# On Commutative Substructures of Special Agebric Structures

PhD thesis booklet

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# 1 Introduction

## 1.1 Objectives and structure of the dissertation

In the dissertation we deal with three topics; all of them require the commutativity of certain substructures of special algebraic structures. The purpose of the examinations on the first topic is to give an upper bound to the dimension of the commutative subalgebras of Grassmann algebras. This is a joint work with Mátyás Domokos, see [DZ15]. Investigations on the second topic are designed to give an upper bound to the size of subsemilattices of finite semilattice indecomposable semigroups, see [Zub16]. The third theme is related to semigroups $S$ whose sets of congruences forms commutative subsemigroups in the semigroup of all binary relations of $S$; these semigroups are called congruence permutable semigroups. This is a joint work with Attila Nagy, see [NZ16].

The dissertation contains an unnumbered Introduction and further four numbered chapters.

Chapter 1 is the Preliminaries, in which we present those notations, basic notions and results which are used in the dissertation.

In Chapter 2, the maximal dimension of commutative subalgebras of Grassmann algebras is in the focus. Our interest in maximal commutative subalgebras was inspired by [Mar15], where the existence of large commutative subalgebras of $E$ is used as an obstruction for embeddability of $E$ into the full matrix algebra $\mathbb{F}^{m\times m}$ for small $m$. The study of commutative subalgebras in non-commutative algebras has a considerable literature. We mention only the theorem of Schur [Sch05] determining the maximal dimension of a commutative subalgebra of $\mathbb{F}^{n\times n}$, see [Jac44] and [Gus76] for alternative proofs.

Let $W$ be a vector space over a field $\mathbb{F}$, and $\pi_1,\dots,\pi_r \in \mathrm{End}_{\mathbb{F}}(W)$ pairwise commuting projections, so $\pi_i^2 = \pi_i$ and $\pi_i\pi_j = \pi_j\pi_i$ for all $i,j \in [r] := \{1,\dots,r\}$. Recall the corresponding direct sum decompositions

$$W = \ker(\pi_j) \oplus \mathrm{im}(\pi_j).$$

Given a subset $J \subseteq [r]$ we set

$$W_J := \bigcap_{j\in J} \ker(\pi_j) \cap \bigcap_{j\notin J} \mathrm{im}(\pi_j).$$

Let $\mathrm{Gras}(W)$ stand for the set of subspaces of $W$, and for $j = 1,\dots,r$ define a map

$$\gamma_j : \mathrm{Gras}(W) \to \mathrm{Gras}(W), \qquad D \mapsto \ker(\pi_j|_D) \oplus \mathrm{im}(\pi_j|_D) \tag{1}$$

where $\pi_j|_D : D \to W$ stands for the restriction of $\pi_j$ to the subspace $D \subseteq W$.

For a subset $J \subseteq [n] := \{1,\dots,n\}$ set $v_J := v_{i_1}\cdots v_{i_k}$, where $J = \{i_1,\dots,i_k\}$ and $i_1 < \cdots < i_k$. Clearly, $\{v_J \mid J \subseteq [n]\}$ is an $\mathbb{F}$-vector space basis of $E$. We shall refer to the elements $v_J \in E$ as ***monomials***.

We prove the following (Theorem 2.4.4; Thesis 1): If $D \subseteq E$ is a subalgebra, and $A := \gamma_1 \dots \gamma_n(D)$, then $A$ is a subalgebra of $E$ spanned as an $\mathbb{F}$-vector space by elements of the form $v_J$, $J \subseteq [n]$, and $\dim(A) = \dim(D)$. Moreover, if $D$ is commutative then $A$ is commutative. If $D^2 = \{0\}$ then $A^2 = \{0\}$.

