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Robust stability of delayed dynamical systems

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Overwiev of the dissertation

The mathematical investigation of time-delayed systems plays an important role in applied mathematics and engineering. Delays are often used to model information propagation in biological and electrical systems, growth rate in population models, economical processes, and also mechanical systems. There exist several applications in mechanics, where the inclusion of the delayed dynamics leads to a better description of the real physical system, e.g., wheel shimmy, machining vibrations, traffic models or human motion control, just to mention a few. Over the years the number of contributions on these research areas has increased significantly and the tendency predicts a strong interest both from the academic and the industrial sides.

Mathematical representations of dynamical systems are always associated with a certain level of model simplification. The simplified models intend to capture the most important features of the real system, however, these reductions often lead to an impaired representation of the real dynamics. One of the most important question is how the model behaves if some of the inputs or parameters are slightly changed.

Uncertainties in the various inputs and parameters have significant effect on the systems' performance. Therefore, it is necessary to investigate that in what extent the systems' behavior depends on the uncertainties. Systems, which guarantee the required performance in case of unmodeled dynamics and uncertainties, are called *robust*. When the stability is preserved in the presence of parameter uncertainties, then the system is said to be *robustly stable* . This dissertation is devoted to the robust stability analysis of a certain class of delayed dynamical systems with engineering applications. Different models with static uncertainties are investigated, i.e., the parameter uncertainties are assumed to be constant in time.

The first topic is the sensitivity analysis of the classical Smith

Predictor, which is one of the simplest type of predictor feedback controller. It is shown that static uncertainties can drastically reduce the applicability of the classical Smith Predictor. A second-order stable plant was investigated in details, where uncertainties arise in the system parameters and time delay. It is demonstrated on this example that the classical Smith Predictor might not be able to increase the domain of stable control parameters, when finite, but small relative uncertainties arise (thesis statement 1).

The second topic is the investigation a human balance control model in the frontal plane, where the uncertainties originate from the anthropometric data of the human body. In this part, the real stability radius is calculated, and the domain of robust control gains was determined. It is shown that this robust domain is located at small control gains (thesis statement 2).

The third topic is the robust stability analysis of machining operations, such as turning and milling. First, the stability of turning operations was investigated and the uncertainty in the frequency response functions (FRFs) was considered. Analytic expressions were derived and robust stability boundaries were determined (thesis statement 3). Then, a robust control design was studied in order to robustly stabilize turning operations using an active vibration damper (thesis statement 4). Finally, milling operations were investigated, where the Floquet theory was applied to time-periodic systems. Using the approximation of the structured singular values and the multi-frequency method, robust stability boundaries were determined (thesis statement 5).

The final topic is the string stability of connected cruise controllers, where human driver uncertainties can detrimentally affect the performance of the system. It is shown that uncertainty in the time delay, control parameters and other systems parameters can be modeled in an uncertain interconnection structure and robust string stability can be guaranteed in order to achieve safe operation (thesis statement 6).

Thesis statement 1

I have analyzed the robust stability of the original Smith predictor employed to a second order plant in case of mismatches in the system parameters and time delay. The results were demonstrated on an asymptotically stable and Lyapunov stable linear system. Based on these examples, it can be stated that the original Smith predictor is sensitive to uncertainties in the system parameters. The results are summarized as follows.

Let a general second order plant be controlled by the original Smith predictor, and the governing equations be written as

$$\ddot{x}(t) + b\dot{x}(t) + ax(t) = \mathbf{K}\big(\mathbf{x}(t-\tau) - \hat{\mathbf{x}}(t-\tau) + \hat{\mathbf{x}}(t)\big),$$
$$\ddot{\hat{x}}(t) + \hat{b}\dot{\hat{x}}(t) + \hat{a}\hat{x}(t) = \mathbf{K}\big(\mathbf{x}(t-\tau) - \hat{\mathbf{x}}(t-\tau) + \hat{\mathbf{x}}(t)\big),$$

