FOURIER ANALYSIS
– EXTREMAL PROBLEMS

Theses for the degree „Doctor of Philosophy”

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I. Overview of the set research tasks

This work deals with two problems from the area of Fourier analysis. Both of them was investigated under the guidance of prof. Szilárd Révész. In the first chapter I demonstrate three-term idempotent counterexamples in the Hardy-Littlewood majorant problem for all $0 < p < 12$ not an even integer. The second chapter covers results of our joint work with Szilárd Révész. We extended the former results on Charathéodory-Fejér type extremal problems from special topological groups to general locally compact Abelian groups, and under certain conditions even to not necessarily Abelian groups.

I.1. Three-term idempotent counterexamples in the Hardy-Littlewood majorant problem. The Hardy-Littlewood majorant problem was raised in the 30’s. We denote, as usual, $T := \mathbb{R}/2\pi\mathbb{Z}$ the one dimensional torus or circle group. Following Hardy and Littlewood [14], $f$ is said to be a majorant to $g$ if $|\hat{g}| \leq \hat{f}$. Obviously, then $f$ is necessarily a positive definite function. The (upper) majorization property (with constant 1) is the statement that whenever $f \in L^p(T)$ is a majorant of $g \in L^p(T)$, then $\|g\|_p \leq \|f\|_p$. Hardy and Littlewood proved this for all $p \in 2\mathbb{N}$ – this being an easy consequence of the Parseval identity. On the other hand Hardy and Littlewood observed that this fails for $p = 3$. Indeed, they took $f = 1 + e_1 + e_3$ and $g = 1 - e_1 + e_3$ (where here and in the sequel we denote $e_k(x) := e(kx)$ and $e(t) := e^{2\pi it}$, as usual) and calculated that $\|f\|_3 < \|g\|_3$.

The failure of the majorization property for $p \notin 2\mathbb{N}$ was shown by Boas [6]. Boas’ construction exploits Taylor series expansion around zero: for $2k < p < 2k + 2$ the counterexample is provided by the polynomials $f, g := 1 + we_{1} \pm wr^{k+2}e_{k+2}$, with $r$ sufficiently small to make the effect of the first terms dominant over later, larger powers of $r$.

Montgomery conjectured that the majorant property for $p \notin 2\mathbb{N}$ fails also if we restrict to idempotent majorants, see [19, p. 144]. (A measure or an integrable function is idempotent if its convolution square is itself: that is, if its Fourier coefficients are all either 0 or 1.) This has been recently proved by Mockenhaupt and Schlag in [18].

1. Theorem (Mockenhaupt & Schlag). Let $p > 2$ and $p \notin 2\mathbb{N}$, and let $k > p/2$ be arbitrary. Then for the trigonometric polynomials $g := (1 + e_k)(1 - e_{k+1})$ and $f := (1 + e_k)(1 + e_{k+1})$ we have $\|g\|_p > \|f\|_p$.

There has been a number of attempts on the Montgomery problem. In particular,Mockenhaupt has already addressed it fifteen years ago, see [17, p. 2, line 15]. Moreover, that time Mockenhaupt worked in the range $2 < p < 4$
and exactly with the polynomials $1 + e_1 \pm e_3$, see also his footnote on p. 32. This attempt is based on an inequality (a discrete and uniform version of the inequality obtained by Hardy and Littlewood only for the continuous case and $p = 3$), which appears in Example 3.4 on p. 33 of [17], with a comment that "This lower bound is established by numerical calculations".

However, there is no convincing argument which would show that this hypothetical inequality would hold for all $p$, and so this preliminary attempt does not lead to a proof. In any case, we may say that Mockenhaupt expressed his view that $1 + e_1 \pm e_{k+2}$, where $2k < p < 2k + 2$, should provide a counterexample in the Hardy-Littlewood majorant problem, (at least for $k = 1, 2$). Our first aim is to analyze this question and execute proper numerical analysis to support this conjecture for $0 < p < 12, p \not\in 2N$. The proof combines delicate calculus with numerical integration and precise error estimates.

I.2. Maximization problems for positive definite functions supported in a given subset of a locally compact group. In this chapter we consider locally compact groups, and the fairly general extremal problem:

2. Problem. Let $\Omega \subset G$ be a given set in the locally compact group $G$ and let $z \in \Omega$ be fixed. Consider a positive definite function $f : G \to \mathbb{C}$ (or $\to \mathbb{R}$), normalized to have $f(0) = 1$ and vanishing outside of $\Omega$. How large can then $|f(z)|$ be?

This extremal problem was investigated in $\mathbb{R}$ and $\mathbb{R}^d$ and for $\Omega$ a 0-symmetric convex body in a paper of Boas and Kac in 1943. Arestov and Berdysheva extended the investigation to $\mathbb{T}^d$, where $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. Kolountzakis and Révész gave a more general setting, considering arbitrary open sets, in all the classical groups above.

Moreover, following observations of Boas and Kac, Kolountzakis and Révész showed how the general problem can be reduced to equivalent discrete problems of "Carathéodory-Fejér type" on $\mathbb{Z}$ or $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$.

