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THE ROLE OF RADIATION DAMPING IN THE DYNAMIC RESPONSE OF STRUCTURES

SUMMARY

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02 January 2020

1. Introduction

In the thesis dynamic analysis of different systems is investigated. First, the study of soil-structure interaction, then the analysis of long rectangular floors is presented. These two systems may seem entirely different, but their behaviour and mathematical representation are similar. In both cases radiation damping occurs, which means energy dissipation in the system due to the wave propagation towards infinity [1]. This phenomenon causes different behaviour than those of discrete systems, therefore the dynamic response will be significantly different than the response of simple mass-spring-dashpot models.

Soil-structure interaction can be taken into account in different ways (Fig. 1). The deformability of the soil can be considered by using elastic support, or spring and dashpot elements. The most accurate approach is the direct method (Fig. 1a), where the soil and the structure are modelled together. In this case nonlinearities can also be considered, however it requires significant computational effort.

The simplified spring-dashpot models are derived from the impedance function of soil with a weightless foundation [1], where the foundation is excited by a harmonic force. The ratio of this force and the displacement of the foundation is the impedance function. This function consists of an amplitude and a phase angle (the shift of the force and displacement). The impedance function can be interpreted as a frequency dependent spring and dashpot elements (Fig. 1b). Its applicability is rather complicated in time domain, therefore to simplify the procedure, the frequency dependent characteristics are often approximated by constant values in practical design [2] (usually the initial values [3]), a spring and dashpot is used for every direction (Fig. 1c). There are more complex models to consider the SSI, more spring, dashpot and mass elements can be used for the different directions (Fig. 1d).

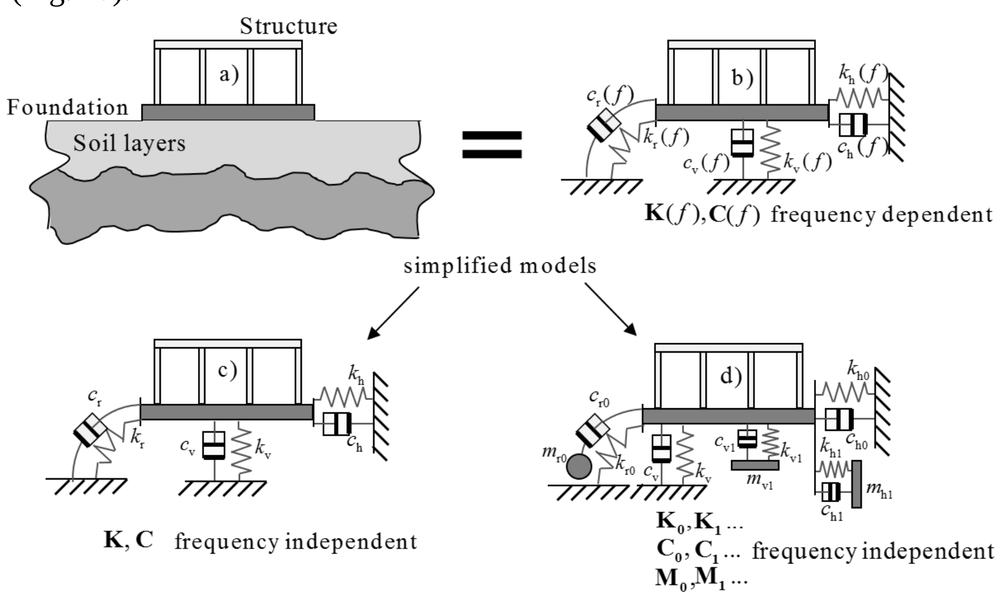


Fig. 1 The modelling levels of soil effect: a) direct approach, b) impedance function, c) one, frequency independent spring and dashpot element for all directions, d) more complex lumped models

Vibration control is a major consideration in the design of light-weight floors [4]. The response (acceleration or speed) of structures subjected to human- or machine-induced vibration is compared to the tolerance limit of human comfort [5–8]. Several recent publications deal with the analysis and design of rectangular plates for both static and dynamic loads [9–12].

The usual design procedure for long plate is to consider a beam with an effective width, than calculate the response similarly as in case of an SDOF system [13]. For example, for the interior part of steel beam composite floors, based on measurements, the following expression is recommended [5, 14]: $b_{\mathrm{eff}} = 2\sqrt[4]{D_{11}/D_{22}}L_y$, with the upper limit of $b_{\mathrm{eff}} \le 2/3 L_x$, where L_y is the span in the joist (y) direction, L_x is the span in the x direction, while D_{11} and D_{22} are the bending stiffnesses for unit width in the x and y directions, respectively. Note, however, that in the excellent Review article [15][15] the authors state that „*the origin of these formulae is not clear, but they mostly appear to be empirical (from parametric FEA) rather than by first principles and they lack consistency across the various forms*".

