PhD Thesis Outline

Projections of self-similar sets and measures and Degrees of random and evolving Apollonian networks

Lajos Vágó

Supervisor

Prof. Károly Simon

Professor of Mathematics

Department of Stochastics,

Institute of Mathematics

MŰEGYETEM 1782

BME

2020

This Thesis covers two topics. The first one is on the geometry of random and deterministic self-similar fractals (Section 1) and it is studied in three of our papers [39, 40, 38]. Then we investigate randomly evolving complex networks in Section 2 based on our paper [25] whose construction is associated to some fractal sets.

1 Projections of self-similar sets and measures

Mandelbrot introduced Fractals to give mathematical description of some phenomena which appears in physics [26, 27], for example, he modeled turbulence using the fractal- or Mandelbrot-percolation set [27]. These inspired the development of fractals, started with a seminar paper of Hutchinson [24]. Since then the literature on fractal geometry grew enormously, the best referenced books are [15, 18, 19]. The simplest fractals are the self-similar ones (deterministic or random), we investigate them from the point of view of their projections to subspaces.

1.1 General notations, basics

An iterated function system (IFS) in $\mathbb{R}^d$ is a finite set of contractions

$$\mathcal{F} = \{\varphi_i : \mathbb{R}^d \to \mathbb{R}^d\}_{i=1}^N. \tag{1.1}$$

A classical theorem of Hutchinson [24] states that an IFS has a unique invariant set which we call *attractor*, that is:

Theorem 1.1 (Hutchinson [24]). *Let $\mathcal{F} = \{\varphi_i\}_{i=1}^N$ be an IFS in $\mathbb{R}^d$. Then there exists a unique nonempty compact set $S \subset \mathbb{R}^d$ such that*

$$S = \bigcup_{i=1}^N \varphi_i(S).$$

The points of S can be labeled with elements of the *code space* $\Sigma = \{1, \dots, N\}^{\mathbb{N}}$ in the following way. For an $\mathbf{i} = (i_1, i_2, \dots) \in \Sigma$ the limit

$$\Pi(\mathbf{i}) := \lim_{n\to\infty} \varphi_{i_1,\dots,i_n}(0) \tag{1.2}$$

exists and is in S, as the functions φ_i are contractions, where $\varphi_{i_1,\dots,i_n}$ denotes $\varphi_{i_n} \circ \cdots \circ \varphi_{i_1}$. We call Π the *natural projection*. On the other hand, every point in S can be obtained this way. Moreover, it is also easy to see that the limit would be the same if we would apply the functions to an arbitrary $\mathbf{x} \in \mathbb{R}^d$. As a consequence, attractors can be easily constructed by applying iteratively the set function $\mathfrak{S}(A) = \bigcup_{i=1}^N \varphi_i(A)$ to an arbitrary compact set A indefinitely.

We call *level-n cylinder sets* the sets $\varphi_{i_1,\dots,i_n}(S)$. In other words, a cylinder consists of points having the same prescribed initial finite word in their code.

Figure 1: The first four approximations of the Cantor set

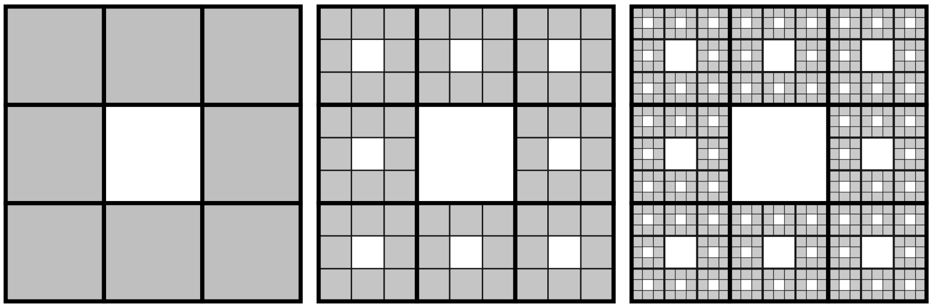

Figure 2: The first three approximations of the Sierpiński carpet

Two common examples of attractors of IFS-s are the *triadic Cantor set* and the *Sierpiński carpet*. The Cantor set (Figure 1) is a subset of the interval $[0, 1]$ defined as the attractor of the IFS

$$\left\{ \frac{x}{3}, \frac{x}{3} + \frac{2}{3} \right\}.$$

To define the Sierpiński carpet (Figure 2) let $\mathbf{t}_1, \dots, \mathbf{t}_8 \in \mathbb{R}^2$ be the 8 elements of the set $\{\{0,1,2\} \times \{0,1,2\} \setminus \{(1,1)\}\}$ in any particular order. Then the Sierpiński carpet is the attractor of the IFS

$$\mathcal{S} := \left\{ \varphi_i(x,y) := \frac{1}{3}(x,y) + \frac{1}{3}\mathbf{t}_i \right\}_{i=1}^{8}. \tag{1.3}$$

In the special case when the contractions are linear functions $\varphi_i(\mathbf{x}) = r_i O_i(\mathbf{x}) + \mathbf{t}_i$ with $0 < r_i < 1$, O_i rotations and $t_i \in \mathbb{R}$ we call the attractor S a *self-similar* set, and we refer to the constants r_i as the *contraction ratio*s and to the t_i as *translation*s. In this work in almost all of the cases we investigate self-similar sets without rotations, like the above mentioned triadic Cantor set and Sierpiński carpet. We also investigate *self-similar measures* on S defined in the following way. Given a probability vector $\mathbf{w} = (w_1, \dots, w_N)$ (that is, $\sum_{i=1}^{n} w_i = 1$ and $w_i \geqslant 0$ $i = 1, \dots, n$) one can define the infinite product measure $\mu = \mathbf{w}^{\mathbf{N}}$ on the code space Σ. Its push forward $\nu = \nu_{\mathbf{w}} = \Pi_* \mu$ is called a *self-similar measure*, that is, for Borel sets $H \subset \mathbb{R}^d$

$$\nu(H) = \mu(\Pi^{-1}(H)). \tag{1.4}$$

Equivalently, ν is the unique compactly supported Radon probability measure with the property that for all $H \subset \mathbb{R}^d$ Borel sets

$$\nu(H) = \sum_{i=1}^{m} p_i \cdot \nu\left(\varphi_i^{-1}(H)\right),$$

see [19].

To measure the "size" of attractors we introduce several dimension notions. The *box* or *Minkovski dimension* $\dim_{\mathrm{B}}$ of a bounded set $S \subset \mathbb{R}$ is defined by

$$\dim_{\mathrm{B}}(S) := \lim_{\delta \to 0+} \frac{N_\delta(S)}{-\log \delta}, \tag{1.5}$$

if the limit exists, where $N_\delta(S)$ is the least number of squares of side length δ needed to cover S. The upper and lower box dimensions $\overline{\dim}_{\mathrm{B}}$ and $\underline{\dim}_{\mathrm{B}}$ are defined in the same way with lim inf and lim sup instead of lim, respectively, which therefore always exist. It is easy to see that the box dimension of simple sets is what one would expect, for example a section has dimension 1, or a filled square has dimension 2. More "rough" sets such as self-similar sets can take any other non-negative real dimensions which are not greater than the dimension of the ambient space. One inconvenient property of box dimension is that the box dimension of the union of countably many sets is not the supremum of the individual box dimensions: E.g., $\dim_{\mathrm{B}}(\mathbb{Q} \cap [0,1])$ is 1, as one needs at least $1/\delta$ sections of length δ to cover this set.