The analogous problem of maximizing $\int_{\Omega} f$ under the same hypothesis was recently well investigated by several authors under the name of "Turán’s extremal problem", although later it turned out that the problem was already considered well before Turán, see the detailed survey [48]. The problem in our focus, was also termed by some as "the pointwise Turán problem", but the paper [39] traced it back to Boas and Kac [25] in the 1940’s and even to the work of Carathéodory [26] and Fejér [28] [29, I, page 869] as early as in the
1910’s. For further use we also introduce the extremal problems

\[ \mathcal{M}(\Omega) := \sup \{ a(1) : a : [1, N] \to \mathbb{R}, \ N \in \mathbb{N}, \ a(n) = 0 \ (\forall n \notin \Omega), \]
\[ T(t) := 1 + \sum_{n=1}^{N} a(n) \cos(2\pi nt) \geq 0 \ (\forall t \in \mathbb{T}) \}, \]

which is called in [39] the Carathéodory-Fejér type trigonometric polynomial problem and

\[ \mathcal{M}_m(\Omega) := \sup \{ a(1) : a : \mathbb{Z}_m \to \mathbb{R}, \ a(0) = 1, \ a(n) = 0 \ (\forall n \notin \Omega), \]
\[ T \left( \frac{r}{m} \right) := \sum_{n \mod m} a(n) \cos \left( \frac{2\pi nr}{m} \right) \geq 0 \ (\forall r \mod m) \}, \]

which is termed in [39] as the Discretized Carathéodory-Fejér type extremal problem.

We use the notation \( f \gg 0 \) for the positive definiteness of a function \( f : G \to \mathbb{C} \) or \( G \to \mathbb{R} \). We introduce two extremal, the possibly widest and smallest function classes:

\[ \mathcal{H}_G(\Omega) := \{ f : G \to \mathbb{C} : f \gg 0, \ f(0) = 1, \ f(x) = 0 \ \forall x \notin \Omega \}, \]
\[ \mathcal{F}_G(\Omega) := \{ f : G \to \mathbb{C} : f \gg 0, \ f(0) = 1, \ f \in C(G), \ \text{supp} \ f \subseteq \Omega \}, \]

where \( \subseteq \) denotes compact inclusion. The respective "Carathéodory-Fejér constants" are then

\[ C^\#_G(\Omega, z) := \sup \left\{ |f(z)| : f \in \mathcal{H}_G(\Omega) \right\}, \quad C^c_G(\Omega, z) := \sup \left\{ |f(z)| : f \in \mathcal{F}_G(\Omega) \right\}. \]

By the above general definition, for \( G = \mathbb{Z} \) and \( G = \mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z} \) the Carathéodory-Fejér constants with \( z := 1 \) — and denoting by \( H \) the fundamental set in place of \( \Omega \) in this case — are

\[ C^\#(H) := C^\#_Z(H, 1) := \sup \{ |\varphi(1)| : \varphi \in \mathcal{H}_Z(\mathbb{Z}) \} \]
\[ := \sup \{ |\varphi(1)| : \varphi : \mathbb{Z} \to \mathbb{C}, \ \varphi \gg 0, \ \varphi(0) = 1, \ \text{supp} \ \varphi \subseteq H \}, \]
\[ C^c(H) := C^c_Z(H, 1) := \sup \{ |\varphi(1)| : \varphi \in \mathcal{F}_Z(\mathbb{Z}) \} \]
\[ := \sup \{ |\varphi(1)| : \varphi : \mathbb{Z} \to \mathbb{C}, \ \varphi \gg 0, \ \varphi(0) = 1, \ \text{supp} \ \varphi \subseteq H, \ \# \text{supp} \ \varphi < \infty \}, \]
\[ C_m(H) := C^\#_{\mathbb{Z}_m}(H, 1) = C^c_{\mathbb{Z}_m}(H, 1) := \sup \{ |\varphi(1)| : \varphi \in \mathcal{H}_{\mathbb{Z}_m}(\mathbb{Z}_m) \} \]
\[ := \sup \{ |\varphi(1)| : \varphi : \mathbb{Z}_m \to \mathbb{C}, \ \varphi \gg 0, \ \varphi(0) = 1, \ \text{supp} \ \varphi \subseteq H \}. \]
Let us write similarly to the complex valued case

\[ K^*_G(\Omega, z) := \sup_{\varphi \in \mathcal{F}^*_G(\Omega)} |\varphi(z)|, \quad K_c^c_G(\Omega, z) := \sup_{\varphi \in \mathcal{F}^c_G(\Omega)} |\varphi(z)|, \]

\[ K^*_G(H) = K^*_Z(H, 1), \quad K_c^c_G(H) = K_c^c_Z(H, 1), \]

\[ K_m(H) := K_m(\Omega, 1) := \sup_{\varphi \in \mathcal{F}^c_m(\Omega)} |\varphi(1)|, \]

where naturally we write for any group, (and so in particular for \( G = \mathbb{Z} \) and \( G = \mathbb{Z}_m \))

\[ \mathcal{F}^*_G(\Omega) := \{ \varphi : G \to \mathbb{R} : \varphi \in \mathcal{F}^*_G(\Omega) \}, \]

\[ \mathcal{F}^c_G(\Omega) := \{ \varphi : G \to \mathbb{R} : \varphi \in \mathcal{F}^c_G(\Omega) \}. \]

As mentioned above, the development started with the extremal problem of Carathéodory and Fejér, originally formulated for positive trigonometric polynomials.