Long plates (and not long orthotropic rib-stiffened plates) behave in a significantly different manner than beams or simple SDOF mass–spring–damper systems. As a consequence, designs based on effective width may lead to 2–4 times higher accelerations than the real values.

2. Problem statement

As it was stated in the Introduction in practical earthquake design to take into account the effect of SSI, usually simple spring-dashpot elements are used. The spring-dashpot element (chosen for example according to the cone-model [2]) with constant characteristics are applicable to the half-space, but when the soil layer is finite in the vertical direction, the layer has resonance points, therefore its dynamic response is significantly different than the response of a spring-dashpot element. According to this, the following questions arise:

- what is the significance and effect of the resonance of the layer,
- what is the maximum error, which can occur by neglecting it,
- in what soil-parameter range can a single spring-dashpot model be used, and in what range it cannot.

After we detect the range where the simple constant-parameter model cannot be used, the goal is to develop a simplified model, which is based on the real physical behaviour (and not on the numerical fitting of parameters) and works in this range. First the problem of a 2D case, a regular soil layer with strip foundation is analysed, for this case we wish to develop a simple model:

- which physically behaves similarly to the soil layer,
- its response is able to produce the same phenomenon (like the presence of cut-off frequency),
- is based on the physical representation of the system,
- its parameters can be calculated by simple formulas.

In case of the vibration of long rectangular floors similar phenomena occurs then in case of a horizontally infinite soil layers. Low-frequency floors are considered, where the harmonic response should be considered in the design. We wish to:

- analyse the harmonic response of long rectangular floors (in case of orthotropic plates the plate can be mechanically long, even if the ratio of it sides is not much bigger than one),
- give the mathematical (physical) representation of the system, which includes the radiation damping,
- compare the solution to the currently available design guides (which are based on the response of an SDOF system).

3. Dynamic response of different systems

The impedance function of an SDOF system, an axially constrained infinite bar, and an infinite beam on elastic foundation is investigated. The impedance function of the infinite systems is derived by Fourier series expansion.

These systems behave significantly differently as a consequence of radiation damping. The maximum values of the amplification factor (which is the amplitude of the impedance function multiplied with the static stiffness) for different damping values are significantly different in the three cases ($D_{0,r}=1/2\xi$ for SDOF systems, $D_{\infty,r,beam}=1/(2\xi)^{0.75}$ for infinite beam on elastic foundation and $D_{\infty,r,bar}=1/(2\xi)^{0.5}$ for axially constrained infinite bar). In case of 1% damping the amplification factor of an SDOF system is more than 7 times bigger, than the amplification factor of an axially constrained infinite bar. The amplification factor and phase angle of the three systems is given in Fig. 2 for ξ=2%.

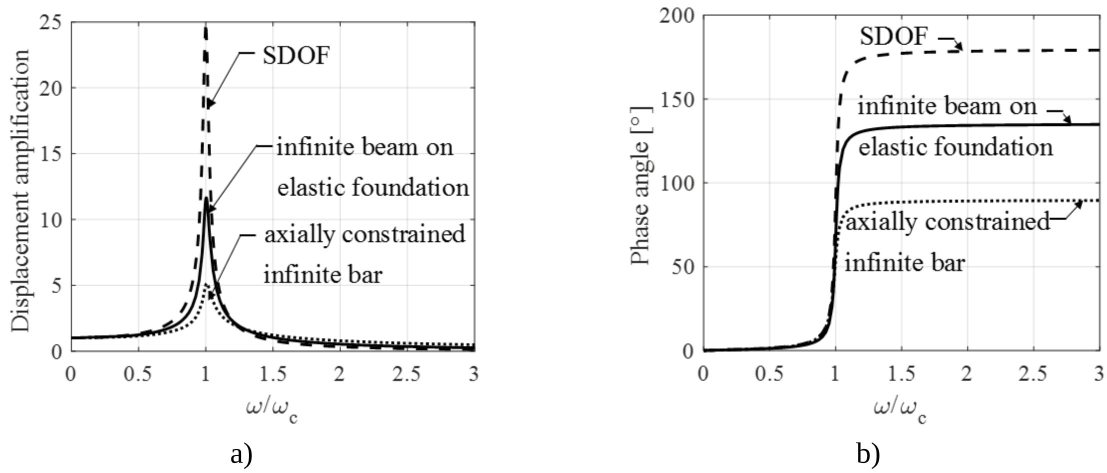


Fig. 2 Amplification factor (a) and phase angle (b) of the SDOF system, axially constrained infinite bar and beam on elastic foundation with damping, ξ=2%

4. Soil-structure interaction

In soil-structure interaction (SSI) there are several modelling methods. For the earthquake resistant design of structures, usually simple mass-spring-dashpot models are used. In this chapter first the effect of resonance of finite soil layers is analysed, and the feasibility of these simple models is discussed. Then a new model, based on the axially constrained infinite bar is given.