Another notion of dimension is *Hausdorff dimension*. To define it first we introduce the so called s-dimensional Hausdorff content $\mathcal{H}^s$ of a bounded set $S \subset \mathbb{R}^d$:

$$\mathcal{H}^s(S) := \lim_{\delta \to 0+} \mathcal{H}^s_\delta(S) \tag{1.6}$$

where

$$\mathcal{H}^s_\delta(S) := \inf\left\{ \sum_{i=1}^{\infty} |A_i|^s \;\middle|\; S \subset \bigcup_{i=1}^{\infty} A_i,\ |A_i| < \delta,\ A_i \subset \mathbb{R}^d \right\},$$

where $|A_i|$ is the diameter of A_i. Then the *Hausdorff dimension* $\dim_{\mathrm{H}}$ of S is defined by

$$\dim_{\mathrm{H}} S := \inf\{s \mid \mathcal{H}^s(S) = 0\} = \sup\{s \mid \mathcal{H}^s(S) = \infty\}.$$

We also define the *Hausdorff dimension of a measure* $\mathfrak{m}$ on $\mathbb{R}^d$ by

$$\dim_{\mathrm{H}} \mathfrak{m} := \inf\left\{\dim_{\mathrm{H}} A : \mathfrak{m}(A) > 0,\ \text{and } A \text{ is a Borel set}\right\}, \tag{1.7}$$

see [19, p. 170] for an equivalent definition.

We will use the following definition of the packing dimension of a set $H \subset \mathbb{R}^d$ [19, p. 23.]:

$$\dim_{\mathrm{P}} H = \inf\{\sup_i \overline{\dim}_{\mathrm{B}} E_i \ : \ H \subset \bigcup_{i=1}^{\infty} E_i\}, \tag{1.8}$$

where $\overline{\dim}_{\mathrm{B}}$ stands for the upper box dimension. Note that originally the packing dimension was defined in an analogous way to the Hausdorff dimension, but replacing the coverings with packings. For convenience we use the above equivalent formulation. The most important properties of the packing dimension can be found in [19].

The *similarity dimension* of a self-similar IFS is the unique $s > 0$ for which

$$\sum_{i=1}^{N} r_i^s = 1. \tag{1.9}$$

The similarity dimension of the corresponding self-similar measure $\nu = \nu_{\mathbf{w}}$ is defined by

$$\dim_{\mathrm{S}}(\nu) := \frac{\sum_{i=1}^{N} p_i \log p_i}{\sum_{i=1}^{N} p_i \log r_i}. \tag{1.10}$$

For given $\mathcal{F}$ it takes its maximum in $p_i = r_i^s$, and the maximum value is the similarity dimension s of $\mathcal{F}$. The self-similar measure associated to these weights is called the *natural measure.*

The following relations hold between the different dimension notions. For any set $S \subset \mathbb{R}^d$

$$\dim_{\mathrm{H}} S \leqslant \overline{\dim}_{\mathrm{B}} S,$$

and if S is self-similar, then

$$\dim_{\mathrm{B}} S = \dim_{\mathrm{H}} S \leqslant \dim_{\mathrm{S}} S.$$

Moreover, additional separation conditions guarantee more. We say that the strong separation condition (SSC) holds, if

$$\varphi_i(S) \cap \varphi_j(S) = \varnothing \ \forall i \neq j.$$

We say that the open set condition (OSC) holds if there is a nonempty open set V such that

$$\varphi_i(V) \subset V \ \forall i, \text{ and}$$
$$\varphi_i(V) \cap \varphi_j(V) = \varnothing \ \forall i \neq j.$$

Clearly self-similar S sets with SSC also satisfy the OSC with V being an open neighborhood of S. For example, the triadic Cantor set (Figure 1) satisfies the SSC, while for the Sierpińsky carpet (Figure 2) only the OSC holds. The following theorem of Hutchinson shows that the above dimensions coincide if the OSC holds.

Theorem 1.2. *[24] Let S be a self-similar set satisfying the OSC. Let $s = \dim_{\mathrm{S}} S$. Then*

$$0 < \mathcal{H}^s(S) < \infty.$$

Moreover,

$$\dim_{\mathrm{B}} S = \dim_{\mathrm{H}} S = s.$$

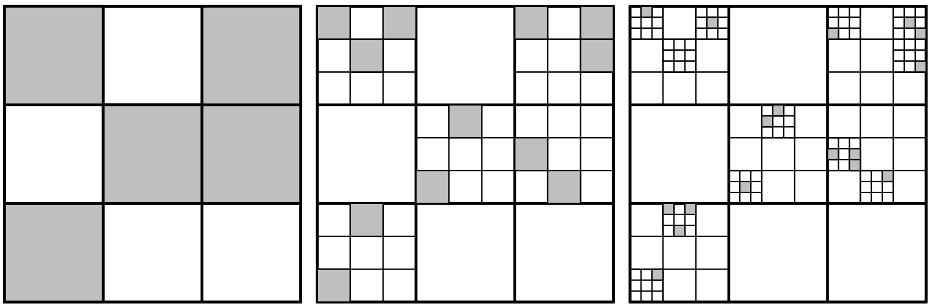

Figure 3: A realization of the first three level sets of the fractal percolation set

Note that if the SSC holds, then each point in S has a unique code in the code space Σ.

In this work we also investigate the *Mandelbrot percolation set*, or *fractal percolation set* (Figure 3), which is a random (statistically) self-similar set defined as follows.

Definition 1.3 (Mandelbrot percolation)**.** Fix an integer $M \geqslant 2$, real numbers $0 \leqslant p_{i,j} \leqslant 1$ for $i, j \in \{0, \ldots, M-1\}$ and let $K = [0,1]^2$. The definition goes iteratively. First, subdivide K into small squares $K_{i,j}$ of side length $1/M$, $i, j \in \{0, \ldots, M-1\}$, and retain all small squares $K_{i,j}$ with probability $p_{i,j}$ independently of each other. The union of the retained closed squares forms the first level set E_1 of the Mandelbrot percolation set. Then repeat the same process on each of the retained small squares to get the second level set E_2, i.e. subdivide the small squares to M^2 smaller squares each, and retain some of them according to the same probability distribution as before, independently of each other and of the former events. Repeat the procedure on the smaller and smaller squares ad infinitum to obtain the Mandelbrot percolation set

$$E = E_\infty = \bigcap_{n=1}^{\infty} E_n.$$

A bit more formal definition of d-dimensional fractal percolation can be found in my dissertation. Based on a branching process argument it is easy to see that if $\sum_{i,j=0}^{M-1} p_{i,j} > 1$ then E is non-empty with positive probability. The name fractal percolation originates from the property that there is a critical probability $p_c < 1$ so that if all $p_{i,j} > p_c$ then the set percolates with positive probability [8], i.e. there is a subset of E which is connected and intersects with both the left and the right side of the unit square. On the other hand, if all $p_{i,j} < p_c$, then the set is totally disconnected almost surely (a.s.).

Another well-known property of the fractal percolation set is that conditioned on non-emptiness the dimension of E_∞ is a.s. constant [34]:

$$\dim_{\mathrm{H}} E = \frac{\log \sum_{i,j=0}^{M-1} p_{i,j}}{\log M}. \tag{1.11}$$

Similarly to deterministic self-similar sets we define the *natural measure* μ on E (conditioned on $E \neq \emptyset$) as the weak limit

$$\mu = \lim_{n\to\infty} \frac{\mathcal{L}_2|_{E_n}}{\mathcal{L}_2(E_n)},$$

where $\mathcal{L}_2$ is the 2-dimensional Lebesgue measure. Mauldin and Williams [31] proved that the above limit exists.

Note that the notions of both the definition of fractal percolation set and its natural measure can be extended from the plane to higher dimensions without difficulty such that we start with the unit cube and then subdivide it to smaller cubes.