3. Theorem (Carathéodory and Fejér). If \( T(t) := 1 + \sum_{n=1}^{N} a(n) \cos(2\pi n t) \geq 0 \ (\forall t \in \mathbb{T}) \), then \( |a_1| < 2 \cos \left( \frac{2\pi}{N+2} \right) \), and the bound is sharp. In other words, \( M([1, N]) = 2 \cos \left( \frac{2\pi}{N+2} \right) \).

For general domains in arbitrary dimension \( d \) the problem was formulated in [39]. With our above notations and general definition we can now recall it simply as follows.

4. Problem (Boas-Kac - type pointwise extremal problem for the space). Find \( K^*_G(\Omega, z) \).

5. Problem (Turán - type pointwise extremal problem for the torus). Find \( K^c_G(\Omega, z) \).

As is easy to see, c.f. [39, Remark 1.4], \( K^c_G(\Omega, z) \geq K^c_G(\Omega, z) \), always.

The extremal value in the above Problem 4. was estimated together with its periodic analogue Problem 5. in the work [22] in dimension \( d = 1 \) for an interval \( \Omega := (-h, h) \). Note that Boas and Kac have already solved the interval (hence dimension \( d = 1 \)) case of Problem 4. in [25], a fact which seems to have been unnoticed in [22].

These problems are not only analogous, but also related to each other, and, in fact, Problem 4. is only a special, limiting case of the more complex Problem 5., see [39, Theorem 6.6]. On the other hand, Boas and Kac have already observed, that Problem 4. (dealt with for \( \mathbb{R} \) in [25]) is connected to trigonometric polynomial extremal problems.
Boas and Kac [25] used the result of Carathéo dory and Fejér to prove the following.

6. **Theorem (Boas-Kac).** Let $\Omega \subset \mathbb{R}^d$ be a convex, symmetric, open, bounded set, i.e. one which generates a corresponding norm $\| \cdot \| = \| \cdot \|_{\Omega}$ on $\mathbb{R}^d$. Consider the Carathéo dory-Fejér extremal problem Problem 2. Then with $\lceil x \rceil$ denoting upper integer part of $x$, we have

$$K^\#_{\mathbb{R}^d}(\Omega, z) = \cos \left( \frac{2\pi}{\lceil 1/\|z\| \rceil + 1} \right).$$

Observe that here $N < 1/\|z\| \leq N+1$ means by convexity and the definition of the norm that $(z) \cap \Omega = \{nz : n \in [-N,N]\}$. Later developments surpassed the assumption of this condition or convexity. Kolountzakis and Révész [39] already considered open symmetric subsets $\Omega \subset \mathbb{R}^d$ or $T^d$.

7. **Theorem (Kolountzakis-Révész).** In $\mathbb{R}^d$ and for any $z \in \mathbb{R}^d$ and $\Omega \subset \mathbb{R}^d$ an open, symmetric neighborhood of $0 \in \mathbb{R}^d$, we have with $H(\Omega, z) := \{k \in \mathbb{Z} : kz \in \Omega\}$ the relation

$$(2) \quad K^\#_{\mathbb{R}^d}(\Omega, z) = K^c_{\mathbb{R}^d}(\Omega, z) = \mathcal{K}(H(\Omega, z)).$$

If $\Omega \subset T^d$ is an open symmetric neighborhood of $0 \in T^d$, and the order of $z$ is infinite (i.e. $z$ has no torsion), then we have with $H(\Omega, z) := \{k \in \mathbb{Z} : kz \in \Omega\}$

$$(3) \quad K^\#_{T^d}(\Omega, z) = K^c_{T^d}(\Omega, z) = \mathcal{K}(H(\Omega, z)).$$

Finally, if the order of $z \in T^d$ is $o(z) = m$, then with $H_m(\Omega, z) := \{k \in \mathbb{Z}_m : kz \in \Omega\}$ we have

$$(4) \quad K^\#_{T^d}(\Omega, z) = K_{T^d}(\Omega, z) = K_m(H_m(\Omega, z)).$$

II. Examinations, methods

II.1. Three-term idempotent counterexamples in the Hardy-Littlewood majorant problem. In the first chapter we examine the functions $F_\pm(x) := 1 + e^{2\pi ix} \pm e^{2\pi i(k+2)x}$, and the integrals $\int_0^1 |F_\pm(x)|^p dx$, where $p \in (2k, 2k+2)$. Let us introduce some notations. We will write $t := p/2 \in [1,2]$ and put $G_\pm(x) := |F_\pm(x)|^2$.

We investigate, as we may in virtue of symmetry on $[-1/2, 1/2]$, the integral $\int_{-1/2}^{1/2} [G_\pm(x) - G_\pm'(x)] dx$.

We apply numerical estimations for integrals, taking in the consideration the calculation error. We approximate the integral in an interval with its expansion around the middle point. The well known Riemann sum approximation
formulae for a function $\Phi$ is
\begin{equation}
\left| \int_0^{1/2} \Phi(x) dx - \frac{1}{2N} \sum_{n=1}^{N} \Phi \left( \frac{n - 1/2}{2N} \right) \right| \leq \min \left( \frac{\|\Phi''\|_\infty}{192N^2}, \frac{\|\Phi'\|_\infty}{16N} \right).
\end{equation}

Investigating the behavior of a function we apply primary function test methods. For better approximation in cases of higher $k$ we improve the numerical integration method by means of invoking a quadrature formula.