4.1. Significance of infinite dimensions and resonance in case of SSI

In case of practical earthquake resistant design the applicability of frequency dependent impedance function is very limited, because its complexity. Rather, engineers are applying constant spring stiffnesses and damping values according to one of the formulas in the literature [16] which are based on the impedance function of a soil half-space, or to calculate a constant spring stiffness by static finite element analysis. None of these are taking into account the possible resonance which may occur in case of the dynamic loading of a finite soil layer.

We investigate the significance of the effect of resonance, and to determine the maximum error, which may occur neglecting it in the design process.

A finite soil layer with a rigid foundation (Fig. 3a) and the simplified (spring-dashpot) models (Fig. 3b) are analysed numerically to determine the effect of resonance. The analyses are limited to horizontal and rocking motion of the foundation. The numerical analysis was performed by the ANSYS computer code. Harmonic and time-history analyses were executed, and different signals were investigated (harmonic excitation, real and artificial earthquake records).

The effect of the different soil parameters on the impedance function of a soil layer is investigated, first for horizontal, then for rocking motion. The curves coincide, when the horizontal axis is normalized by h/v_s, where the h is the thickness of the soil layer, v_s is the shear wave velocity in the soil, and the vertical axis is normalized by the static stiffness. The different impedance curves and their comparison to an SDOF simplified model is given in the dissertation, the curves are given for different b/h ratios, where b is the half width of the strip foundation.

a) b)

Fig. 3 a) Finite soil layer with thickness h and rigid foundation, b) simplified model with constant spring stiffness and dashpot element

As it is shown in the dissertation significant error may occur when the simplified model is used and when the dominant frequency of the earthquake is close to the first eigenfrequency of the soil-structure system. The frequency content of typical earthquakes (analysing the 44 far-field record of [17]) is in the range of $0.45<f<2.82$ 1/s. Fig. 4 shows the resonance-sensitive zones according to the dominant frequency content of the analysed records of the h and v_s parameters of the soil for different masses (m).

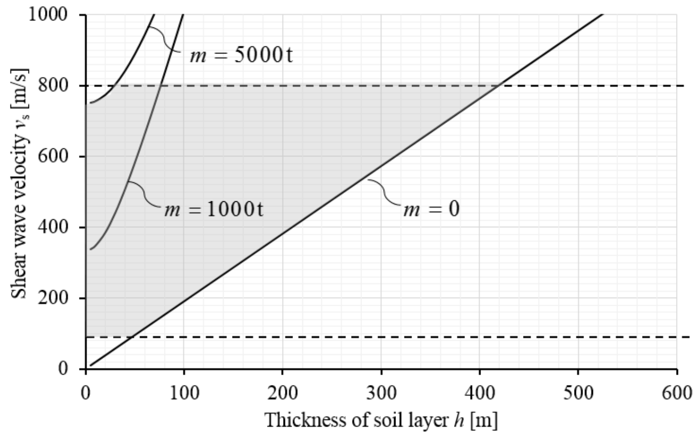


Fig. 4 Parameter range (v_s and h), where the natural frequency of the system is in the range of the dominant frequency of earthquakes

4.2. New simple model in SSI to take into account the radiation damping and resonance

In this chapter a rigid object resting on the surface of the ground is considered. The object is infinite in one (y) direction (e.g. strip foundation). The dimension of the soil in the vertical direction is finite, the height is denoted by h while it can be infinite or finite (l) in the x direction (Fig. 3a). The rock under the soil is excited by earthquakes on lines z=0 and $x=\pm l/2$. The slip between the rigid foundation and the soil layer and between the soil and the rock is neglected.

The differential equation of the approximate model of the layer is derived by Rayleigh-Ritz method (Fig. 5). It may be observed that this is equivalent to the differential equation of an axially constrained infinite bar, only the constants are different. With the aid of these constants the soil layer can be represented as an axially constrained infinite bar.

Furthermore, to consider the effect of the size of the foundation. by matching the potential energy of the two systems, we obtain an axially constrained bar, with a spring and mass at the end (Fig. 5 c), the formula for the spring constant and mass is given in the dissertation.

When $2b \ll h$ (approximately b<5h), the SDOF system at the end of the bar can be neglected.