The geometry of random and deterministic self-similar sets may be extremely complex if separation conditions like OSC or SSC do not hold. In such cases it is often challenging to claim something about the dimension or geometry of a particular set. Instead, we investigate parametric families of self-similar sets or measures depending regularly on the parameter and make statements holding for typical sets or measures of that family, where we mean typical in terms of either measure or topology.

Let

$$\mathcal{F}_\alpha := \left\{ \varphi_i^\alpha(x) := r_{\alpha,i} \cdot x + t_i^{(\alpha)} \right\}_{i=1}^m, \quad \alpha \in A, \tag{1.12}$$

be a one-parameter family of self-similar IFS on $\mathbb{R}$ and let μ be a measure on the symbolic space $\Sigma := \{1,\dots,m\}^{\mathbb{N}}$. We denote the family of its push forward measures by $\{\nu_\alpha\}_{\alpha\in A}$ and by $\{\nu_{\alpha,\mathbf{w}}\}_\alpha$ the parametrized self-similar measure corresponding to a given probability vector $\mathbf{w}$.

The most common parametric families are linear projections of a given self-similar set or measure.

Definition 1.4. Fix $\alpha \in (0, \pi/2)$. We will use the following linear projections (see Figure 4):

- Let Π_α be the linear projection mapping from $\mathbb{R}^2$ to the x-axis Δ in angle α, i.e. parallel to the vector $(1, \tan\alpha)$. Sometimes we identify Δ with $\mathbb{R}$.
- We also consider projection proj_α which is the orthogonal projection to the line going through the origin and making angle α with the x-axis.

Note that in [35] Π_α is defined in a slightly different way, namely it projects to the diagonal of the unit square K. Here we modify it to map to the x-axis because it is easier to generalize it to projections of high dimensional Mandelbrot percolation.

Both the Π_α and proj_α projected images of a planar self-similar set or measure without rotations are also self-similar on the line with the same contraction ratios, and this way they form a family parametrized by α.

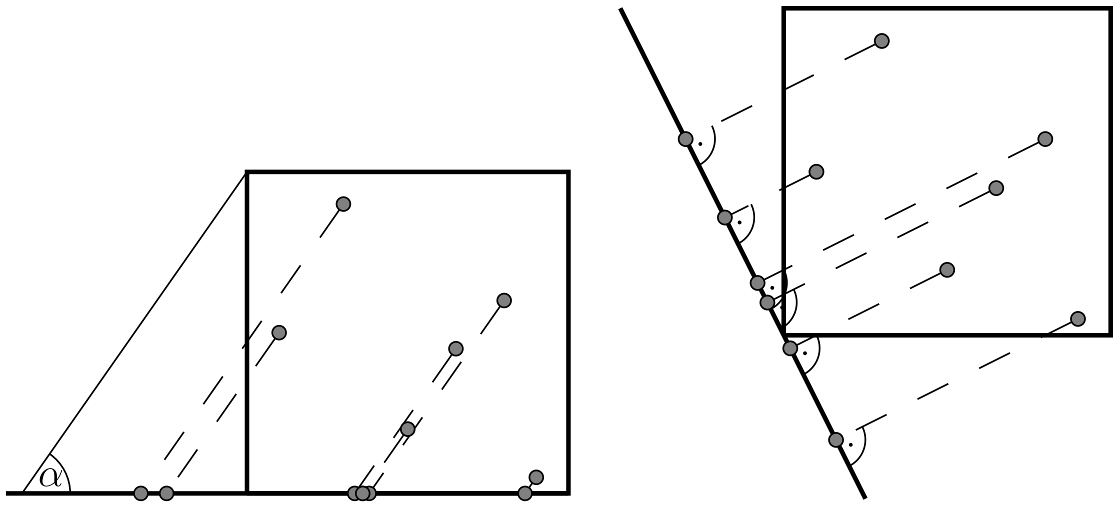

Figure 4: Π_α (left) and proj_α (right) projections.

1.2 Problems, main results

In this section we summarize our contribution to the understanding of self-similar sets and measures based on the papers [39, 40, 38] which are joint work with Károly Simon. We also review the most relevant related results from the literature clearly separated by subsection titles ending with *Problems, related literature* and *Our contribution.*

1.2.1 Projections of Mandelbrot percolation sets: Problems, related literature

It is natural to expect that if we pick a set of Hausdorff dimension greater than 1, then its projection to a line, which is 1-dimensional, has to be big. A classical, fundamental theorem of Marstrand [29] confirms this expectation for Lebesgue typical projection angles. Let $\mathcal{L}$ stand for the one dimensional Lebesgue measure.

Theorem 1.5 (Marstrand [29]). *Let $A \subseteq \mathbb{R}^2$ be a Borel set. Then*

(i) if $\dim_{\mathrm{H}} A \leqslant 1$, then $\dim_{\mathrm{H}}(\text{proj}_\alpha A) = \dim_{\mathrm{H}} A$ for $\mathcal{L}$ almost all α;

(ii) or if $\dim_{\mathrm{H}} A > 1$, then $\mathcal{L}(\text{proj}_\alpha A) > 0$ for $\mathcal{L}$ almost all α.

Later, Mattila [30] extended the results of Marstrand [29] to projections mapping from $\mathbb{R}^d$ to $\mathbb{R}^k$ with $1 \leqslant k \leqslant d-1$. In the last decades, many works investigated properties of projections of *fractal sets*, either complementing, or strengthening in special cases the results of Marstrand. For a review on this topic, see [14]. Here we shortly highlight the directions of research to which our results contribute, all corresponding to projections of random and deterministic self-similar sets and measures.

The first line of research in this work investigates projections of the fractal percolation set E. By Marstrand's theorem 1.5 and (1.11) we know in particular that a.s., conditioned on $E \neq \varnothing$ for $\mathcal{L}$ almost all (a.a.) $\alpha \in [0, \pi)$:

$$\text{(i) } \dim_{\mathrm{H}} \operatorname{proj}_{\alpha} E = s, \text{ if } s \leqslant 1, \text{ and}$$
$$\text{(ii) } \mathcal{L}(\operatorname{proj}_{\alpha} E) > 0 \text{ if } s > 1,$$

where $s = \frac{\log \sum_{i,j=0}^{M-1} p_{i,j}}{\log M}$ is the a.s. dimension of $E \neq \varnothing$. That is, from the point of view of Lebesgue measure and Hausdorff dimension, Lebesgue typical orthogonal projections of E are as expected. However, from this result we can not get any information about projections in any given angle α. A classical result for a fixed α is due to Falconer and Grimmett [16, 17]. Their result is for linear projections mapping from dimension d to k, but for simplicity we state it only for the planar fractal percolation.

Theorem 1.6 (Falconer and Grimmett [16]). *Suppose that for all $i \in \{0, \ldots, M-1\}$*

$$\sum_{j=0}^{M-1} p_{i,j} > 1.$$

Then a.s., conditioned on $E \neq \varnothing$, $\operatorname{proj}_0 E$ have nonempty interior.

On the other hand, if for an $i \in \{0, \ldots, M-1\}$ we have

$$\sum_{j=0}^{M-1} p_{i,j} < 1,$$

then a.s. the interior of $\operatorname{proj}_0 E$ is empty.

Note that for $p \geqslant p_c$ (the critical probability for percolation is defined on the bottom of page 5) it is not hard to see that a.s. the projection of E has nonempty interior given $E \neq \varnothing$. To see this let us denote by $p^* > 0$ the probability that the set E percolates. By statistical self-similarity every small squares $K_{\mathbf{i}_n, \mathbf{j}_n}$ of any level n percolate with the same positive probability p^*, given $K_{\mathbf{i}_n, \mathbf{j}_n} \neq \varnothing$. Moreover, events on disjoint small squares are independent of each other. Therefore, a.s. conditioned on $E \neq \varnothing$ there will be some small squares of the fractal percolation set that percolate, which results in an interval of $\operatorname{proj}_0 E$. However, for p in the range $(1/M, p_c)$ the a.s. dimension of E is > 1 but E is still totally disconnected. This range is non-empty because p_c is quite close to 1, e.g., $0.993 > p_c(M = 2) > 0.881$ and $0.94 > p_c(M = 3) > 0.784$ as shown in the paper of Don [11].