where $\mathbf{x}(t) = [x(t), \dot{x}(t)]^\top$ **is the state,** $\hat{\mathbf{x}}(t) = [\hat{x}(t), \dot{\hat{x}}(t)]^\top$ **is the predicted state,** $\mathbf{K} = [-k_\mathrm{p}, -k_\mathrm{d}]$**,** k_p **and** k_d **are control gains,** a**,** b **are system parameters,** τ **is the time delay, and** $\hat{a}$**,** $\hat{b}$**,** $\hat{\tau}$ **are their estimations, respectively. In case of a marginally stable plant (**$a > 0$**,** $b = 0$**, Lyapunov stable, but not asymptotically stable), the domain of asymptotically stable control parameters disappears for infinitesimally small overestimation of parameter** a **(**$0 < \hat{a} - a < \varepsilon$**). The stable domain does not change for infinitesimal error in the estimation of the delay** τ**. In case of an asymptotically stable plant (**$b > 0$**), the stable domain does not change for infinitesimal small error of the estimation of parameters** a **and** τ**. In case of large enough finite error of parameter estimations, the domain of the stable control parameters with the Smith predictor becomes smaller, than the domain of the conventional state feedback controller without the Smith predictor.**

Related publications: [1], [2], [3].

Thesis statement 2

I have investigated a human balancing model in the frontal plane with mediolateral balance control. The robust stability boundaries associated with unstructured complex-valued perturbations can be found in the available literature. As an extension, I have determined the more realistic real structured stability radius, and showed that the most robust control gains differ from the ones obtained by the previous method. The results are summarized as follows.

Let the linearized equation of motion around the equilibrium position of the single-degree-of-freedom model of the mediolateral balance control of a human balancing model be written as

$$I\ddot{x}(t) + Gx(t) = -k_{\mathrm{p}}x(t-\tau) - k_{\mathrm{d}}\dot{x}(t-\tau),$$

where k_{p} and k_{d} are control gains of the proportional-derivative controller, $I > 0$ is the equivalent inertia, $G < 0$ is the equivalent stiffness, $x(t)$ is the generalized coordinate, and τ is the reaction delay. The domain of the robust stable control gains shrinks with increasing uncertainty in the system parameters I and G. The most robust stable control gains are located in the left bottom part of the stable region in the plane $(k_{\mathrm{p}}, k_{\mathrm{d}})$, i.e., when $k_{\mathrm{p}} \gtrsim -G$ and $k_{\mathrm{d}} \gtrsim -G\tau$.

Related publications: [4].

Thesis statement 3

Stability lobe diagrams of turning operations in the plane on machining parameters can be calculated based on the frequency response function of the tool. I have modeled the uncertainty of measurements and modal parameters by an envelope centered around the nominal FRF. Based on the cross-section of the envelopes, I have derived the following conditions to calculate the robust stability boundaries.

A conservative estimation on the robustness of turning operations in case of static uncertainty of the frequency response functions (FRF) can be given based on an envelope centered around the nominal frequency response function. Let the nominal complex FRF be denoted by $H(\omega) = H_{\mathrm{Re}}(\omega) + \mathrm{i}H_{\mathrm{Im}}(\omega)$, moreover, let the envelope be determined by a rotated elliptical cross section with main and minor axes $w_1(\omega)$, $w_2(\omega)$, and let $\alpha(\omega)$ be the angle between the main axis of the ellipse and the real axis. The robust machining parameters can be determined with the formula of the safety factor

$$SF_{\mathrm{c}} = \min_{\omega \geq 0} \left(\left(\frac{\tilde{H}_{\mathrm{cr,Re}}(\omega) \cos\big(\alpha(\omega)\big) - \tilde{H}_{\mathrm{cr,Im}}(\omega) \sin\big(\alpha(\omega)\big)}{w_1(\omega)} \right)^2 + \left(\frac{\tilde{H}_{\mathrm{cr,Re}}(\omega) \sin\big(\alpha(\omega)\big) + \tilde{H}_{\mathrm{cr,Im}}(\omega) \cos\big(\alpha(\omega)\big)}{w_2(\omega)} \right)^2 \right)^{1/2},$$