The main result of Chapter 2 is Theorem 2.6.1 (Thesis 2) which gives in particular the maximal dimension of a commutative subalgebra of $E$, and gives some partial results on their structure. It turns out when $n$ is even, then all maximal (with respect to inclusion) subalgebras have the same dimension. When $n$ is odd, then there are maximal commutative subalgebras of different dimension. As we show in Theorem 2.4.4 any commutative subalgebra of $E$ one can associate via a simple linear algebric process to an equidimensional commutative subalgebra spanned by monomials (products of generators). This result has some independent interest, and also makes it possible to make a tight connection between our question and the Erdős-Ko-Rado Theorem on intersecting set families.

In Chapter 3, subsemilattices of finite semilattice indecomposable semigroups are in the focus. A congruence $\alpha$ on a semigroup $S$ is called a semilattice congruence if the factor semigroup $S/\alpha$ is a semilattice. In this case we also say that $S$ is the semilattice of the $\alpha$-classes of $S$. A semigroup $S$ is said to be semilattice indecomposable (s-indecomposable) if its universal relation is the only semilattice congruence on $S$. It is known that every semigroup is a semilattice of s-indecomposable semigroups. In Chapter 3, we give a new characterization the finite s-indecomposable semigroups (Theorem 3.2.1; Thesis 3): A finite semigroup $S$ is s-indecomposable if and only if, $\mathbb{C}[S/K_S]/J(\mathbb{C}[S/K_S])$ (where $K_S$ denote the kernel of $S$) has exactly one 1-dimensional ideal.

In the literature of the theory of semigroups there are many papers about s-indecomposable semigroups (see, for example, the papers, [Chr69], [Nag84], [Nag85], [Nag92], [Nag92-2], [Nag93], [Nag98], [NJ04], [Nor88], [PuW71], [Tam82], [TK54], and the books [Gri01], [Nag01]). Some of them deal with the s-indecomposable semigroups without idempotents, the others investigate the s-indecomposable semigroups containing at least one idempotent. In this chapter we deal with finite s-indecomposable semigroups in terms of what can be said about the maximal size of their subsemilattices. The answer is known in special classes of semigroups. As $ef = fe$ implies $e = f$ for every idempotent elements $e$ and $f$ of a completely simple semigroup, the cardinality of the subsemilattices in a completely simple semigroup is one. In the classes of semigroups investigated in [Chr69], [Nag84], [Nag85], [Nag92], [Nag92-2], [Nag93], [Nag98], [NJ04], [Nor88] and [TK54], the finite s-indecomposable semigroups are ideal extensions of special completely simple semigroups by nilpotent semigroups. Thus their idempotents are in the completely simple part, and so the cardinality of their subsemilattices is one.

The situation is more interesting in general. In Theorem 3.4.2 (Thesis 4), we show that if $Y$ is a subsemilattice of a finite semilattice indecomposable semigroup $S$, then $|Y| \leq 2\left\lfloor \frac{|S|-1}{4} \right\rfloor + 1$. We also show that, for every positive integer $n$, there are finite semilattice indecomposable semigroups $S$ which contain a subsemilattice $Y$ such that $|Y| = 2\left\lfloor \frac{|S|-1}{4} \right\rfloor + 1$. These semigroups are characterized in that special case when $|S| = 4k+1$ ,$k$ is a non-negative integer (these are $B_2$-combinatorial semigroups). We prove (Theorem 3.5.5; Thesis 5) that a semigroup $S$ is a $B_2$-combinatorial semigroup if and only if $S$ has a zero and, for every nonzero element $a$ of $S$, the principal factor $J(a)/I(a)$ is isomorphic to the semigroup $B_2$.