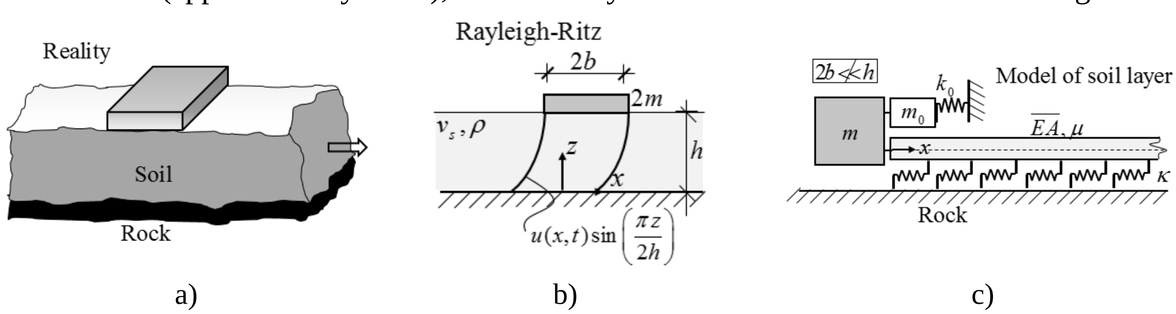
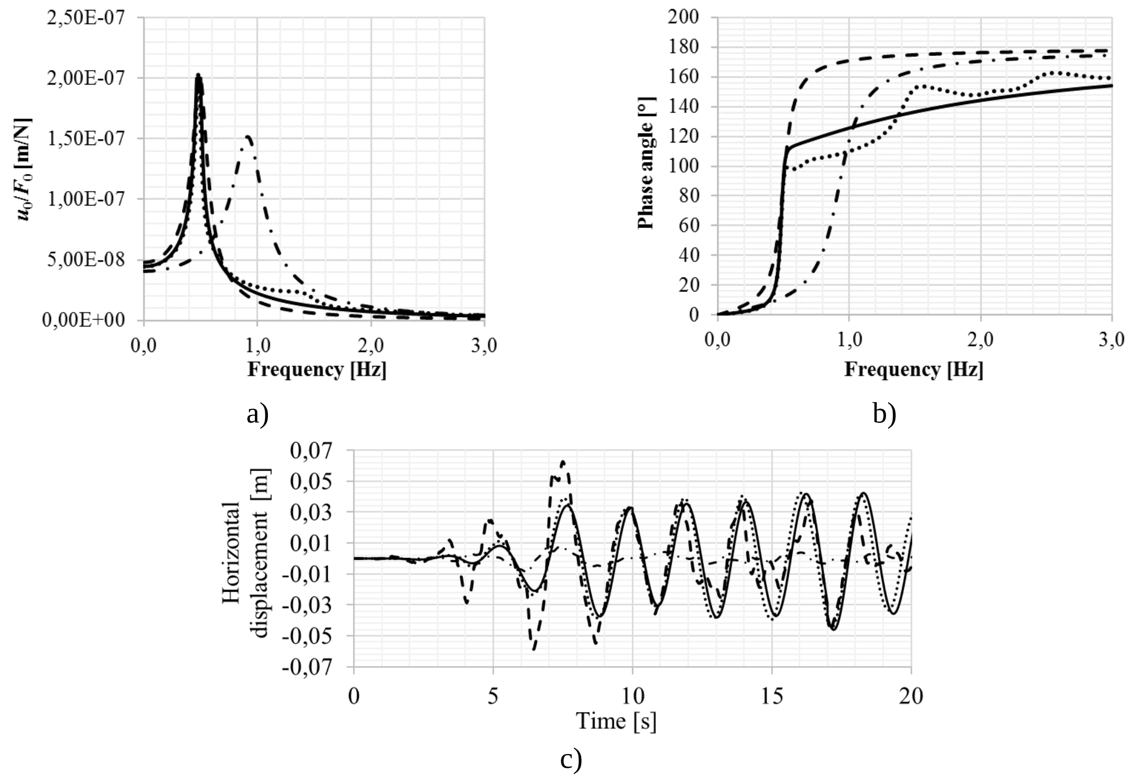


Fig. 5 Simplified models of a soil layer with an object, a) reality, b) 2D model, c) 1D model

The mathematical derivation of the DE, the steady-state and transient solutions for harmonic force and base excitations are given in the dissertation. The eigen frequency of the system is derived analytically and also with the Dunkerley approximation.

As an example we consider a flat, rigid structure. The total mass is $2m$=720 t, the width is $2b$=20 m, and it is resting on a soil layer with a total thickness of h=50 m and shear wave velocity v_s=100 m/s^2 (Fig. 5b). The parameters of our replacement model for one meter width are calculated by the derived formulas. The impedances of the direct approach and our simple model (Fig. 5c) are compared to the impedances of SDOF systems (Fig. 3b). First the spring and dashpot elements are calculated according to the literature [18], then the spring and dashpot element and an additional mass are calculated in such a way, that the resonance frequency and the static stiffness of the SDOF system is the same as those of the soil layer. The curves without damping are showed in the dissertation, with damping in Fig. 6a and b. The differences in the models are also presented by time-history analysis for an earthquake record [17] in Fig. 6c.