8. Lemma. Let $\Phi$ be a four times continuously differentiable function on $[0,1/2]$. Then we have
\begin{equation}
\left| \int_0^{1/2} \Phi(x) dx - \sum_{n=1}^{N} \left\{ \Phi \left( \frac{2n - 1}{4N} \right) \frac{1}{2N} + \Phi'' \left( \frac{2n - 1}{4N} \right) \frac{1}{192N^3} \right\} \right| \leq \frac{\|\Phi''\|_\infty}{60 \cdot 2^{10}N^4}.
\end{equation}

We have to estimate some higher order integrals of $G_\pm$.

9. Lemma. Let $\rho \in \mathbb{N}$ with $1 \leq \rho \leq k + 1$. Then we have
\begin{equation}
G_\rho^\rho = |F_\rho^\rho|^2 = \left| \sum_{\nu=0}^{\rho(k+2)} a_\pm(\nu) e_\nu \right|^2
\end{equation}
with
\begin{equation}
a_\pm(\nu) := (\pm 1)^\mu \left( \frac{\rho}{\mu} \right) \left( \frac{\rho - \mu}{\lambda} \right),
\end{equation}
where $\mu := \left\lfloor \frac{\nu}{k+2} \right\rfloor$ and $\lambda := \nu - \mu(k+2)$ is the reduced residue of $\nu \mod k+2$.

Therefore,
\begin{equation}
\int_0^{1/2} |G_\rho|^{\rho} = \frac{1}{2} \sum_{\nu=0}^{\rho(k+2)} |a_\pm(\nu)|^2.
\end{equation}

In particular, $\int_0^{1/2} |G_\rho|^{\rho} = \int_0^{1/2} |G_{-\rho}|^{\rho}$ for all $0 \leq \rho \leq k + 1$ and thus $d(k) = d(k+1) = 0$.

We have to analyze the functions
\begin{equation}
H(x) := H_{t,j,\pm}(x) := G_{\pm}(x) \log^j G_{\pm}(x) \quad (x \in [0,1/2]) \quad (t \in [k,k+1], j \in \mathbb{N}).
\end{equation}

To find the maximum norm of $H_{t,j,\pm}$, we in fact look for the maximum of an expression of the form $v^t \log v^j$, where $v = G(x)$ ranges from zero (or, if $G \neq 0$, from some positive lower bound) up to $\|G\|_{\infty} \leq 9$. For that, a direct calculus provides the following.
10. Lemma. For any $s > 0$ and $m \in \mathbb{N}$ the function $\alpha(v) := \alpha_{s,m}(v) := v^s |\log v|^m$ behaves on $[0, \infty)$ the following way. It is nonnegative, continuous, continuously differentiable, (apart from possibly 0 in case $s \leq 1$), has precisely two zeroes at 0 and 1, and it has one single critical point $v_0 = \exp(-m/s)$. Consequently, it has exactly one local maximum point at $v_0$ where its local maximum is $(\frac{m}{es})^m$, furthermore, the function increases in $[0, v_0]$ and also on $[1, \infty)$, and decreases on $[v_0, 1]$. Therefore for any finite interval $[a, b] \subset [0, \infty)$ we have

$$\max_{[a,b]} \alpha(v) = \begin{cases} 
\alpha(b) & \text{if } a < b \leq v_0, \\
\alpha(v_0) & \text{if } a \leq v_0 < b \leq 1, \\
\max\{\alpha(v_0), \alpha(b)\} & \text{if } a \leq v_0, 1 < b, \\
\alpha(a) & \text{if } v_0 < a < b \leq 1, \\
\max\{\alpha(a), \alpha(b)\} & \text{if } v_0 < a < 1 < b, \\
\alpha(b) & \text{if } 1 \leq a < b.
\end{cases}$$

(11)

To apply the quadrature formula, we need the higher (till 4) derivatives of $H_{t,j,\pm}(x)$.

$$H^{(n)} = \sum_{m=0}^{n} \binom{n}{m} (G^t)^{(m)} (\log^j G)^{(n-m)}.$$  

(12)

In case of $k = 5$ we combine the above fourth order quadrature also with total variation. Denote $\text{Var}(\psi, [a,b])$ the total variation of the function $\psi$ on $[a,b]$, and in particular let $\text{Var}(\psi) := \text{Var}(\psi, \mathbb{T})$, we formulate here

11. Corollary. Denote by $Z := Z_\pm$ the set of local maximum points of $G = G_\pm$. For any positive parameter $t > 0$ we have $\text{Var}(G^t) < 2 \sum_{\zeta \in Z} G^t(\zeta)$. In particular, if $k = 5$, $\text{Var}(G_\pm) < 74$.