In Chapter 4, the semigroup algebras of congruence permutable semigroups are in the focus. A semigroup is said to be a congruence permutable semigroup if $\alpha \circ \beta = \beta \circ \alpha$ is satisfied for every congruences $\alpha$ and $\beta$ on $S$, where $\circ$ is the usual composition of binary relations. It is known that a semigroup $S$ is congruence permutable if and only if the set $Con(S)$ of all congruences on $S$ forms a semigroup under the operation $\circ$; in this case the semigroup $(Con(S); \circ)$ is necessarily commutative. The examined problem is the following. Let $S$ be a semigroup and $\mathbb{F}$ a field. For an arbitrary congruence $\alpha$ on $S$, let $\mathbb{F}[\alpha]$ denote the kernel of the extended canonical homomorphism $\mathbb{F}[S] \to \mathbb{F}[S/\alpha]$. By Lemma 5 of Chapter 4 of [Okn91], for every semigroup $S$ and every field $\mathbb{F}$, the mapping

$$\varphi_{\{S;\mathbb{F}\}} Con(\mathbb{F}[S]) \to Con(S)$$

$$J \mapsto \varrho_J$$

is a surjective $\wedge$-homomorphism such that $\varrho_{\mathbb{F}[\alpha]} = \alpha$ for every congruence $\alpha$ on $S$. As a homomorphic image of a semigroup is also a semigroup, and $\alpha \circ \beta = \alpha \vee \beta$ is satisfied for every congruences $\alpha$ and $\beta$ of a congruence permutable semigroup, the following assertions are obvious. If $S$ is a semigroup such that, for a field $\mathbb{F}$, $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism, then $S$ is a congruence permutable semigroup. Moreover, if $S$ is a congruence permutable semigroup, then $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism if and only if $\varphi_{\{S;\mathbb{F}\}}$ is a $\vee$-homomorphism, that is, $ker_{\varphi_{\{S;\mathbb{F}\}}}$ is $\vee$-compatible.

We show that the converse of the first assertion is not true in general: let a congruence permutable semigroup $S$, the mapping $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism or not depends on the field $\mathbb{F}$. We show that if $S = C_4$ is the cyclic group of order 4, then $\varphi_{\{C_4;\mathbb{F}_3\}}$ is not a $\circ$-homomorphism, where $\mathbb{F}_3$ is the field of 3 elements. At the same time, we show that $\varphi_{\{C_4;\mathbb{F}_2\}}$ is a $\circ$-homomorphism, where $\mathbb{F}_2$ is the field of 2 elements.

By the above, it is a natural idea to find all couples $(S, \mathbb{F})$ of congruence permutable semigroups $S$ and fields $\mathbb{F}$, for which the mapping $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism. We prove the following results.

Theorem 5.1.1 (Thesis 6): Let $S$ be a congruence permutable semilattice. Then, for an arbitrary field $\mathbb{F}$, $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism.

Theorem 6.0.1 (Thesis 7): Let $S$ be a congruence permutable rectangular band. Then, for an arbitrary field $\mathbb{F}$, $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism.

Thus, in the class of all semilattices and the class of all rectangular bands, the congruence permutability of semigroups is not only necessary but also sufficient condition to be the mapping $\varphi_{\{S;\mathbb{F}\}} : Con(\mathbb{F}[S] \mapsto Con(S)$ a $\circ$-homomorphism.

## 1.2 Theses of the dissertation

**1 Thesis** *(Theorem 2.4.4) Let $D \subseteq E$ be a subalgebra (not necessarily unitary), and set $A := \gamma_1 \dots \gamma_n(D)$. Then the following hold:*

*(i) $A$ is a subalgebra of $E$,*

*(ii) as an $\mathbb{F}$ vector space it is spanned by elements in the form $v_J$, $J \subseteq [n]$,*

*(iii) $\dim(A) = \dim(D)$,*

*(iv) if $D$ is commutative then so is $A$,*

*(v) if $D^2 = \{0\}$ then $A^2 = \{0\}$.*

**2 Thesis** *(Theorem 2.6.1) Write $k$ for the lower integer part of $n/4$.*

*(i) Let $A$ be a commutative subalgebra of $E$ of maximal dimension. Then*

$$\dim(A) = \dim(E_{\overline{0}}) + |\mathcal{F}|,$$

*where $\mathcal{F} \subseteq 2^{[n]}$ is an odd intersecting family of maximal possible size. Hence*

$$\dim(A) = \begin{cases} 3 \cdot 2^{n-2} & \text{if } n \text{ is even;} \\ 2^{n-1} + \sum_{l=k}^{2k} \binom{n}{2l+1} & \text{if } n = 4k+1; \\ 2^{n-1} + \sum_{l=k}^{2k} \binom{n}{2l+3} + \binom{n-1}{2k} & \text{if } n = 4k+3. \end{cases}$$