To handle individual non-coordinate axis directions took more than 20 years. Recently, Rams and Simon [36, 35] proved the surprising fact that, under mild conditions, for projections of the fractal percolation set E there is no exceptional angle in Theorem 1.5 at all a.s. if $E \neq \varnothing$. The result in [35], which corresponds to part (ii) of Theorem 1.5, relies on a condition called $A(\alpha)$. As they show, this condition holds for all $\alpha \in (0, \pi/2)$ when $p_{i,j} \equiv p$ and $p > 1/M$ (i.e. homogeneous retaining probabilities and $s > 1$).

Theorem 1.7 (Rams and Simon [35]). *Suppose that Condition $A(\alpha)$ holds for all $\alpha \in (0, \pi/2)$. Then a.s., simultaneously for* all $\alpha \in (0, \pi/2)$, *orthogonal projections* $\text{proj}_\alpha(E)$ *have nonempty interior, conditioned on $E \neq \varnothing$.*

Note that the methods used in [35] allow to extend the above theorem relatively easily to sets $D \subset (0, \pi/2)$ of αs to obtain non-empty interior simultaneously for all $\alpha \in D$, Theorem 1.10 below contains this extension.

The authors of [35] also show a non-trivial inhomogeneous example for which Condition $A(\alpha)$ holds for all $\alpha \in (0, \pi/2)$. Namely, if $M = 3$, then even if the retaining probability of the middle square $p_{1,1} = 0$, but for all other (i, j) we have $p_{i,j} > 1/2$, then Condition $A(\alpha)$ holds for all $\alpha \in (0, \pi/2)$, and hence a.s. there is an interval in all orthogonal projections. Later we refer to sets constructed with $M = 3$, $p_{1,1} = 0$ and $p_{i,j} = p > 0$ for $(i, j) \neq (1, 1)$ as random Sierpiński carpets.

Now let us focus on our contributions in this area.

Projections of Mandelbrot percolation sets: Our contribution

First we consider an example of inhomogeneous fractal percolation sets for which the extension of Theorem 1.7 applies for an interval D of directions α. The set investigated is the Cantor-like random carpet which is constructed with $M = 3$ and inhomogeneous retaining probability setting $p_{1,j} = 0$ for all $j = 0, \ldots, 2$ and $p_{i,j} > 1/2$ for all other (i, j).

Theorem 1.8 ([39]). *Consider the Cantor-like random carpet E constructed as above. Then Condition $A(\alpha)$ holds for all $\alpha \in (\pi/4, 3\pi/4)$. As a consequence of Theorem 1.7, a.s. if $E \neq \varnothing$ there is an interval in all of the sets* $\text{proj}_\alpha(E)$, $\alpha \in (\pi/4, \pi/2)$. *On the other hand,* $\text{proj}_0(E)$ *contains no intervals as it is contained in the triadic Cantor set.*

We emphasize that this set behaves differently from homogeneous fractal percolations. Almost surely, conditioned on non-emptiness, the projections of the homogeneous set are totally disconnected if its similarity dimension is < 1 or if its dimension is > 1 then projections in *all* directions have an interval. In contrast, the Cantor-like random carpet above has exceptional directions for the existence of intervals in the projection and also has an interval of directions for which the projected set has nonempty interior.

In addition, we also provide complementing examples which admit several non-trivial exceptional directions, i.e. fractal percolation sets with $\dim_{\mathrm{H}} > 1$ for which a.s. the proj_α projections have empty interior for some α.

Theorem 1.9 ([39]). *Consider random Sierpiński carpets E_ε constructed with $M = 3$, $p_{1,1} = 0$ and $p_{i,j} = 3/8 + \varepsilon$ for all other (i, j) with $\varepsilon > 0$. Note that in this case*

$\dim_{\mathrm{H}} E_\varepsilon > 1$ *a.s. if* $E_\varepsilon \neq \varnothing$. *Then for all* $\alpha \in (0, \pi/2)$ *such that* $\tan \alpha \in \mathbb{Q}$ *there is an* $\varepsilon > 0$ *such that a.s.* $\mathrm{proj}_\alpha E_\varepsilon$ *contains no intervals.*

As an immediate corollary, for any n natural number we can find an $\varepsilon > 0$ and distinct directions $\alpha_1, \dots, \alpha_n$ such that with $p_{i,j} = 3/8 + \varepsilon$ these directions are exceptional for the existence of intervals in the projection, although the dimension is > 1.

Finally, we extend the results of [35] to orthogonal projections mapping from $\mathbb{R}^d$ to $\mathbb{R}^k$ with $1 \leqslant k \leqslant d-1$. Let us denote such projections with proj_α, the same way as before, although the parameter α is not an angle any more. The range of proj_α is denoted by $S_\alpha \subset \mathbb{R}^d$. Similarly to Theorem 1.7 we rely on the high-dimensional analogue of Condition A, which we show to hold under minimal conditions in case of homogeneous retaining probabilities, as well as in some non-trivial inhomogeneous cases.

Theorem 1.10. *[38] Fix* $d \geqslant 2$ *and* $1 \leqslant k < d$ *and suppose that Condition* $A(\alpha)$ *holds for all* $\alpha \in D$, *where* D *is any set of* α *such that* S_α *is not a coordinate plane. Then for almost all realizations of* E *such that* $E \neq \varnothing$, *simultaneously for* all α *orthogonal projections* $\mathrm{proj}_\alpha(E)$ *have nonempty interior (relative to the topology of* $\mathbb{R}^k$*).*

1.2.2 Overlapping cylinders: Problems, related literature

A serious challenge in studying projected random or deterministic self-similar sets lays in the structure of so-called *overlaps*, i.e., in the structure of the intersections of cylinder sets. Projections $\mathrm{proj}_\alpha S$ of a self-similar set S to lines when $\dim_{\mathrm{H}} S > 1$ always result in intersections. In some cases intersections are so severe that the Hausdorff dimension of the projected set drops, i.e. $\dim_{\mathrm{H}}(\mathrm{proj}_\alpha S) < \dim_{\mathrm{H}} S \leqslant 1$, or $\dim_{\mathrm{H}}(\mathrm{proj}_\alpha S) < 1$ although $\dim_{\mathrm{H}} S > 1$. This happens, e.g., for proj_α projections of the Sierpiński carpet with $\tan \alpha \in \mathbb{Q}$ [28]. Therefore, these angles are among the exceptions allowed by Marstrand's theorem 1.5. A key question is than how do the structure of overlaps in exceptions of Marstrand's theorem look like when we consider projections of self-similar sets?

We say that there is an *exact overlap* if we can find two distinct $\mathbf{i} = (i_1, \dots, i_k)$ and $\mathbf{j} = (j_1, \dots, j_\ell)$ finite words such that

$$\varphi_{i_1} \circ \cdots \circ \varphi_{i_k} = \varphi_{j_1} \circ \cdots \circ \varphi_{j_\ell}. \tag{1.13}$$

Clearly having exact overlaps is one mechanism that can cause dimension drop. The following two questions have naturally arisen for a long time (e.g. Question 1 below appeared as [33, Question 2.6]):

Question 1 Is it true that a self-similar measure on the line has Hausdorff dimension strictly smaller than the minimum of 1 and its similarity dimension only if we have exact overlap?

Question 2 Is it true for a self-similar measure ν on the line having similarity dimension greater than one, that ν is singular only if there is exact overlap?