where

$$\tilde{H}_{\mathrm{cr,Re}}(\omega) = -\frac{1}{2\kappa} - H_{\mathrm{Re}}(\omega),$$
$$\tilde{H}_{\mathrm{cr,Im}}(\omega) = \frac{\sin(\omega\tau)}{2\kappa\big(1-\cos(\omega\tau)\big)} - H_{\mathrm{Im}}(\omega)$$

denote the critical complex perturbation, $\tau = 60/\Omega$ is the regenerative time delay, Ω is the speed of rotation of the workpiece given in rpm, and κ is the specific cutting force coefficient. Any pair of machining parameters (Ω^*, κ^*) are guaranteed to be robust against bounded static uncertainty in the frequency response functions, if the system without uncertainties is stable and $SF_{\mathrm{c}} < 1$.

Related publications: [5], [6].

Thesis statement 4

I have analyzed the robust stability of turning operations with a proportional-derivative controller and feedback delay. The uncertainties of dynamical measurements are directly taken into account in the complex frequency response functions, while robust stability was analyzed using the structured singular value analysis. The results are summarized as follows.

Robust stability of the dynamical model of turning operations subjected to a proportional-derivative controller can be analyzed by constructing the M-$\boldsymbol{\Delta}$ uncertain interconnection structure and structured singular value analysis. Let $\mathrm{H}(\omega)$ denote the nominal frequency response function matrix of the tool between the tool-tip and actuation point of the controller, moreover let $\tilde{\mathrm{H}}(\omega)$ denote its complex static uncertainty matrix. The corresponding M-$\boldsymbol{\Delta}$ structure is written as

$$
\begin{aligned}
\mathbf{M}(\omega_{\mathrm{c}}) &= \mathbf{K}(\omega_{\mathrm{c}})\big(\mathbf{I} - \mathbf{H}(\omega_{\mathrm{c}})\mathbf{K}\left(\omega_{\mathrm{c}}\right)\big)^{-1}, \\
\boldsymbol{\Delta}(\omega_{\mathrm{c}}) &= \tilde{\mathbf{H}}(\omega_{\mathrm{c}}),
\end{aligned}
$$

where I is the identity matrix, ω_{c} is the vibration frequency on the stability boundary, moreover

$$
\mathbf{K}(\omega_{\mathrm{c}}) = \begin{bmatrix} w f_q'(h_0)\big(\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{c}}\tau_1} - 1\big) & 0 \\ 0 & -\left(k_{\mathrm{p}} + k_{\mathrm{d}}\mathrm{i}\omega_{\mathrm{c}}\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{c}}\tau_2} \end{bmatrix},
$$

and w is the depth of cut, $\mathrm{i}^2 = -1$, $f_q'(h_0)$ is the slope of the specific cutting force characteristics, h_0 is the nominal chip thickness, τ_1 is the regenerative time delay, k_{p}, k_{d} are the proportional and derivative control gains, and τ_2 is the time delay of the controller.

Related publications: [7].

Thesis statement 5

Dynamical model of milling operations is described by time-periodic delay-differential equations. I have combined the multi-frequency solution and the structured singular value analysis in order to determine robust stability boundaries with uncertain dynamics. The results are summarized as follows.

In case of machining operation described by time-periodic delay differential equations, where the principal period (T) equals to the regenerative time delay (τ), the structure of the truncated infinite-dimensional matrix $\mathrm{D}(\omega_\mathrm{c})$ given by the multi-frequency method can be factorized as

$$\mathbf{D}(\omega_\mathrm{c}) = \mathbf{I} - \mathbf{U}(\omega_\mathrm{c})\mathbf{E}(\omega_\mathrm{c})\mathbf{W},$$