*(ii) If $n$ is even, then all maximal commutative subalgebras of $E$ have the same dimension, but they are not all isomorphic for $n > 2$.*

*(iii) If $n = 4k+1$, then $E_{\overline{0}} \oplus (\bigoplus_{n/2<i \text{ odd}} E_i)$ is the only maximal dimensional commutative subalgebra of $E$.*

*(iv) If $n = 4k+3$, then the maximal dimensional commutative subalgebras of $E$ are exactly the subspaces of the form*

$$E_{\overline{0}} \oplus (\bigoplus_{n/2<i \text{ odd}} E_i) \oplus C,$$

*where $C \subset E_{2k+1}$ is a square zero subspace of dimension $\binom{n-1}{2k}$.*

*(v) When $n$ is odd, then exist maximal commutative subalgebras that are not maximal dimensional commutative subalgebras in $E$.*

**3 Thesis** *(Theorem 3.2.1) A finite semigroup $S$ is s-indecomposable if and only if $\mathbb{C}[S/K_S]/J(\mathbb{C}[S/K_S])$ has exactly one 1-dimensional ideal.*

**1.2.1 Definition** *A finite semigroups $S$ is $B_2$-**combinatorial** if s-indecomposable, $|S| = 4k+1$ and it has a subsemilattice $Y$ with size $2\left\lfloor \frac{|S|-1}{4} \right\rfloor + 1 = 2k+1$.*

**4 Thesis** *(Theorem 3.4.2) Let $S$ be an s-indecomposable finite semigroup.*

*(i) If $Y$ is a subsemilattice of $S$, then $|Y| \leq 2\left\lfloor \frac{|S|-1}{4} \right\rfloor + 1$.*

*(ii) For every positive integer $n$, there is a semigroup $S$ such that $|S| = n$ and $S$ has a subsemilattice $Y$ of $S$ such that $|Y| = 2\left\lfloor \frac{|S|-1}{4} \right\rfloor + 1$.*

**5 Thesis** *(Theorem 3.5.5) Let $S$ be a finite semigroup. Then (i) and (ii) are equivalent:*

*(i) $S$ is a $B_2$-combinatorial semigroup,*

*(ii) $S$ has a zero and for every non-zero element $a$ of $S$, the principal factor $J(a)/I(a)$ is isomorphic to the semigroup $B_2$.*

**6 Thesis** *(Theorem 5.1.1) Let $S$ be a congruence permutable semilattice. Then, for an arbitrary field $\mathbb{F}$, $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism.*

**7 Thesis** *(Theorem 6.0.1) Let $S = L \times R$ be a congruence permutable rectangular band where $L$ is a left zero semigroup, $R$ is a right zero semigroup. Then, for an arbitrary field $\mathbb{F}$, $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism.*

# 2 Summary of research results

In this section we summarize the results of chapters of the dissertation. The titles of subsections are same as the titles of the chapters of the dissertation. The numbering of theorems are also follow the numbering of the dissertation.

## 2.1 Commutative subalgebras of Grassmann algebras

In Chapter 2 of the dissertation the maximal dimension of commutative subalgebras of Grassmann algebras is determined. It is shown that for any commutative subalgebra $A$ of a Grassmann algebra $E$, there exists a commutative subalgebra of $E$ which is spanned by monomials and has the same dimension as $A$. It follows that the maximal dimension of a commutative subalgebra can be expressed in terms of the maximal size of an intersecting family of subsets of odd size in a finite set.