········ FE model ——— present model - - - - spring-dashpot model (by matching the resonance point)
— · — spring-dashpot [18]

Fig. 6 a) amplitude, b) phase angle of the impedance function with damping, c) solution of the different models for an earthquake record ([17] record no. 32)

4.3. Identification of model parameters of the new model in SSI

In this chapter, again, a rigid object resting on the surface of the ground is investigated. Both 2D (e.g. strip foundations) and 3D problems (e.g. shallow foundations) are considered. The depth of the soil above the rock is finite and may vary with the horizontal coordinates, while the horizontal dimensions can be infinite or finite (Fig. 7). The rock under the soil is excited by earthquakes and the horizontal response of the structure is investigated. Our hypothesis is that the soil can be reasonably well represented by the simple model, which was derived for regular 2D problems (Fig. 5), i.e. an axially constrained infinite bar connected parallelly to a mass-spring system. The advantage of using this model is that it contains only very few parameters and results in a 1D problem instead of a 3D one. We wish to validate this hypothesis. It will be also shown that the number of the independent parameters of the model is three (Fig. 7):

- the static stiffness of the model *K*,
- the eigen (cut-off) frequency ω_c,
- the ratio η which shows the contribution of the sub models. (For η=0 it gives the mass-spring system, while for η=1 the axially constrained infinite bar.)

Furthermore a simple identification method is developed to determine the parameters of the model.

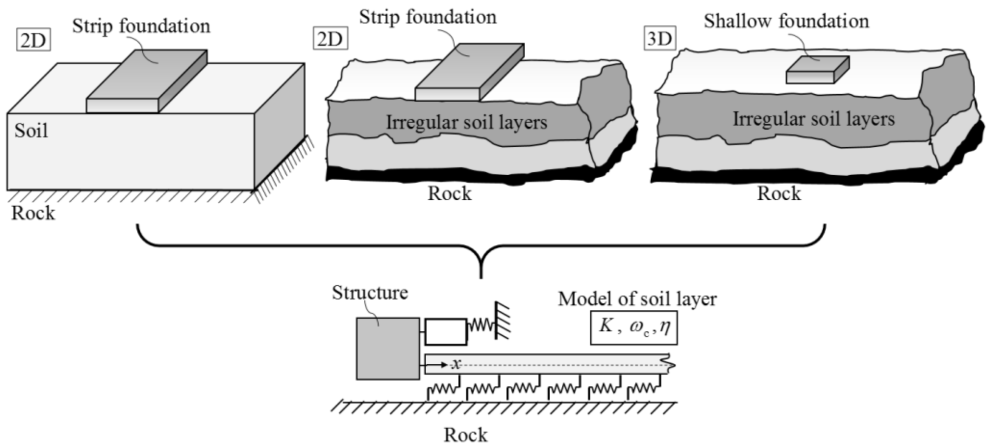
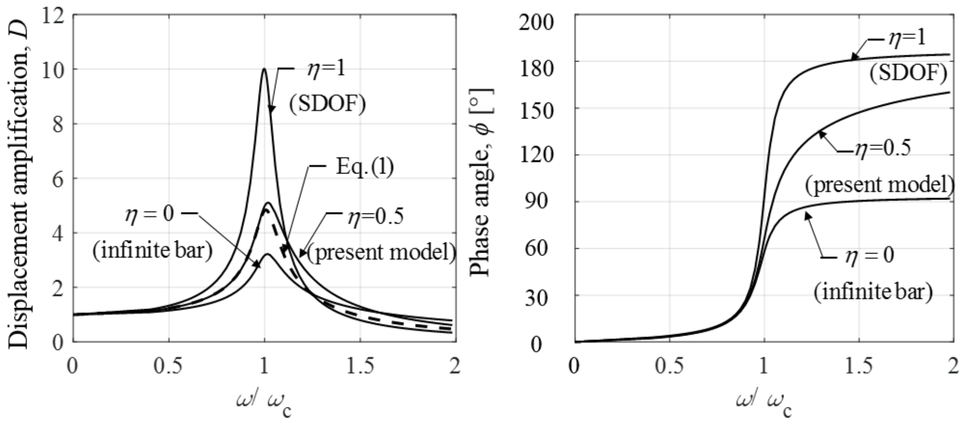


Fig. 7 Examples of 2D and 3D problems, and the model (the model parameters are: *K*, ω_c, η)

The amplification factor is derived analytically and also an approximate solution is given:

$$D_{\max} \approx \frac{1}{(1-\eta)2\xi + \eta\sqrt{2\xi}} \quad (1)$$

The solution in case of damping, characterized by the damping ratio ξ (<1), is given for SDOF systems and for axially constrained infinite bars in the dissertation. The calculated dynamic displacement amplification factors and phase angles for ξ=0.05 are shown in Fig. 8. The curves due to the mass-spring stiffness is identified by η=1, while those of the infinite bar by η =0. The analytical solutions are shown by continuous lines, while the approximate solution by dashed line.



a) b)

Fig. 8 Amplification factor and phase angle for ξ=0.05 damping ratio

To calculate the model parameters the following identification process should be considered:
Step 1: ω_c is obtained directly from the impedance curve.
Step 2: The A and B ordinates of the impedance curve are determined (Fig. 9). From Eq. (1), we obtain: $\eta = \dfrac{2\xi - 1/D_{max}}{2\xi - \sqrt{2\xi}}$, $D_{max} = \dfrac{B}{A}$.