where

$$\begin{aligned}
\mathbf{U}(\omega_\mathrm{c}) &= \mathrm{diag}\big[\mathbf{H}(-s\hat{\Omega}+\omega_\mathrm{c}), \mathbf{H}((-s+1)\hat{\Omega}+\omega_\mathrm{c}), \dots, \\
&\qquad \dots, \mathbf{H}(s\hat{\Omega}+\omega_\mathrm{c})\big], \\
\mathbf{E}(\omega_\mathrm{c}) &= \mathrm{diag}\Big[\mathbf{I}\big(1-\mathrm{e}^{-\mathrm{i}(-s\hat{\Omega}+\omega_\mathrm{c})\tau}\big), \mathbf{I}\big(1-\mathrm{e}^{-\mathrm{i}((-s+1)\hat{\Omega}+\omega_\mathrm{c})\tau}\big), \dots, \\
&\qquad \dots, \mathbf{I}\big(1-\mathrm{e}^{-\mathrm{i}(s\hat{\Omega}+\omega_\mathrm{c})\tau}\big)\Big], \\
\mathbf{W} &= \begin{bmatrix} \mathbf{G}_0 & \mathbf{G}_{-1} & \cdots & \mathbf{G}_{-s} \\ \mathbf{G}_1 & \mathbf{G}_0 & \cdots & \mathbf{G}_{-s+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{G}_s & \mathbf{G}_{s-1} & \cdots & \mathbf{G}_0 \end{bmatrix},
\end{aligned}$$

and $\det(\mathbf{D}(\omega_\mathrm{c})) = 0$ gives the possible stability boundaries. The quantities in the formulas above are: $\hat{\Omega} = 2\pi/T$, $\mathrm{H}(\omega)$ is the frequency response function matrix measured at the tool-tip, I is the identity matrix with appropriate dimensions, $\mathrm{i}^2 = -1$, G_k is the k-th Fourier component of the time-periodic matrix, ω_c

is the chatter frequency, and $s \in \mathbb{Z}^+$ is the number of Fourier components considered in the approximated system. Let the additive static uncertainty of the measured frequency response function matrix be denoted by $\tilde{\mathrm{H}}(\omega)$. Robust stability of the time-periodic system with respect to uncertainties in the frequency response functions can be analyzed by constructing the M-$\boldsymbol{\Delta}$ uncertain interconnection structure and structured singular value analysis, where

$$\begin{aligned}
\mathbf{M}(\omega_\mathrm{c}) &= \mathbf{E}(\omega_\mathrm{c})\mathbf{W}\big(\mathbf{I} - \mathbf{U}(\omega_\mathrm{c})\mathbf{E}(\omega_\mathrm{c})\mathbf{W}\big)^{-1}, \\
\boldsymbol{\Delta}(\omega_\mathrm{c}) &= \operatorname{diag}\big[\tilde{\mathbf{H}}(-s\hat{\Omega} + \omega_\mathrm{c}), \tilde{\mathbf{H}}((-s+1)\hat{\Omega} + \omega_\mathrm{c}), \ldots, \\
&\qquad \ldots, \tilde{\mathbf{H}}(s\hat{\Omega} + \omega_\mathrm{c})\big].
\end{aligned}$$

Related publications: [8], [9].

Thesis statement 6

I have analyzed the effect of parameter uncertainties on the performance of connected vehicles system. I have applied the Rekasius substitution in order to include uncertainty of the time delay and formulate the uncertain interconnection structure. Robust string stability and robust plant stability can be analyzed by formulating the the linear fractional transformation model and structured singular value analysis. The results are summarized as follows.

Consider the model of a human-driven vehicle following a leader vehicle (leader-follower model) in the form

$$\begin{aligned}
\dot{h}(t) &= v_l(t) - v_f(t)\,, \\
\dot{v}_f(t) &= (\alpha + \tilde{\alpha})\Big(V\big(h(t-(\tau+\tilde{\tau}))\big) - v_f(t-(\tau+\tilde{\tau}))\Big) + \\
&\qquad (\beta + \tilde{\beta})\big(v_l(t-(\tau+\tilde{\tau})) - v_f(t-(\tau+\tilde{\tau}))\big),
\end{aligned}$$

where $h(t)$ is the headway, $v_l(t)$ is the velocity of the leader vehicle, $v_f(t)$ is the velocity of the human-driven vehicle following the leader, τ is the reaction time delay, α and β are control gains and $V(h)$ is a range policy function (e.g. constant time-gap spacing policy). The derivative of $V(h)$ at the uniform equilibrium flow (h^*, v^*) equals to $V'(h^*) = \kappa + \tilde{\kappa}$ (where $1/V'(h^*)$ is the time-gap). The parameters of the linearized system are κ, α, β and τ, where the additive uncertainty of these parameters ($\tilde{\kappa}$, $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\tau}$) denote the static uncertainty of the parameters of the human-driven vehicle. The transfer function between the velocity fluctuations of the follower and leader vehicles is written as

