with the time delay. The stability domain appears in the region of greater values of the control force as the inclination of the slope increases. This gives a limitation for successful balancing on inclined surfaces in the presence of realistic control force saturation.

1.2 There exists a critical time delay in the system of the controlled pendulum on inclined surfaces. Above this critical value the stabilization of the upper equilibrium of the pendulum is impossible. The critical time delay is determined by the length of the pendulum and the inclination of the slope

\[ \tau_{cr} = \sqrt{\frac{L(1 + 4H^2)}{6g(1 + H^2)}} \ln \left( 3 + \frac{\sqrt{5}}{2} \right), \]

where \( L \) is the length of the pendulum, \( g \) is the gravitational acceleration and \( H \) is the inclination of the slope or the tangent of the angle of the slope. Since \( \tau_{cr} (H \to \infty) = 2\tau_{cr} (H = 0) \), the critical time delay may increase twice as much as that is on horizontal surface, but this means a limited benefit only because the control force which is necessary for successful balancing increases without any limit with the inclination of the slope.

1.3 In case of balancing on inclined surface the frequency of the self-excited oscillation arising after the loss of stability can vary between zero and 25% of the sampling frequency as in the case of balancing on horizontal surface.

**Thesis 2**

An improved balancing model, the “artificial labyrinth”, is constructed according to the structure and functioning of the human balancing organ. Using this model, digital balancing is successful without knowing a reference vertical direction. The stability conditions of the upper equilibrium of this model and the critical sampling delay is determined.

2.1 The simplest solution to detect the vertical direction is the use of another pendulum attached to the top of the inverted one. The control force is proportional to the angle
and the angular velocity of the two pendula. However, this system cannot be stabilized even if the digital effects are neglected.

The new model is based on the principles of gyroscopic stabilization and the structure and functioning of the human balancing organ. A second inverted pendulum and a disc with a torsional viscosity is attached to the upper end of the pendulum. Using this model, balancing is successful even on inclined surfaces without knowing a reference vertical direction. The control force is proportional to the (relative) angular velocity of the two pendula as well as that of the large pendulum and the disc

\[ Q = P(\dot{\varphi} + \dot{\gamma}) + D(\dot{\varphi} - \dot{\beta}), \]

where \( \varphi, \gamma \) and \( \beta \) are the absolute angle of the large pendulum, small pendulum and the disc, respectively. Balancing is based on the analogy with gyroscopic stabilization. Double statically unstable equilibria can be balanced by gyroscopes or analogous forces if negative damping is also applied to the system. This explains that both parts of the control force are proportional to angular velocities and that the necessary conditions of stability are \( P > 0 \) and \( D < 0 \), where the negative \( D \) appears as negative damping. The constructed model, the “artificial labyrinth” can be seen in the figure.

2.2 The stability conditions concerning the “artificial labyrinth” determine a stability domain in the plane of the control parameters. The time delay and the inclination of the slope have great influence on the stability domain. The control force must keep the pendulum on the surface, so it should be proportional to the angle of the surface. Although, only the relative angle between the pendulum and the surface can be detected, the size of the stability domain is greater if this relative angle is used, because this way, the control force has a part proportional to the angle of the pendulum which improves stability.

2.3 There exists a critical time delay in the system of the “artificial labyrinth” used on inclined surfaces. Above this critical value the stabilization of the upper equilibrium of the two pendula is impossible. The critical time delay depends on the parameters describing the mechanical model of the labyrinth. If all the other parameters are fixed, the critical time delay is proportional to the square root of the length of the small pendulum and it has a maximum if the torsional damping is varied. At a certain value over the optimum of the torsional damping, the critical time delay decreases to 0. It means that both the small pendulum and the disc with the torsional damping is necessary for successful balancing.

Thesis 3

The stability conditions of the upper equilibrium of the controlled inverted pendulum on a cart and the critical spring stiffness of the driving belt is determined. Considering backlash at the driving, parameter domains are determined where the upper equilibrium of the pendulum and periodic oscillations around it are stable.

3.1 A model of the digitally controlled inverted pendulum on a cart is considered where the control force is provided by a motor placed on the cart. The motor drives one of the wheels of the cart through an elastic teeth belt. The stiffness of the belt has great influence on the stability conditions. There exists a critical spring stiffness of the driving belt. Below this critical value the stabilization of the upper equilibrium of the pendulum is impossible. The critical spring stiffness is a decreasing function of the length of the
pendulum, or, in other words, the shorter the pendulum is the more rigid the belt is necessary for successful balancing.