Step 3: The stiffnesses are: $K = \dfrac{1}{A}, \;\; k_0 = K(1-\eta), \;\; k_s = K\eta.$

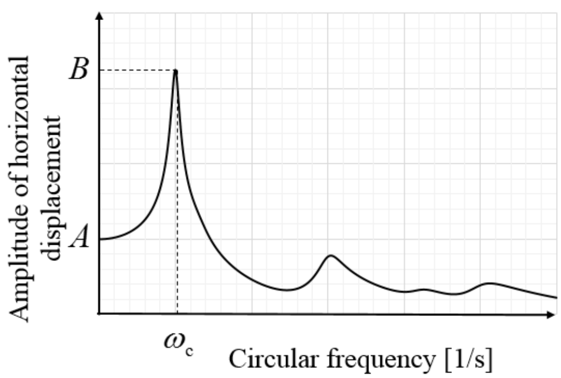


Fig. 9 Three representative parameters of the impedance curve

Irregular soil layer are considered with a strip foundation. The impedance curves (Fig. 10, solid lines), the model parameters, and the impedance of our model (Fig. 10, dotted line) are calculated according to the given identification process.

The problem of the irregular soil layers is investigated, where the weight of the object is 200 t, and it is subjected to earthquake record no. 32 of [17], the corresponding curves are given in the dissertation. It can be seen that the present model can reasonably approximate the results of the direct 2D analysis.

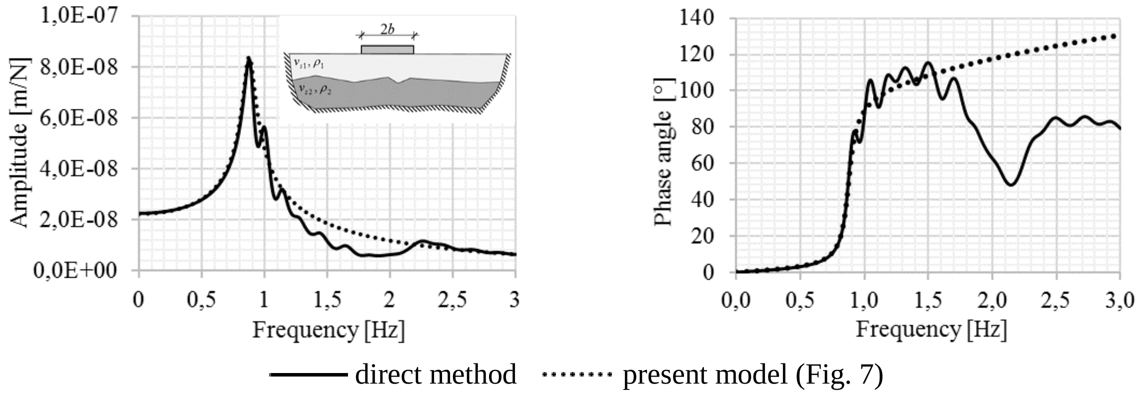
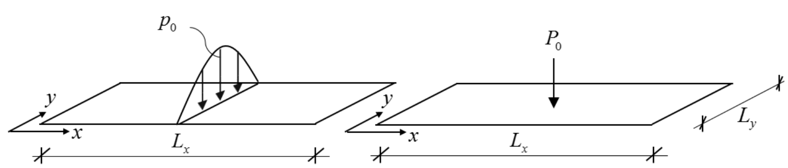


Fig. 10 Impedance curves for irregular soil layers (v_{s1} =100 m/s, ρ_1=1800 kg/m^3, v_{s2} =250 m/s, ρ_2=1950 kg/m^3)