12. Lemma. Let $B_r, D_r > 0; 1 \leq t_r \leq T$ and $j_r \geq 0$ for $r = 0, 1, \ldots, R$. Assume

$$|\varphi^{IV}(x)| \leq \sum_{r=0}^{R} \left\{ B_r G^{t_r}(x) \cdot |\log G(x)|^{j_r} + D_r G^{t_r}(x) |G'(x)| \cdot |\log G(x)|^{j_r} \right\}.$$  

(13)
Then for arbitrary $N \in \mathbb{N}$ the quadrature formula
\begin{equation}
\left| \int_0^{1/2} \phi - \sum_{n=1}^N \left\{ \psi \left( \frac{2n-1}{4N} \right) \frac{1}{2N} + \psi'' \left( \frac{2n-1}{4N} \right) \frac{1}{192N^3} \right\} \right| 
\leq \frac{1}{60 \cdot 2^{10} N^5} \sum_{r=1}^R \left\{ B_r Q_N(G, t_r, j_r) + D_r Q_N^*(G, t_r, j_r) \right\}
\end{equation}
holds true with
\begin{equation}
Q_N(G, t, j) := \chi(j \neq 0) \left( \max_{[0,1/9]} v^j \log v^j \right) N + \log^j 9 \left\{ N \int_T G^t + \frac{1}{2} \text{Var}(G^t) \right\}
\end{equation}
and
\begin{equation}
Q_N^*(G, t, j) := \chi(j \neq 0) \left( \max_{[0,1/9]} v^j \log v^j \right) \left\{ \frac{14}{9} N + 1700 \right\}
\end{equation}
(16) + \log^j 9 \left\{ \frac{N}{t+1} \text{Var}(G^{t+1}) + 88 \text{Var}(G^t) + 1700 \sqrt{\int_T G^{2t}} \right\}.

II.2. Maximization problems for positive definite functions supported in a given subset of a locally compact group. In the second chapter we use the properties of positive definite functions. For general $G$ group positive definite functions are defined by the property that
\begin{equation}
\forall n \in \mathbb{N}, \forall x_1, \ldots, x_n \in G, \forall c_1, \ldots, c_n \in \mathbb{C} \quad \sum_{j=1}^n \sum_{k=1}^n c_j c_k^* f(x_j x_k^{-1}) \geq 0.
\end{equation}
Properties $(f : G \to \mathbb{C})$: $f(e) \geq 0$, where $e$ is the unit element of $G$; $f = \tilde{f}$, where $\tilde{f}(x) := f(x^{-1})$; $|f(z)| \leq f(e)$ for all $z \in G$. All characters $\gamma \in \hat{G}$ of a LCA group $G$ are positive definite.

13. Lemma. Let $f, g : G \to \mathbb{C}$ be arbitrary positive definite functions. Then we have

(i) If $H \leq G$ is a subgroup of $G$, and $h := \chi_{HG}$, that is, $g|_H$ on $H$ and vanishing elsewhere, then also $h \gg 0$.

(ii) $\tilde{f} \gg 0$, $f^*(x) := f(x^{-1}) \gg 0$ and $\Re f \gg 0$.

(iii) If $\alpha, \beta > 0$ are arbitrary positive constants, then $\alpha f + \beta g \gg 0$.

(iv) For arbitrary $n \in \mathbb{N}$, complex numbers $a_j \in \mathbb{C}$ ($j = 1, \ldots, n$) and group elements $y_j \in G$ ($j = 1, \ldots, n$), the derived function $F(x) := \sum_{j=1}^n \sum_{k=1}^n a_j a_k f(y_j^{-1} x y_k) \gg 0$. 
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(v) \( fg \gg 0 \).

14. Theorem (Herglotz). Let \( \psi : \mathbb{Z} \to \mathbb{C} \) be a sequence on \( \mathbb{Z} \). Then \( \psi \gg 0 \) (i.e. \( \psi \) is positive definite) if and only if there exists a positive Borel measure \( \mu \) on \( \mathbb{T} \) such that

\[
\psi(n) = \int_{\mathbb{T}} e^{2\pi i n t} d\mu(t) \quad (n \in \mathbb{Z}).
\]

Furthermore, in case \( \text{supp} \psi \subset [-N; N] \) we have \( \psi \gg 0 \) if and only if \( T(t) := \bar{\psi}(t) = \sum_{n=-N}^{N} \psi(n) e^{2\pi i nt} \geq 0 \) \( (t \in \mathbb{T}) \), and then \( d\mu(t) = T(-t) dt \) and \( \psi(n) = \int_{\mathbb{T}} T(t)e^{-2\pi i nt} dt \).

Fundamental tool in topological groups is the Haar measure, which is a non-negative regular and translation invariant Borel measure \( \mu_G \), existing and being unique up to a positive constant factor in any LCA group, see [49, p. 1,2]. As a direct consequence of uniqueness, we also have \( \mu_G(E) = \mu_G(-E) \) for all Borel measurable set \( E \), [49, 1.1.4].

We consider the convolution of functions with respect to the Haar-measure \( \mu_G \), that is

\[
(f \ast g)(x) := \int_G f(y)g(x-y)dy = \int_G f(x+z)g(-z)dz
\]

defined for all functions \( f, g \in L^1(\mu_G) \), or pairs of functions \( f \in L^p(\mu_G), g \in L^q(\mu_G) \) with \( 1/p + 1/q = 1 \).