Most of the experts believe that the answer to Question 1 is positive and it has been confirmed in some special cases [22]. On the other hand, a result of Nazarov, Peres and Shmerkin indicated that the answer to Question 2 should be negative. Namely, they constructed in [32] a planar self-affine set having dimension greater than one, such that the angle-α projection of its natural measure was singular for a dense G_δ set of parameters α. However, this was not a family of self-similar measures.

The analogous question in high dimensions turns out to be simple, namely there are planar self-similar measures with similarity dimension > 2 that are singular but have no exact overlap (see e.g. [4]), and such a planar example can be easily extended to any dimensions by taking its cross-product with $[0,1]^k$.

Overlapping cylinders: Our contribution

We answer Question 2 negatively, which is the strongest result of the Thesis. Here we only state the main result in its simplest, less general form, and we discuss the topic in more detail with additional results there.

Theorem 1.11 ([40]). *Consider angle-α projections* proj_α *of the Sierpiński carpet S. Let ν_α stand for the natural measure of* $\text{proj}_\alpha S$. *Note that* $\dim_{\text{H}} S > 1$. *We obtain that*

$$\{\alpha \in (0,\pi/2) : \nu_\alpha \perp \mathcal{L}\text{eb}\} \text{ is a dense } G_\delta \text{ set} \tag{1.14}$$

and

$$\dim_{\text{H}}\left(\{\alpha \in (0,\pi/2) : \nu_\alpha \not\ll \mathcal{L}\text{eb}\}\right) = 0. \tag{1.15}$$

Note that for self-similar measures ν we either have $\nu \ll \mathcal{L}\text{eb}$ or $\nu \perp \mathcal{L}\text{eb}$. The second part, (1.15) is basically due to Shmerkin and Solomyak [37] with slight modifications (the so-called non-degeneracy condition of [37, Theorem A] doesn't hold, for details see the proof of Theorem 1.11). Clearly, it also implies that for the Sierpiński carpet the exceptional set of angles in Theorem 1.5 is not only of 0 measure, but of 0 Hausdorff dimension as well. On the other hand, (1.14) shows that this set of exceptions is huge in the topological sense.

As we mentioned before, (1.14) also answers Question 2 negatively. To see this note that the set of angles α such that the self-similar set $\text{proj}_\alpha S$ has an exact overlap must be countable. Actually, if S is the Sierpinsky carpet, $\text{proj}_\alpha S$ can have an exact overlap only if $\tan\alpha \in \mathbb{Q}$. However, a dense G_δ set in $(0,\pi/2)$ is uncountable, hence there are angles α such that $\nu_\alpha \perp \mathcal{L}$ but it is not caused by exact overlap.

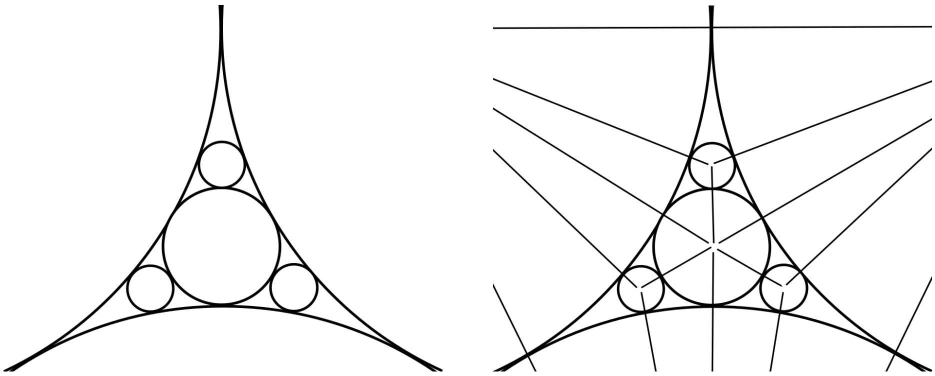

Figure 5: Apollonian circle packing after a few steps with 7 circles and the corresponding Apollonian Network with 7 nodes (circles), straight lines represent edges.

2 Degrees of random and evolving Apollonian networks

Now let us turn our attention to two randomly evolving graph models belonging to the class of Apollonian networks. The following results are based on the paper [25] which is a joint work with István Kolossváry and Júlia Komjáthy.

2.1 Background and notations

The construction of deterministic and random Apollonian networks originates from the problem of Apollonian circle packing: starting with three mutually tangent circles, we inscribe in the interstice formed by the three initial circles the unique circle that is tangent to all of them: this fourth circle is known as the inner Soddy-circle. Iteratively, for each new interstice its inner Soddy-circle is drawn. After infinite steps the result is an Apollonian gasket [7, 21].

An Apollonian network (AN) is the resulting graph if we place a vertex in the center of each circle and connect two vertices if and only if the corresponding circles are tangent, see Figure 5. This model was introduced independently by Andrade et al. [2] and Doye and Massen [12] as a model for networks arising in real-life such as the network of internet cables or links, collaboration networks or protein interaction networks. Apollonian networks can serve as a model for these networks since their main characteristic properties can be observed also in the examples above: a power-law degree distribution, a high clustering coefficient, and small distances, usually referred to as the small-world property. Moreover, by construction, Apollonian networks also show hierarchical structure: a property that is very commonly observed in e.g. social networks.

It is straightforward to generalize Apollonian packings to arbitrary $d \geqslant 2$ with mutually tangent d dimensional hyperspheres. Analogously, if each d-hypersphere

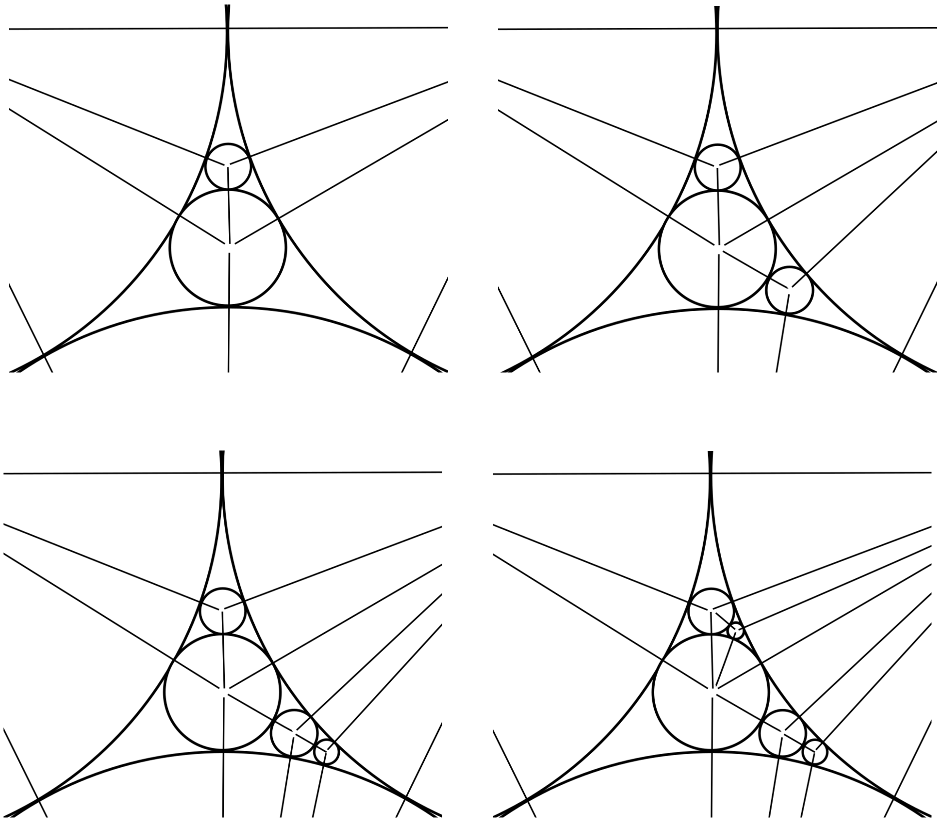

Figure 6: A realization of the first few steps of RAN_2.

corresponds to a vertex and vertices are connected by an edge if the corresponding d-hyperspheres are tangent, then we obtain a d-dimensional AN (see [41, 42]).