## 2.2 Square zero subspaces

For a subset $J \subseteq [n] := \{1, \ldots, n\}$ set $v_J := v_{i_1} \cdots v_{i_k}$, where $J = \{i_1, \ldots, i_k\}$ and $i_1 < \cdots < i_k$. Clearly, $\{v_J \mid J \subseteq [n]\}$ is an $\mathbb{F}$-vector space basis of $E$. We shall refer to the elements $v_J \in E$ as ***monomials***. The Grassmann algebra is graded:

$$E = \bigoplus_{k=0}^{\infty} E_k \text{ where } E_k = \mathrm{Span}_{\mathbb{F}}\{v_J \mid I \subseteq [n], \quad |J| = k\}$$

(of course, for $k > n$ we have $E_k = \{0\}$). Sometimes we pay attention to the $\mathbb{Z}/2\mathbb{Z}$-grading induced by the above $\mathbb{Z}$-grading:

$$E = E_{\overline{0}} \oplus E_{\overline{1}} \text{ where } E_{\overline{0}} := \bigoplus_{k \text{ is even}} E_k, \quad E_{\overline{1}} := \bigoplus_{k \text{ is odd}} E_k$$

The defining relations of $E := E$ imply the multiplication rules $v_J v_K = (-1)^{|J|\cdot|K|} v_K v_J$ and when $J \cap K \neq \emptyset$, then $v_J v_K = 0$. It follows that $E_{\overline{0}}$ is contained in the center of $E$, and the elements of $E_{\overline{1}}$ anticommute: $ab = -ba$ for any pair $a, b \in E_{\overline{1}}$. In particular, $a, b \in E_{\overline{1}}$ commute if and only if $ab = 0$. So the commutative subalgebras of $E$ have a natural connection with square zero subspaces. For subspaces $C, D \subseteq E$ we write $CD$ for the subspace $\mathrm{Span}_{\mathbb{F}}\{cd \mid c \in C, d \in D\}$, and we call a subspace $D \subseteq E$ a ***square zero subspace*** if $D^2 = 0$, that is, if $cd = 0$ for all $c, d \in D$. A commutative subalgebra $A$ of $E$ is called ***maximal*** if there is no commutative subalgebra of $E$ properly containing $A$. Similarly, a square zero subspace of $E_{\overline{1}}$ is called ***maximal*** if it is not properly contained in a square zero subspace of $E_{\overline{1}}$.

**2.2.1 Proposition** *([DZ15])*

*(i) If $D \subseteq E_{\overline{1}}$ is a square zero subspace, then $K := E_{\overline{0}} D \subseteq E_{\overline{1}}$ is also a square zero subspace and $E_{\overline{0}} \oplus K$ is a commutative subalgebra of $E$.*

*(ii) The map $D \mapsto E_{\overline{0}} \oplus D$ gives a bijection between the maximal square zero subspaces in $E_{\overline{1}}$ and the maximal commutative subalgebras of $E$.*

**2.2.2 Remark** It is interesting to compare the above “structure theorem" with the case of the matrix algebra $\mathbb{F}^{n\times n}$, where by a theorem of Schur [Sch05], a commutative subalgebra of maximal possible dimension is also of the form

$$Z(\mathbb{F}^{n\times n}) \oplus D = \mathbb{F} I_n \oplus D,$$

where $D$ is a subspace with $D^2 = 0$, $I_n$ stands for the identity matrix and $Z(\mathbb{F}^{n\times n})$ denote the center of $\mathbb{F}^{n\times n}$. However, unlike for $E$, in $\mathbb{F}^{n\times n}$ not all maximal subalgebras are of this form.

Maximal square zero subspaces in $E_{\overline{1}}$ can be characterized in terms of certain bilinear maps on $E_{\overline{1}}$ defined as the composition of the multiplication map $E_{\overline{1}} \times E_{\overline{1}} \to E_{\overline{0}}$ and a projection from $E_{\overline{0}}$ to one of its homogeneous components. Recall that any $x \in E$ can be uniquely written as

$$x = \sum_{J \subseteq [n]} x_J v_J, \tag{2}$$

where $x_J \in \mathbb{F}$. For every $x \in E$ and $J \subseteq [n]$ the $x_J$ denote the coefficient of $v_J$ in the equation 2.