We also consider convolution of (bounded, complex valued, regular Borel) measures and convolution of such measures and functions as well.

\[
\mu \ast \lambda(E) = \int_G \mu(E-y)d\lambda(y).
\]

One can equivalently define convolution of measures by the relation

\[
\int_G f d(\mu \ast \lambda) := \int_G \int_G f(x+y)d\mu(x)d\lambda(y)
\]

\[
= \int_G \int_G f(x+y)d\lambda(y)d\mu(x) \quad (f \in L^\infty(G)).
\]

Also convolutions of measures with functions or functions with measures can be obtained the same way. It is easy to see that for any \( f \in L^1(\mu_G) \) and \( \nu \) a measure on \( G \), we have the formula

\[
f \ast \nu(x) = \nu \ast f(x) = \int_G f(x-y)d\nu(y).
\]

Special cases: for any \( u, v \in G \) the formula \( \delta_u \ast \delta_v = \delta_{u+v} \) holds true (where \( \delta_u \) denotes the Dirac measure (unit point mass) at \( u \in G \)): for \( \int_G \phi d(\delta_u \ast \delta_v) = \)
\[ \int_G \int_G \phi(x+y)d\delta_u(x)d\delta_v(y) = \phi(u+v) = \int_G \phi d\delta_{u+v}. \]

Also, if \( \phi \in L^\infty(G) \) and \( u \in G \), then we have in view of (23)

\[ \delta_u \ast \phi(x) = \int_G \phi(x-y)d\delta_u(y) = \phi(x-u). \]

If for some Borel measurable \( A \) we take the characteristic function, \( \phi := \chi_A \), we obtain similarly

\[ \delta_u \ast \chi_A(x) = \chi_A(x-u) = \chi_{A+u}(x). \]

For \( \chi_A, \chi_B \) with \( A, B \) Borel measurable sets with finite measure, these lead to

\[ \chi_A \ast \chi_B(x) = \int_G \chi_A(y)\chi_B(x-y)dy = \int_G \chi_A\chi_{(x-B)}d\mu_G \]

\[ = \int_G \chi_{A \cap (x-B)}d\mu_G = \mu_G(A \cap (x-B)). \]

We also use the special cases of the following lemmas:

15. Lemma. Let \( f \in L^2(\mu_G) \) be arbitrary. Then the "convolution square" of \( f \) exists, moreover, it is a continuous positive definite function, that is, \( f \ast \tilde{f} \geq 0 \) and belongs to \( C(G) \).

As a converse in case of \( \Z \), a consequence of the Riesz-Fejér Theorem is the next.

16. Lemma. Let \( \psi : \Z \to \C \) be a finitely supported positive definite sequence. Then there exists another sequence \( \theta : \Z \to \C \), also finitely supported, such that \( \theta \ast \tilde{\theta} = \psi \). Moreover, if \( \text{supp} \psi \subset [-N, N] \), then we can take \( \text{supp} \theta \subset [0, N] \).

A slightly less strict analog of the existence of a convolution square-root also holds in \( \Z_m \).

17. Lemma. If \( \psi : \Z_m \to \C, \psi \geq 0 \) on \( \Z_m \), then there exists \( \theta : \Z_m \to \C \) with \( \theta \ast \tilde{\theta} = \psi \).

In case of non-Abelian groups we extend the results under certain conditions, involving the following new definition.

18. Definition. We say that \( z \in G \) is a round element of the locally compact group \( G \), if for all open neighborhood \( U \) of the unit \( e \) there exists another open neighborhood \( V \subset U \) of \( e \) such that \( \mu_G(zVz^{-1}\triangle V) = 0 \). Furthermore, the group \( G \) itself is called round, if all elements \( z \in G \) are round according to the above.

Note that the symmetric difference of two subsets of \( A, B \subset G \) are defined by
\[ A \triangle B := (A \setminus B) \cup (B \setminus A) \]
19. Proposition. A point $z \in G$ is round if and only if for any neighborhood $U$ of the unit element $e$ there exists a nonzero, continuous, positive definite function $f \gg 0$ such that $\text{supp} f \subseteq U$ and $f$ is invariant under conjugation by $z$: $f(z^{-1}xz) = f(x)$ for all $x \in G$.

III. New scientific results

III.1. Three-term idempotent counterexamples in the Hardy-Littlewood majorant problem. We discuss the following reasonably documented conjecture.

1.1.1. Conjecture. For all $p$ not an even integer, there are three-term idempotent counterexamples in the Hardy-Littlewood majorant problem.

In fact, we address the more concrete form, going back to the examples of Hardy-Littlewood and Boas and discussed also by Mockenhaupt [17].

1.1.2. Conjecture. Let $2k < p < 2k + 2$, where $k \in \mathbb{N}$ arbitrary. Then the three-term idempotent polynomial $P_k := 1 + e_1 + e_{k+2}$ has smaller $p$-norm than $Q_k := 1 + e_1 - e_{k+2}$.

We solve this problem for $k = 0, \ldots, 5$.

1.2.1. Proposition. Let $F(x, y) := e(4y) + e(x + 2y) + e(2x + y)$. Then, for $p > 2$, taking the marginal integral function $f(y) := f_p(y) := \int_0^1 |F(x, y)|^pdx$, we have that (mod 1) $f$ has a unique, strict maximum at 0. Conversely, for $0 < p < 2$ it has strict global maximum at $\frac{1}{2}$.