The network arising by this construction is deterministic. Zhou et al. [45] proposed to randomise the dynamics of the model such that in one step only one interstice is picked u.a.r. and filled with a new circle. This construction yields a d dimensional *random Apollonian network* (RAN) [43], see Figure 6. We give a detailed definition below. Using heuristic and rigorous arguments the results in [1, 9, 10, 13, 20, 43, 45] show that RANs have the above mentioned main features of real-life networks.

A different random version of the original Apollonian network was introduced by Zhang et al. [44], called *Evolutionary Apollonian networks* (EAN) where in every step *every* interstice is picked and filled independently of each other with probability q. If an interstice is not filled in a given step, it can be filled in the next step again. We call q the *occupation parameter*. For $q = 1$ we get back the deterministic AN model. It is conjectured in [44] that an EAN with parameter q, as $q \to 0$, should show similar topological behavior to RANs. To make this statement rigorous, instead of looking at a sequence of evolving EAN-s with decreasing parameters, we slightly modify the model and investigate the asymptotic behavior of a single EAN when q might differ in each step of the dynamics. That is, we consider a series $\{q_n\}_{n=1}^{\infty}$ of occupation parameters so that q_n applies for step n of the dynamics, and assume that q_n tends to 0. In this setting, the interesting question is to determine the correct rate for q_n that achieves the observation that EAN shows similar behavior as RAN when the parameter tends to zero.

In the followings we give the precise definition of the RAN and EAN models and introduce the necessary notations before we state our main results.

Random Apollonian networks

A random Apollonian network $\mathrm{RAN}_d(n)$ in d dimensions can be constructed as follows. The graph at step $n = 0$ consists of $d + 2$ vertices, embedded in $\mathbb{R}^d$ in such a way that $d+1$ of them forms a d-dimensional simplex, and the $(d+2)$-th vertex is located in the interior of this simplex, connected to all of the vertices of the simplex. This vertex in the interior forms $d+1$ d-simplices with the other vertices: initially we set the status of these d-simplices 'active', and call them *active cliques*. For $n \geqslant 1$, pick an active clique C_n of $\mathrm{RAN}_d(n-1)$ *uniformly at random* (u.a.r.), insert a new vertex v_n in the interior of C_n and connect v_n with all the vertices of C_n. The newly added vertex v_n forms new cliques with each possible choice of d vertices of C_n. Set the status of the clique C_n 'inactive', and the status of the newly formed d-simplices 'active'. The resulting graph is $\mathrm{RAN}_d(n)$. At each step n a $\mathrm{RAN}_d(n)$ has $n + d + 2$ vertices and $nd + d + 1$ active cliques.

Evolutionary Apollonian networks

Given a sequence of occupation parameters $\{q_n\}_{n=1}^{\infty}$, $0 \leqslant q_n \leqslant 1$, an evolutionary Apollonian network $\mathrm{EAN}_d(n, \{q_n\}) = \mathrm{EAN}_d(n)$ in d dimensions can be constructed iteratively as follows. The initial graph is the same as for a $\mathrm{RAN}_d(0)$ and there are $d+1$ active cliques. For $n \geqslant 1$, choose each active clique of $\mathrm{EAN}_d(n-1)$ independently of each other with probability q_n. The set of chosen cliques $\mathcal{C}_n$ becomes inactive (the non-picked active cliques stay active) and for every clique $C \in \mathcal{C}_n$ we place a new vertex $v_n(C)$ in the interior of C that we connect to all vertices of C. This new vertex $v_n(C)$ together with all possible choices of d vertices from C forms $d+1$ new cliques: these cliques are added to the set of active cliques for every $C \in \mathcal{C}_n$. The resulting graph is $\mathrm{EAN}_d(n)$. The case $q_n \equiv q$ was studied in [44] where it was further suggested that for $q \to 0$ the graph is similar to a $\mathrm{RAN}_d(n)$. We prove their conjecture by showing that EANs obey the same power law exponent as RANs if $q_n \to 0$ and $\sum_{n=0}^{\infty} q_n = \infty$.

Remark 2.1. Note that both in the RAN and EAN models, there is a one-to-one correspondence between cliques and vertices/future vertices: vertex v corresponds to the clique C that became inactive when v was placed in the interior of the d-simplex corresponding to C. In this respect, we call vertices that are already present in the graph *inactive vertices*, and we refer to active cliques as *active vertices*: this notation means that these vertices are not yet present in the graph, but might become present in the next step of the dynamics.

Additional notations

Let us introduce some additional notation to be able to state our results on the degree distribution of RANs and EANs.

We define $D_v(n)$ as the degree of vertex v after the n-th step. Let us denote by $\widetilde{N}_k(n)$ and $\widetilde{p}_k(n)$ the number and the empirical proportion of inactive vertices with degree k at time n respectively in a $\mathrm{RAN}_d(n)$, i.e.

$$\widetilde{p}_k(n) := \frac{\widetilde{N}_k(n)}{n+d+2} := \frac{1}{n+d+2} \sum_{i=1}^{n+d+2} \mathbb{1}\{D_i(n) = k\}. \tag{2.1}$$

Analogously, for the graph $\mathrm{EAN}_d(n, \{q_n\})$ we use the notations $N_k(n)$ and $p_k(n)$ defined by

$$p_k(n) := \frac{N_k(n)}{N(n)} = \frac{1}{N(n)} \sum_{i \in V(n)} \mathbb{1}\{D_i(n) = k\}, \tag{2.2}$$

where $V(n)$ denotes the set of vertices after n steps and $N(n) = |V(n)|$.

We say that a sequence of events $\mathcal{E}_n$ happens *with high probability* (whp) if $\lim_n \mathbb{P}(\mathcal{E}_n) = 1$. Note that 'with high probability' is the same as 'asymptotically almost surely'. We further define for an event A and a σ-algebra $\mathcal{F}$ the conditional probability $\mathbb{P}(A|\mathcal{F}) = \mathbb{E}[\mathbb{1}_A|\mathcal{F}]$, where $\mathbb{1}_A$ is the indicator of the event A, i.e., it takes value 1 if A holds and 0 otherwise. We will sometimes replace $\mathcal{F}$ by a list of random variables, in this case we drop the σ-algebra notation and list the random variables in the conditioning, and this means conditional on the σ-algebra generated by this list of random variables.

2.2 Results

All of the results are from [25].

2.2.1 Degree distribution

Our first result states that for a $\mathrm{RAN}_d(n)$ the empirical distribution $\widetilde{p}_k(n)$ tends to a proper distribution in the ℓ_∞-metric:

Theorem 2.2 (Degree distribution for RANs)**.** *For all $d \geqslant 2$ there exist a probability distribution $\{p_k\}_{k=d+1}^{\infty}$ and a constant c for which*

$$\lim_{n\to\infty} \mathbb{P}\left(\max_k |\widetilde{p}_k(n) - p_k| \geqslant c\sqrt{\frac{\log n}{n}} \right) = 0.$$

Further, p_k follows a power law with exponent $(2d-1)/(d-1) \in (2, 3]$, more precisely

$$p_k = \frac{d}{2d+1} \frac{\Gamma(k-d+\frac{2}{d-1})}{\Gamma(1+\frac{2}{d-1})} \frac{\Gamma(2+\frac{d+2}{d-1})}{\Gamma(k+1-d+\frac{d+2}{d-1})} = C(d)k^{-\frac{2d-1}{d-1}}(1+o_k(1)), \tag{2.3}$$

where $o_k(1)$ denotes a quantity that tends to zero as $k \to \infty$, $C(d)$ is constant that depends on d and $\Gamma(x)$ is the Gamma function.