3.2 Backlash always appears at the contact of machine elements. Considering backlash in the model of the inverted pendulum on a cart the stability conditions do not change in their mathematical point but the stability domain can be divided three parts. In the first part, the upper equilibrium of the pendulum is stable. Changing the control parameters stable periodic solutions appear at homoclinic bifurcation points. Then the upper equilibrium of the pendulum or periodic motions around it are stable. Trajectories spiral to a stable fix point or a stable limit cycle depending on the initial conditions. Further change in the control parameters leads to other bifurcation points where the upper equilibrium of the pendulum loses its stability but the periodic motions are stable. If the control parameters are chosen outside all of these domains then neither the upper equilibrium of the pendulum nor the periodic oscillations are stable.

Thesis 4

The stability conditions of the upper equilibrium of the digitally controlled inverted pendulum on a cart and the critical sampling delay is determined. Backlash and sampling delay together result in stable but not periodic motions with finite amplitude.

4.1 The sampling delay has great influence on the stability conditions of the inverted pendulum on a cart driven by a DC motor. The size of the stability domain in the plane of the control parameters decreases with the sampling delay. There exists a critical sampling delay above that the stabilization of the upper equilibrium of the pendulum is impossible. This critical delay is proportional to the square root of the length of the pendulum. It is also proportional to the spring stiffness but there is an asymptote which corresponds to the result gained for the system with rigid belt.

4.2 Considering backlash at the driving and sampling delay together in the digitally controlled inverted pendulum on a cart the stability conditions do not change in their mathematical point but the stability domain can be divided two parts. In the first part, the upper equilibrium of the pendulum is stable. Changing the control parameters bifurcation points appear which form the boundary between the two subdomains of the stability domain. The upper equilibrium of the pendulum loses its stability and stable but not periodic motions with finite amplitude appear. If the control parameters are chosen outside all of these domains then neither the upper equilibrium of the pendulum nor the motions with finite amplitude are stable.

Thesis 5

A one dimensional map is considered which has analogous mathematical properties as digitally controlled piecewise linear systems. The arising motion is more complicated than quasiperiodic but it does not satisfy all the conditions of chaos. Due to its properties this motion may be called “marginally chaotic”.

5.1 The following one dimensional (half DOF) continuous model has analogous mathematical properties as the digitally controlled inverted pendulum on a cart driven by a DC motor

\[ x' (T) - a \tau x (T) = -\psi (x) \tau P x (j), \quad T \in [j, j + 1), \]
\[
\psi(x) = \begin{cases} 
0 & |x| < 1 \\
1 & |x| \geq 1 
\end{cases},
\]

where \( a \) is the mechanical parameter, \( P \) is a control parameter, while \( \tau \) and the piecewise constant function \( \psi \) considers sampling delay and backlash, respectively. The following one dimensional map is derived from this model

\[
x(j + 1) = Ax(j) - B\Phi(x(j)),
\]

\[
\Phi(x(j)) = \begin{cases} 
0 & |x| < 1 \\
x(j) & |x| \geq 1 
\end{cases},
\]

where \( A = e^{a\tau}, B = - (1 - e^{a\tau}) P/a \) and \( \Phi(x(j)) = \psi(x(j)) x(j) \). If \( \Phi \equiv x(j) \) (no backlash), then the map is stable (convergent) if and only if \( P > a \). It implies that \( A > 1 \) and \( A - B < 1 \). In what follows, this parameter domain is considered when backlash described by \( \Phi \) also appears.

5.2 The map has no stable fix point, its only unstable fix point is the zero, however, the set \( \mathcal{A} = \{ A - B, A \} \) is an attractor. Periodic solutions of the map exist if and only if \( A \) and \( B \) satisfy the following condition

\[
B = A - \frac{1}{n-k A^k},
\]

where \( k, n \in \mathbb{Z}, n \geq 2, 1 \leq k \leq n-1 \) and \( n \) is the period. It implies that periodic solutions exist only for countable number of infinitely many parameters \( A \) and \( B \). The interesting property of these periodic solutions that they are neither attractors nor repellers, their Lyapunov exponents are 0.

5.3 There are uncountable number of infinitely many parameter values where stable but not periodic solutions exist in the attractor \( \mathcal{A} \). The orbits of this map are topologically transitive but their Lyapunov exponents are 0. It means that the sensitive dependence on initial conditions does not fulfill. Orbits neither approach nor stretch each other. The motion may be called “marginally chaotic”.

4 Publications

4.1 Journal and conference papers


### 4.2 Conference and seminar lectures