5. Harmonic response of long rectangular plates

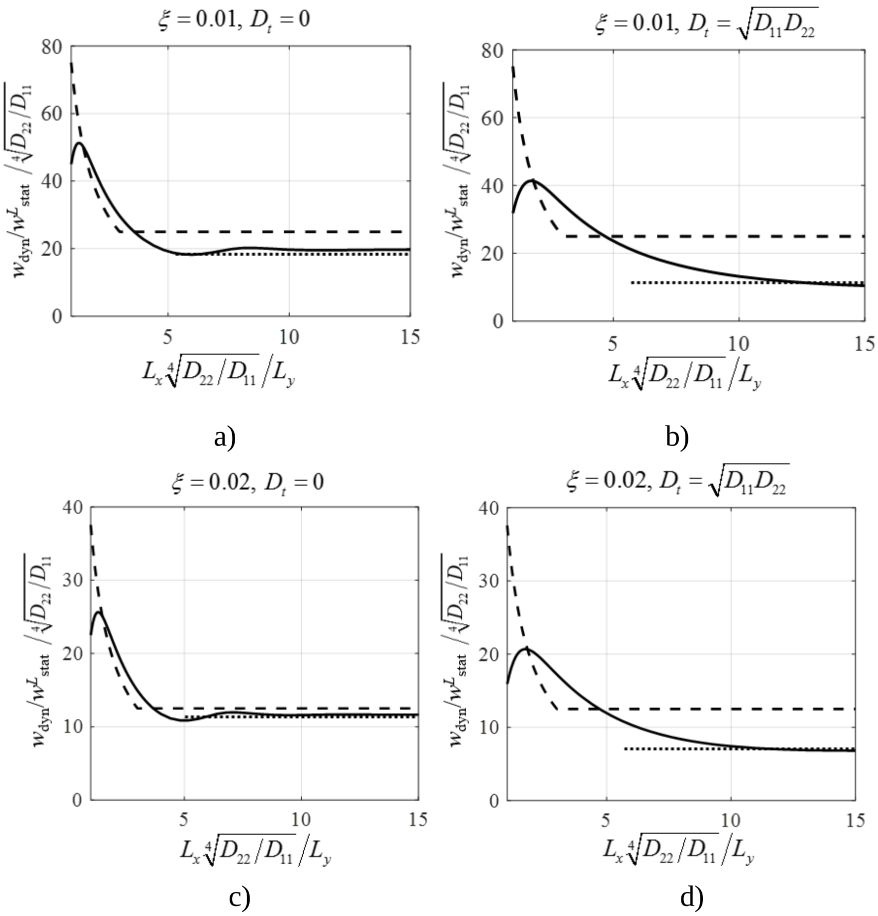
Rectangular orthotropic plates are considered, which are significantly "longer" in the x direction than perpendicular to it. (For orthotropic plates [19] $L_x >> \sqrt[4]{D_{11}/D_{22}}\,L_y$.) The edges parallel to the x coordinate are simply supported. The damping of the structure is characterized by the damping ratio, ξ. The plate is subjected to either a sinusoidal line load or to a concentrated load (Fig. 11). The load is periodic, and it is represented in time by its Fourier series expansion [20]. We wish to determine the steady-state response of the floor, the maximum displacement and acceleration, and investigate the applicability of the concept of effective width. To solve the above problem for the trigonometrical load an analytical solution, while for the concentrated load the FE analysis is applied.



a) b)

Fig. 11 Long plate subjected to a trigonometrical line load (a) and to a concentrated load (b)

The impedance function and amplification factor is derived by Fourier series expansion. The peak value of the amplification factor can be determined numerically. For a plate without torsional stiffness and for an isotropic (or Huber orthotropic) plate the amplification factors at resonance can be approximated as $D_{d,r}^{0}\approx 1/(2\xi)^{0.75}$ and $D_{d,r}^{H}\approx 1/(2\xi)^{0.6}$ respectively. The dynamic response of a long plate at resonance can be calculated according to these amplification factors and the effective width, which is determined by matching the static deflection of a beam to the infinite plate. The derived functions are showed in Fig. 12 for different damping ratios, the formulas given by the design guide [14] are also illustrated. They only give proper results for plates with zero torsional stiffness and damping ratio ξ=2% and for the case of resonance.



······· our approximate solution ——— exact solution ---- formulas of the design guide [14]

Fig. 12 Dynamic displacements of a plate subjected to a concentrated load at resonance (ξ=1, 2%). The displacements are normalized by the static response of a L_y wide plate strip $w_{stat}^L=P_0L_y^3/48D_{22}$ and by the ratio of the stiffnesses $(D_{22}/D_{11})^{0.25}$. On the left: the plate has no torsional stiffness while on the right $D_t=(D_{11}\,D_{22})^{0.5}$.

6. Discussion

In the thesis two different problems (soil-structure interaction in earthquake design and harmonic response of long plates) are investigated, which are connected by their mathematical representations. The two cases have similar phenomenon, in both cases radiation damping occurs, which causes an entirely different behaviour than that of a SDOF system. First the basic systems are analysed, an axially constrained infinite bar, which can be used in the modelling of a soil layer, and an infinite beam on elastic foundation, the behaviour of which is identical to the long rectangular floors without torsional stiffness. A plate with torsional stiffness can be represented by the combination of the two models.

7. Main contributions

[21] investigated the transverse vibration of an infinite string on elastic foundation without damping, and derived its complex impedance function. Mathematically this solution is equivalent to the longitudinal harmonic vibration of an axially constrained undamped infinite bar. This is presented in [22].

Main contribution 1

I derived the longitudinal response of an axially constrained infinite bar subjected to harmonic force excitation at a point of the bar taking into account the effect of damping. I showed numerically that at resonance the dynamic amplification factor can be approximated by $1/\sqrt{2\xi}$, where ξ is the damping ratio.