Remark 2.3. To get the asymptotic behaviour of p_k above we used the property that $\Gamma(t+a)/\Gamma(t) = t^a(1+o(1))$.

For the theorem that describes the degree distribution of the graph $\mathrm{EAN}_d(n, \{q_n\})$ we need the following additional analytic assumption on the sequence $\{q_n\}_{n\in\mathbb{N}}$.

Assumption 2.4. Assume, as before, that $q_n \to 0$ and $\sum_{i=1}^{\infty} q_n \to \infty$. We assume further that there exist constants c_1, C_1 (that depend only on the sequence $\{q_n\}_{n=1}^{\infty}$) such that

$$c_1 \leqslant \frac{\sum_{i=1}^{n} q_i^2 \prod_{j=1}^{i}(1+dq_j)}{q_n \prod_{j=1}^{n}(1+dq_j)} \leqslant C_1, \tag{2.4}$$

and for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \exp\left\{-\varepsilon q_n \mathrm{e}^{d\sum_{j=0}^{n} q_j}\right\} < \infty. \tag{2.5}$$

Theorem 2.5 (Degree distribution for EANs)**.** *Let $d \geqslant 2$ and $\{q_n\}_{n=0}^{\infty}$ be probabilities such that Assumption 2.4 is satisfied. Then the degree distribution tends to the same asymptotic degree distribution $\{p_k\}_{k=d+1}^{\infty}$ as in the case of $\mathrm{RAN}_d(n)$ given in (2.3). More precisely, there exists a constant $C_0 > 0$ and a random variable $\eta < \infty$ such that for any $k \in \mathbb{N}$*

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{i<k} |p_i(n) - p_i| \geqslant C_0 k! \eta^k q_n\right) = 0.$$

In particular, the degree distribution converges pointwise for all $i < k = k(n)$ for any $k(n)$ that satisfies $C_0 k! \eta^k q_n \to 0$ as $n \to \infty$.

The following lemma gives classes of sequences $\{q_n\}_{n\in\mathbb{N}}$ that satisfy Assumption 2.4.

Lemma 2.6 (Regularly varying sequences)**.** *Let $L(x)$ denote a slowly varying function at infinity, that is, for every $c > 0$, $\lim_{x\to\infty} L(cx)/L(x) = 1$. Then Assumption 2.4 is satisfied in the following cases*

1. *$q_n = L(n)/n^{\alpha}$, for some $\alpha \in (0,1)$,*

2. *$q_n = (b + o(n^{-\delta}))/n$, with $b > 1/d$, for some arbitrary $\delta > 0$.*

Remark 2.7. Clearly, case (2) in Lemma 2.6 does not cover all possible regularly varying functions with $\alpha = 1$ for which Assumption 2.4 holds: one can check that the assumption holds for other cases as well, e.g., $q_n = b + b'/\log n$ with any $b > 1/d$ and arbitrary $b' \in \mathbb{R}$ works. On the other hand, some cases where $q_n = L(n)/n$ and $L(n) \to \infty$ fail, e.g. $q_n = b/(n \log n)$ does not satisfy (2.4).

Next we describe the clustering coefficient of RANs and EANs.

2.2.2 Clustering coefficient

The clustering coefficient of a vertex is the proportion of the number of existing edges between its neighbors, compared to the number of all possible edges between those. Here we investigate the clustering coefficient of the whole graph, which is the average of clustering coefficients over the vertices. Since these are direct consequences of the formula for the limiting degree distributions p_k, we state them as corollaries. This corollary is similar to the result in [43, Section 4.2.], but now, that we have established the degree distribution, it has a rigorous proof. This is based on the observation that the clustering coefficient of a vertex with degree k is *deterministic* both in RANs and EANs and equals

$$\frac{d(2k-d-1)}{k(k-1)} \sim \frac{2d}{k}.$$

The reason for this formula is the following: when the degree of vertex v increases by one by adding a new vertex w in one of the active cliques v is contained in, then the number of edges between the neighbors of v increases exactly by d, since the newly added vertex w is connected to the d other vertices in the clique. It was observed in simulations and heuristically proved in [43], that the average clustering coefficient of these networks converges to a strictly positive constant. Our next corollary determines the exact value of these constants for the two models:

Corollary 2.8 (Clustering coefficient)**.** *The average clustering coefficient of* $\mathrm{RAN}_d(n)$ *converges to a strictly positive constant as* $n \to \infty$*, given by*

$$\begin{aligned} Cl_d &:= \sum_{k=d+1}^{\infty} \frac{d(2k-d-1)}{k(k-1)} p_k \\ &= \sum_{k=d+1}^{\infty} \frac{d(2k-d-1)}{k(k-1)} \cdot \frac{d}{2d+1} \frac{\Gamma(k-d+\frac{2}{d-1})}{\Gamma(1+\frac{2}{d-1})} \frac{\Gamma(1+\frac{2d+1}{d-1})}{\Gamma(k-d+\frac{2d+1}{d-1})}. \end{aligned} \tag{2.6}$$

Further, the clustering coefficient of $\mathrm{EAN}_d(n, \{q_n\})$ *converges to the same value as in* (2.6) *if* $q_n \to 0$ *and* $\sum_{n\in\mathbb{N}} q_n = \infty$*.*

2.3 Related literature

Several results related to the asymptotic degree distribution of Apollonian networks are known. It is not hard to see that if a vertex belongs to k active cliques, then the chance for that vertex to get a new edge is proportional to k: this argument shows that these models belong to the wide class of *Preferential attachment models* [3, 5, 6]. As a result, some of the classical methods can be adapted to this model.

Using heuristic arguments, Zhang and his co-authors [43] obtained that the asymptotic degree exponent should be $\frac{2d-1}{d-1} \in (2, 3]$, which is in good agreement with their simulations. Parallel to writing the paper [25], we noticed that Frieze and Tsourakakis [20] very recently derived rigorously the exact asymptotic degree distribution of the two-dimensional RAN. Even though our work is independent of theirs, the methods are similar: this is coming from the fact that both of the methods used there and in our work are an adaptation of standard methods given in [6, 23]. So, to avoid repetition we decided to only sketch some parts of the proof and include the part that does not overlap with their work, i.e. the degree distribution of EANs.

What is entirely new in [25] is that we study the EAN model rigorously. For the degree distribution of EANs only heuristic arguments were known before. Zhang, Rong and Zhou [44] studied the graph series $\mathrm{EAN}_d(n)$ with (fixed) occupation parameter q. They derived the asymptotic degree exponent using heuristic arguments, and the result fits well with the simulations. They also suggested that as $q \to 0$ the model $\mathrm{EAN}_d(n)$ converges to $\mathrm{RAN}_d(n)$ in some sense. We confirm their claim by deriving the asymptotic degree distribution of $\mathrm{EAN}_d(n, \{q_n\})$ with $\{q_n\}$ such that $q_n \to 0$ and $\sum_{n=0}^{\infty} q_n = \infty$, obtaining the same degree distribution. This way we can give an analytic proof of the idea of Zhang and his co-authors.

3 References

[1] Marie Albenque and Jean-Francois Marckert. Some families of increasing planar maps. *Electron. J. Probab.*, 13:no. 56, 1624–1671, 2008.