Define a bilinear map $\phi$ as follows:

- if $n$ is even then $\Phi : E_{\overline{1}} \times E_{\overline{1}} \to E_n$, $\Phi(a, b) = (ab)_{[n]} v_{[n]}$;

- if $n$ is odd then $\Phi : E_{\bar{1}} \times E_{\bar{1}} \to E_{n-1}$, $\Phi(a,b) = \sum_{J \in \binom{[n]}{n-1}} (ab)_J v_J$.

Here $\binom{[n]}{k}$ stands for the set of $k$-element subsets of $[n]$. Since the multiplication map on $E_{\bar{1}}$ is skew-symmetric, the bilinear map $\Phi$ is also skew-symmetric. Since

$$\dim(\mathrm{im}(\Phi)) = \begin{cases} 1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd,} \end{cases}$$

when $n$ is even, the $\mathrm{im}(\Phi) = E_n$ can be identified with $\mathbb{F}$, so $\Phi$ is a skew-symmetric bilinear form. It is a non-degenerate form, since if $x_J \neq 0$ for some $x \in E_{\bar{1}}$ and $J \subseteq [n]$, then $v_{[n]\setminus J} \in E_{\bar{1}}$ and $\Phi(x, v_{[n]\setminus J}) = x_J \neq 0$. It means if $n$ is even then $(E_{\bar{1}}, \Phi)$ is a symplectic vector space.

Given a subspace $D \subseteq E_{\bar{1}}$ we write

$$D^{\perp} := \{x \in E_{\bar{1}} : \Phi(x,w) = 0 \ \forall w \in D\}.$$

**2.2.3 Proposition** *([DZ15]) A subspace $D \subseteq E_{\bar{1}}$ is a maximal square zero subspace in $E_{\bar{1}}$ if and only if $D$ is an $E_{\bar{0}}$-submodule and $D = D^{\perp}$.*

**2.2.4 Corollary** *([DZ15])*
*If $n \geq 2$ is even, then any maximal commutative subalgebra of $E$ has dimension $3 \cdot 2^{n-2}$.*

## 2.3 Commuting projections

Let $W$ be a vector space over a field $\mathbb{F}$, and $\pi_1, \dots, \pi_r \in \mathrm{End}_{\mathbb{F}}(W)$ pairwise commuting projections for some positive $r$. So $\pi_i^2 = \pi_i$ and $\pi_i \pi_j = \pi_j \pi_i$ for all $1 \leq i, j \leq r$. Recall the corresponding direct sum decompositions $W = \ker(\pi_j) \oplus \mathrm{im}(\pi_j)$. Given a subset $J \subseteq [r]$ we set

$$W_J := \bigcap_{j \in J} \ker(\pi_j) \cap \bigcap_{j \notin J} \mathrm{im}(\pi_j).$$

Let $\mathrm{Gras}(W)$ stand for the set of subspaces of $W$, and for $j = 1, \dots, r$ define a map

$$\gamma_j : \mathrm{Gras}(W) \to \mathrm{Gras}(W), \qquad D \mapsto \ker(\pi_j|_D) \oplus \mathrm{im}(\pi_j|_D) \tag{3}$$

where $\pi_j|_D : D \to W$ stands for the restriction of $\pi_j$ to the subspace $D \subseteq W$. Note that for $A, D \in \mathrm{Gras}(W)$ we have $\gamma_j(A) + \gamma_j(D) \subseteq \gamma_j(A+D)$ (this inclusion is proper in general). It is also obvious that if $A \subseteq D$, then $\gamma_j(A) \subseteq \gamma_j(D)$.