1.2.1. Remark. Note that $f_p(0) < f_p(1/2)$ for $0 < p < 2$ is exactly Conjecture 1.1.2 for $k = 0$.

For other values of $k$ we solve the problem by numerical methods.

1.3.1. Proposition. Let $F_{\pm}(x) := 1 + e(x) \pm e(3x)$ and consider the $p$th marginal integrals $f_{\pm}(p) := \int_0^1 |F_{\pm}(x)|^pdx$ as well as their difference $\Delta(p) := f_-(p) - f_+(p) = \int_0^1 |F_-(x)|^p - |F_+(x)|^pdx$. Then for all $p \in (2, 4)$, $\Delta(p) > 0$.

1.4.1. Proposition. Let $F_{\pm}(x) := 1 + e(x) \pm e(4x)$ and consider the $p$th marginal integrals $f_{\pm}(p) := \int_0^1 |F_{\pm}(x)|^pdx$ as well as their difference $\Delta(p) := f_-(p) - f_+(p) = \int_0^1 |F_-(x)|^p - |F_+(x)|^pdx$. Then for all $p \in (4, 6)$, $\Delta(p) > 0$.

Similarly, the cases $k = 3, 4, 5$ are also proved in the dissertation.
III.2. Carathéodory-Fejér type extremal problems. First we extend the theorem of Kolountzakis and Révész to all LCA groups. Note that these theorems are equivalence statements, which greatly decrease the complexity of the problems when reducing them to given extremal problems on $\mathbb{Z}$ or $\mathbb{Z}_m$, yet they do not necessarily solve them in the sense of providing the exact numerical value of the extremal quantity. For points $z \in G$ with infinite order the problem becomes equivalent to the trigonometric polynomial extremal problem of the sort (1).

2.5.1. Theorem. Let $G$ be any locally compact Abelian group and let $\Omega \subset G$ be an open (symmetric) neighborhood of 0. Let also $z \in \Omega$ be any fixed point with $o(z) = \infty$, and denote $H(\Omega, z) := \{k \in \mathbb{Z} : kz \in \Omega\}$. Then we have
\[ C^c_G(\Omega, z) = K^c_G(\Omega, z) = C^\#_G(\Omega, z) = K^\#_G(\Omega, z) = C(H(\Omega, z)). \] (28)

2.5.1. Corollary. For $G$ any locally compact Abelian group, $\Omega \subset G$ any open (symmetric) neighborhood of 0, and $z \in \Omega$ any fixed point with $o(z) = \infty$, we have $C^c_G(\Omega, z) = K^c_G(\Omega, z) = C^\#_G(\Omega, z) = K^\#_G(\Omega, z)$, the common value of which can thus be denoted simply by $C_G(\Omega, z)$.

If $z \in G$ is cyclic (has torsion), the situation is analogous:

2.5.2. Theorem. Let $G$ be any locally compact Abelian group and let $\Omega \subset G$ be an open (symmetric) neighborhood of 0. Let also $z \in \Omega$ be any fixed point with $o(z) = m < \infty$, and denote $H_m(\Omega, z) := \{k \in \mathbb{Z}_m : kz \in \Omega\}$. Then we have
\[ C^\#_G(\Omega, z) = C^c_G(\Omega, z) = C_m(H_m(\Omega, z)) \]
and
\[ K^\#_G(\Omega, z) = K^c_G(\Omega, z) = K_m(H_m(\Omega, z)). \]

2.5.2. Corollary. For $G$ any locally compact Abelian group, $\Omega \subset G$ any open (symmetric) neighborhood of 0, and $z \in \Omega$ any fixed point with $o(z) < \infty$, we still have $C^c_G(\Omega, z) = C^\#_G(\Omega, z)$ and $K^c_G(\Omega, z) = K^\#_G(\Omega, z)$, the common value of which can thus be denoted by $C_G(\Omega, z)$ and $K_G(\Omega, z)$, respectively.

In case of non-Abelian groups we assume some restrictions.

2.6.1. Condition. Assume that there is a natural number $N \in \mathbb{N}$ satisfying
\[ \langle z \rangle \cap \Omega = \{z^n : -N \leq n \leq N\}, \text{ where } \langle z \rangle := \{z^n : n \in \mathbb{Z}\}. \] (29)

To the best of our knowledge, this is the first attempt to deal with such extremal problems – including other extremal problems of the "Turán type", as
mentioned above – in the wider generality of not necessarily Abelian locally compact groups.

First, we can formulate a direct generalization of the Carathéodory-Fejér result with the same conditions holding in the general group \( G \).

2.6.2. Theorem. Let \( G \) be any locally compact topological group, with unit element \( e \) and let \( \Omega \subset G \) be an open (symmetric) neighborhood of \( e \). Let also \( z \in \Omega \) be any fixed point with \( o(z) = \infty \), and assume that Condition 2.6.1 is satisfied with a certain \( N \). Then we have

\[
C_G^c(\Omega, z) = C_G^\#(\Omega, z) = K_G^c(\Omega, z) = K_G^\#(\Omega, z) = \cos \left( \frac{2\pi}{N+2} \right).
\]

If \( z \in G \) is cyclic (has torsion), the next result is an extension of Theorem 2.5.2 to not necessarily commutative locally compact groups in case Condition 2.6.1 holds.