[2] José S. Andrade, Hans J. Herrmann, Roberto F. S. Andrade, and Luciano R. da Silva. Apollonian Networks: Simultaneously Scale-Free, Small World, Euclidean, Space Filling, and with Matching Graphs. *Phys. Rev. Lett.*, 94:018702, 2005.

[3] A.-L. Barabási and R. Albert. Emergence of scaling in random networks. *Science*, 286(5439):509–512, 1999.

[4] Balázs Bárány, Michał Rams, and Károly Simon. Dimension theory of some non-markovian repellers part I: A gentle introduction.

[5] B. Bollobás and O. Riordan. The diameter of a scale-free random graph. *Combinatorica*, 24:5–34, 2004.

[6] B. Bollobás, O. Riordan, J. Spencer, and G. Tusnády. The degree sequence of a scalefree random graph process. *Random Structures and Algorithms*, 18:279–290, 2001.

[7] D.W. Boyd. The Sequence of Radii of the Apollonian Packing. *Mathematics of Computation*, 19:249 – 254, 1982.

[8] J. T. Chayes, L. Chayes, and R. Durrett. Connectivity properties of mandelbrot's percolation process. *Probab. Theory Related Fields*, 77:307–324, 1988.

[9] C. Cooper, A. Frieze, and R. Uehara. The height of random k-trees and related branching processes. *Random Struct. Alg.*, 45:675–702, 2014.

[10] A. Darrasse and M. Soria. Degree distribution of RAN structures and Boltzmann generation fo trees. In *Conference on Analysis of Algorithms, AofA 07, DMTCS Proceedings*, pages 1–14, 2007.

[11] H. Don. New methods to bound the critical probability in fractal percolation. *Random Structures & Algorithms*, 47:710–730, 2015.

[12] J.P.K Doye and C.P Massen. Self-similar disk packings as model spatial scale-free networks. *Phys. Rev. E*, 71:016128, 2005.

[13] E. Ebrahimzadeh, L. Farczadi, P. Gao, A. Mehrabian, C. M. Sato, N. Wormald, and J. Zung. On the Longest Paths and the Diameter in Random Apollonian Networks. *Electronic Notes in Discrete Mathematics*, 43:355–365, 2013.

[14] K. Falconer, J. Fraser, and X. Jin. Sixty years of fractal projections. *Fractal Geometry and Stochastics V. Progress in Probability, Birkhäuser, Cham*, 70, 2015.

[15] K. J. Falconer. *The Geometry of Fractal Sets.* Cambridge Tracts in Mathematics. Cambridge University Press, 1985.

[16] K. J. Falconer and G. R. Grimmett. On the geometry of random cantor sets and fractal percolation. *J. Theoret. Probab.*, 5:465–485, 1992.

[17] K. J. Falconer and G. R. Grimmett. On the geometry of random cantor sets and fractal percolation. correction. *J. Theoret. Probab.*, 7:209–210, 1994.

[18] Kenneth Falconer. *Fractal geometry - mathematical foundations and applications.* Wiley, 1990.

[19] Kenneth Falconer. *Techniques in fractal geometry.* John Wiley & Sons, Ltd., Chichester, 1997.

[20] Alan Frieze and Charalampos E. Tsourakakis. On certain properties of random Apollonian networks. In *Proceedings of the 9th international conference on Algorithms and Models for the Web Graph*, WAW'12, pages 93–112. Springer-Verlag, 2012.

[21] Ronald L. Graham, Jeffrey C. Lagarias, Colin L. Mallows, Allan R. Wilks, and Catherine H. Yan. Apollonian circle packings: number theory. *Journal of Number Theory*, 100(1):1 – 45, 2003.

[22] Michael Hochman. On self-similar sets with overlaps and inverse theorems for entropy. *Ann. of Math. (2)*, 180(2):773–822, 2014.

[23] Remco van der Hofstad. *Random Graphs and Complex Networks*. 2014.

[24] John E. Hutchinson. Fractals and self similarity. *Indiana Univ. Math. J.*, 30 (5):713–747, 1981.

[25] István Kolossváry, Júlia Komjáthy, and Lajos Vágó. Degrees and distances in random and evolving apollonian networks. *Advances in Applied Probability*, 48(3):865–902, 2016.

[26] Benoit B. Mandelbrot. *Fractals*. Freeman San Francisco, 1977.

[27] Benoit B. Mandelbrot. *The fractal geometry of nature*. Freeman, San Francisco, CA, 1982.

[28] Anthony Manning and Károly Simon. Dimension of slices through the sierpinski carpet. *Transactions of the American Mathematical Society*, 365(1):213–250, 2013.

[29] J. M. Marstrand. Some fundamental geometrical properties of plane sets of fractional dimensions. *Proc. Lond. Math. Soc.*, 4:257–302, 1954.

[30] P. Mattila. Hausdorff dimension, orthogonal projections and intersections with planes. *Ann. Acad. Sci. Fenn. A Math.*, 1:227–244, 1975.

[31] R. Daniel Mauldin and S. C. Williams. On the hausdorff dimension of some graphs. *Transactions of the American Mathematical Society*, 298(2):793–803, 1986.

[32] Fedor Nazarov, Yuval Peres, and Pablo Shmerkin. Convolutions of cantor measures without resonance. *Israel Journal of Mathematics*, 187(1):93–116, 2012.

[33] Yuval Peres and Boris Solomyak. Problems on self-similar sets and self-affine sets: an update. In *Fractal Geometry and Stochastics II*, pages 95–106. Springer, 2000.

[34] J. Peyriére. Mandelbrot random beadsets and birthprocesses with interaction. *IBM Research Report*, pages RC–7417, 1978.

[35] M. Rams and K. Simon. Projections of fractal percolations. *Erg. Th. and Dyn. Sys.*, 35:530–545, 2015.

[36] M. Rams and Károly Simon. The dimension of projections of fractal percolations. *J. Stat. Phys.*, 154:633–655, 2014.

[37] Pablo Shmerkin and Boris Solomyak. Absolute continuity of self-similar measures, their projections and convolutions. *Transactions of the American Mathematical Society*, 368:5125–5151, 2015.

[38] Károly Simon and Lajos Vágó. Projections of mandelbrot percolation in higher dimensions. In Christoph Bandt, Michael Barnsley, Robert Devaney, Kenneth J. Falconer, V. Kannan, and Vinod Kumar P.B., editors, *Fractals, Wavelets, and their Applications*, volume 92, pages 175–190. Springer Proceedings in Mathematics & Statistics, 2014.

[39] Károly Simon and Lajos Vágó. Fractal percolations. *Banach Center Publications*, 115:183–196, 2018.

[40] Károly Simon and Lajos Vágó. Singularity versus exact overlaps for self-similar measures. *Proc. Amer. Math. Soc.*, 147(5):1971–1986, 2019.

[41] Zhongzhi Zhang, Lichao Chen, Shuigeng Zhou, Lujun Fang, Jihong Guan, and Tao Zou. Analytical solution of average path length for Apollonian networks. *Phys. Rev. E*, 77:017102, 2008.

[42] Zhongzhi Zhang, Francesc Comellas, Guillaume Fertin, and Lili Rong. High-dimensional Apollonian networks. *Journal of Physics A: Mathematical and General*, 39(8):1811, 2006.

[43] Zhongzhi Zhang, Lili Rong, and Francesc Comellas. High-dimensional random Apollonian networks. *Physica A: Statistical Mechanics and its Applications*, 364:610 – 618, 2006.

[44] Zhongzhi Zhang, Lili Rong, and Shuigeng Zhou. Evolving Apollonian networks with small-world scale-free topologies. *Phys. Rev. E*, 74:046105, 2006.

[45] Tao Zhou, Gang Yan, and Bing-Hong Wang. Maximal planar networks with large clustering coefficient and power-law degree distribution. *Phys. Rev. E*, 71:046141, 2005.