Related publication: [P1]

Performing a literature survey no results were found for the impedance function of infinite bars on elastic foundation.

Main contribution 2

I derived the transverse response of an infinite bar on elastic foundation subjected to vertical harmonic force excitation at a point of the beam with and without damping. I showed numerically that at resonance the dynamic amplification factor can be approximated by $1/(2\xi)^{0.75}$ where ξ is the damping ratio.

Related publications: [P4], [P6]

In the literature, usually simple spring-dashpot models are used [2] to take into account the effect of soil-structure interaction (SSI). It is observed that these models can be inaccurate [23], however it is not given in which parameter range it must not be used.

Main contribution 3

I made a sensitivity analysis for the impedance function of horizontally infinite, vertically finite soil layers with strip foundation on it, and determined the parameter range, where the simplified spring-dashpot model must not be used for earthquake resistant design.

The corresponding diagrams are given in the dissertation (Fig. 41 and Fig. 42.).

Related publications: [P3], [P9], [P10], [P11]

There are models in the literature to consider SSI in finite and infinite soil layers (lumped parameter models, echo constants), but they depend on several parameters, often without direct physical meaning and the application of these models is rather complex. Although Wolf [22] in his book on SSI presents the mechanical model of an axially constrained bar, no recommendation is given, how and when it could be used for SSI calculation.

Main contribution 4

I derived a new simple model based on the physical representation of the horizontally infinite, vertically finite soil layer with strip foundation on it (2D problem): an axially constrained infinite bar connected to a mass-spring system parallelly.

4.1. I derived the differential equation (DE) of the new model by the Rayleigh-Ritz method.

4.2. I derived explicit expressions for the calculation of the model parameters.

4.3. I derived the solution of the DE of the model for harmonic force excitation.

4.4. I derived the solution of the DE of the model for harmonic base excitation.

The derived formulas are given in Section 5.2.1 of the dissertation (Eqs. (38), (47), (48), (50)).

Related publications: [P2], [P7], [P8]

Main contribution 5

I extended my simple model for 3D problems and for irregular soil layers. I proved that the model consists only three independent parameters, ω_c cut-off frequency, K static stiffness and ratio η, which shows the contribution of the mass-spring system and the axially constrained infinite bar. I gave a simple identification procedure to determine the model parameters of the axially constrained infinite bar connected parallelly to a mass spring system based on the impedance curve of the soil layers.

Related publications: [P1]

According to the literature [24] and design guides [14] the harmonic analysis of floors should be performed based on a SDOF system. This is not valid for long floors, therefore to obtain a good agreement with the measurements at the resonance point, [14] presented an empirical formula for the effective width, which has no physical meaning [15].

Main contribution 6

I determined the response of long floors subjected to harmonic excitation.

6.1. I derived an analytical solution for floors subjected to a sine line load.

6.2. I derived the impedance curve and showed numerically that at resonance the dynamic amplification factor can be approximated by $1/(2\xi)^{0.75}$ for long floors without torsional stiffness and by $1/(2\xi)^{0.6}$ for long floors with Huber orthotropy, where ξ is the damping ratio.

6.3. I extended the results for concentrated loads using finite element analysis.

Related publications: [P4], [P6]

The author's publications where the main contributions were published

Research articles

[P1] Pap, Z. B., Kollár, L. P. " Modeling of SSI of horizontally vibrating structures by infinitely long constrained bars". *submitted*

[P2] Pap, Z. B., Kollár, L. P. " Model of Soil-Structure Interaction of Structures Resting on Finite Depth Soil Layers ". *Periodica Polytechnica Civil Engineering*, 2019; December. DOI: https://doi.org/10.3311/PPci.14459.

[P3] Pap, Z. B., Kollár, L. P. "Effect of Resonance in Soil-Structure Interaction for Finite Soil Layers". *Periodica Polytechnica Civil Engineering*, pp. 678–684. 2018. DOI:https://doi.org/10.3311/PPci.11960

[P4] Pap, Z. B., Kollár, L. P. " Dynamic Response of Long Rectangular Floors Subjected to Periodic Force Excitation". *Materials*, 12(9), 2019. DOI:https://doi.org/10.3390/ma12091417

[P5] Kollár, L. P., Pap, Z. B., "Modal Mass of Floors Supported by Beams" Structures, pp. 119-130, 13, 2018.

Conference papers

[P6] Pap, Z. B., Kollár, L. P. "Födémek vizsgálata periodikus gerjesztésre". *XIII. Magyar Mechanikai Konferencia*, Miskolc, Magyarország, 2019.

[P7] Pap, Z. B., Kollár, L. P. "Rezonancia jelentősége és modellezése a talaj-szerkezet kölcsönhatásban véges kiterjedésű talaj esetén". *XIII. Magyar Mechanikai Konferencia*, Miskolc, Magyarország, 2019.

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