$$T(s) = \frac{\big((\kappa+\tilde{\kappa})(\alpha+\tilde{\alpha})+(\beta+\tilde{\beta})s\big)\mathrm{e}^{-s\tau}\dfrac{1-s\tilde{\vartheta}(s)}{1+s\tilde{\vartheta}(s)}}{s^2+\big((\kappa+\tilde{\kappa})(\alpha+\tilde{\alpha})+(\alpha+\tilde{\alpha}+\beta+\tilde{\beta})s\big)\mathrm{e}^{-s\tau}\dfrac{1-s\tilde{\vartheta}(s)}{1+s\tilde{\vartheta}(s)}},$$

where the uncertainty of the time delay is modeled by the Rekasius substitution with restriction to $s = \mathrm{i}\omega$, moreover ω is the frequency and $\tilde{\vartheta}(\mathrm{i}\omega) = \omega^{-1}\tan(0.5\omega\tilde{\tau})$ is a frequency-dependent real parameter. The follower vehicle is robust string stable if $|T(\mathrm{i}\omega)| < 1$, $\omega > 0$ for all allowed uncertainties. Robust string stability with respect to uncertain parameters in the follower's model can be analyzed by constructing the M-$\boldsymbol{\Delta}$ uncertain interconnection structure and analyzing the structured singular values, where ($s = \mathrm{i}\omega$ and in case of the Rekasius substitution $0 \leq \omega < \pi/|\tilde{\tau}|$)

$$\mathbf{M}(s) = \left[\begin{array}{c|c}\mathbf{M}_{1,1}(s) & \mathbf{M}_{1,2}(s)\\ \hline \mathbf{M}_{2,1}(s) & M_{2,2}(s)\end{array}\right],$$

$$\boldsymbol{\Delta}(s) = \mathrm{diag}[\tilde{\kappa}, \tilde{\alpha}, \tilde{\beta}, \tilde{\vartheta}(s), \delta^{\mathrm{c}}], \quad \delta^{\mathrm{c}} \in \mathbb{C}, |\delta^{\mathrm{c}}| < 1,$$

moreover

$$\mathbf{M}_{1,1}(s) =$$

$$\frac{1}{D(s)}\begin{bmatrix} -\alpha e^{-s\tau} & -e^{-s\tau} & -e^{-s\tau} & -2 \\ s^2+\beta s e^{-s\tau} & -(\kappa+s)e^{-s\tau} & -(\kappa+s)e^{-s\tau} & 2(\kappa+s) \\ -\alpha s e^{-s\tau} & -s e^{-s\tau} & -s e^{-s\tau} & 2s \\ \alpha s^3 e^{-s\tau} & s^3 e^{-s\tau} & s^3 e^{-s\tau} & -s^3+s\big(\kappa\alpha+s(\alpha+\beta)\big)e^{-s\tau} \end{bmatrix},$$

$$\mathbf{M}_{1,2}(s) = \frac{1}{D(s)}\begin{bmatrix} s+\alpha e^{-s\tau} \\ \kappa s-\beta s e^{-s\tau} \\ s^2+\alpha s e^{-s\tau} \\ (\kappa\alpha s^2+\beta s^3)e^{-s\tau} \end{bmatrix},$$

$$\mathbf{M}_{2,1}(s) = \frac{1}{D(s)}\left[\alpha s e^{-s\tau}\ s e^{-s\tau}\ s e^{-s\tau}\ -2s\,(\kappa\alpha+\beta s)e^{-s\tau}\right],$$

$$M_{2,2}(s) = \frac{(\kappa\alpha+\beta s)e^{-s\tau}}{D(s)},$$

$$D(s) = s^2+\big(\kappa\alpha+s(\alpha+\beta)\big)e^{-s\tau}.$$

Related publications: [10], [11].

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