2.6.3. Theorem. Let \( G \) be any locally compact topological group, with unit element \( e \) and let \( \Omega \subset G \) be an open (symmetric) neighborhood of \( e \). Let also \( z \in \Omega \) be any fixed point with \( o(z) = m < \infty \), and assume that Condition 2.6.1 is satisfied with some \( N \leq m \). Then we have

\[
C_G^c(\Omega, z) = C_G^\#(\Omega, z) = C_m([-N, N])
\]

and

\[
K_G^c(\Omega, z) = K_G^\#(\Omega, z) = K_m([-N, N]).
\]

For \( o(z) = \infty \) we get the exact value \( \cos \left( \frac{2\pi}{N+2} \right) \) of the extremal constant, while for \( o(z) = m < \infty \) we only obtain an equivalence, but not the concrete value.

In order to formulate some other condition than Condition 2.6.1, we use the notion of roundness, defined above.

2.9.1. Theorem. Let \( G \) be any locally compact topological group, with unit element \( e \) and let \( \Omega \subset G \) be an open (symmetric) neighborhood of \( e \). Let also \( z \in \Omega \) be any torsion-free point (i.e. \( o(z) = \infty \)) which is round in the sense of Definition 18. Then with \( H := H(\Omega, z) := \{ k \in \mathbb{Z} : z^k \in \Omega \} \) we have

\[
C_G^c(\Omega, z) = C_G^\#(\Omega, z) = K_G^c(\Omega, z) = K_G^\#(\Omega, z) = \mathcal{C}(H).
\]

This result extends Theorem 2.5.1.

2.9.1. Corollary. For \( G \) a locally compact group, \( \Omega \subset G \) an open (symmetric) neighborhood of \( e \), and \( z \in \Omega \) any fixed round point with \( o(z) = \infty \), we have \( C_G^c(\Omega, z) = K_G^c(\Omega, z) = C_G^\#(\Omega, z) = K_G^\#(\Omega, z) \), the common value of which can thus be denoted simply by \( C_G(\Omega, z) \).
If \( z \in \Omega \) is a cyclic (torsion) element, then the situation is described by the next result.

**2.9.2. Theorem.** Let \( G \) be any locally compact topological group, with unit element \( e \) and let \( \Omega \subset G \) be an open (symmetric) neighborhood of \( e \). Let also \( z \in \Omega \) be any cyclic point with \( o(z) = m < \infty \), and let \( H_m := H_m(\Omega, z) := \{ k \in \mathbb{Z}_m : z^k \in \Omega \} \). Then we have

\[
C^c_G(\Omega, z) = C^\#_G(\Omega, z) = C_m(H_m) \quad \text{and} \quad K^c_G(\Omega, z) = K^\#_G(\Omega, z) = K_m(H_m).
\]

Note that here we did not assume any extra condition. Still, we obtain the full strength of the result, and thus an unconditional extension of the result of Theorem 2.5.2. Dropping the roundedness condition, assumed for torsion-free \( z \), is possible here, for we have proved that any cyclic element is round.

**2.9.2. Corollary.** For \( G \) a locally compact group, \( \Omega \subset G \) an open (symmetric) neighborhood of \( e \), and \( z \in \Omega \) any fixed point with \( o(z) < \infty \), we still have \( C^c_G(\Omega, z) = C^\#_G(\Omega, z) \) and \( K^c_G(\Omega, z) = K^\#_G(\Omega, z) \), the common value of which can thus be denoted by \( C_G(\Omega, z) \) and \( K_G(\Omega, z) \), respectively.

**2.10.1. Corollary.** Let \( G \) be any locally compact group and let \( \Omega \subset G \) be an open (symmetric) neighborhood of \( e \). Let also \( z \in \Omega \) be any fixed point with \( o(z) = m \), and denote \( H_m(\Omega, z) := \{ k \in \mathbb{Z}_m : z^k \in \Omega \} \). Then we have

\[
\cos(\pi/m)C_G(\Omega, z) \leq K_G(\Omega, z) \leq C_G(\Omega, z).
\]

Also the "\( m = \infty \)" case holds true at least for \( z \in G \) round. That is, for round and torsion-free elements \( z \in \Omega \) we have

\[
K_G(\Omega, z) = C_G(\Omega, z).
\]

The known duality results on \( \mathbb{Z} \) can be extended to the more general situation.

**2.10.2. Corollary.** For any locally compact group \( G \), open set \( \Omega \subset G \) and round (in particular, any cyclic) \( z \in \Omega \) we have \( K(\Omega)K(\Omega^*) = \frac{1}{2} \) where \( \Omega^* \) is any symmetric open set with \( Z \cap \Omega \cap \Omega^* = \{0, z, -z\} \) and \( (\Omega \cup \Omega^*) \supset Z \), with \( Z := \langle z \rangle \), i.e. \( \{ z^k : k \in \mathbb{Z} \} \) or \( \{ z^k : k \in \mathbb{Z}_m \} \), respectively.
IV. Bibliography

IV.1. References in Chapter 1


**IV.2. References in Chapter 2**


IV.3. **THE DISSERTATION IS BASED ON THE AUTHOR’S FOLLOWING PAPERS**


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