**2.3.1 Lemma** *([DZ15]) Take $D \in \mathrm{Gras}(W)$ and denote $A := \gamma_1 \dots \gamma_r(D)$. Then we have the equalities*

*(i)* $\dim(A) = \dim(D)$;

*(ii)* $A = \bigoplus_{J \subseteq [r]} (A \cap W_J)$.

**2.3.2 Remark** *([DZ15])* Note that though $\pi_1, \dots, \pi_r$ commute, the maps $\gamma_1, \dots, \gamma_r$ do not necessary commute, i.e. $\gamma_i \gamma_j(D)$ may be different from $\gamma_j \gamma_i(D)$.

## 2.4 Projections on the Grassmann algebra

We shall apply Lemma 2.3.1 for the case when $W = E$ is the Grassmann algebra. For $i = 1, \dots n$, define the linear map $\pi_i : E \to E$ by

$$\pi_i(x) := \sum_{i \notin J \subseteq [n]} x_J v_J$$

(see (2) for the notation). Then $\pi_i^2 = \pi_i$ and $\pi_i \pi_j = \pi_j \pi_i$. Define $\gamma_i : \mathrm{Gras}(E) \to \mathrm{Gras}(E)$ as in equation (3).

Observe that

$$\ker(\pi_i) = v_i E = E v_i,$$
$$\mathrm{im}(\pi_i) = \mathrm{Span}_{\mathbb{F}}\{v_J | J \subseteq [n], i \notin J\}.$$

Clearly, $W_J = E_J$ is the 1-dimensional subspace spanned by $v_J$. An extra feature now is that the maps $\pi_i$ are algebra homomorphisms, moreover from

$$\ker(\pi_i) = v_i E = E v_i \text{ and } v_i^2 = 0$$

we get

$$\ker(\pi_i)^2 = \{0\}. \tag{4}$$

**2.4.1 Proposition** *([DZ15]) Let $D, A \in \mathrm{Gras}(E)$ be subspaces and $1 \leq i \leq n$. The following hold for $\gamma_i$:*

- *(i) $\gamma_i(A)\gamma_i(D) \subseteq \gamma_i(AD)$.*
- *(ii) If $D$ is a subalgebra of $E$, then $\gamma_i(D)$ is also a subalgebra.*
- *(iii) If $D^2 = \{0\}$, then $\gamma_i(D)^2 = \{0\}$.*
- *(iv) If $D$ is a right ideal [left ideal] in $E$, then $\gamma_i(D)$ is also a right ideal [left ideal] in $E$.*
- *(v) If $D$ is a commutative subalgebra of $E$, then $\gamma_i(D)$ is a commutative subalgebra of $E$.*

**2.4.2 Remark** *([DZ15])* Note that in the situation of Proposition 2.4.1 (ii) the algebra $\gamma_i(D)$ is not necessarily isomorphic to the algebra $D$.

Combining Lemma 2.3.1, Proposition 2.4.1 and the fact that $E_J = \mathbb{F}v_J$ we obtain the following:

**2.4.3 Remark** *([DZ15])* Note that in the situation of Proposition 2.4.1 (ii), the algebra $\gamma_i(D)$ is not necessarily isomorphic to the algebra $D$.

Combining Lemma 2.3.1, Proposition 2.4.1 and the fact that $E_J = \mathbb{F}v_J$ we obtain the following:

**2.4.4 Theorem** *([DZ15]) Let $D \subseteq E$ be a subalgebra (not necessarily unitary), and set $A := \gamma_1 \dots \gamma_n(D)$. Then the following hold:*

*(i)* *$A$ is a subalgebra of $E$,*

*(ii)* *as an $\mathbb{F}$ vector space it is spanned by elements in the form $v_J$, $J \subseteq [n]$,*

*(iii)* $\dim(A) = \dim(D)$,

*(iv)* *if $D$ is commutative then so is $A$,*

*(v)* *if $D^2 = \{0\}$ then $A^2 = \{0\}$.*

**2.4.5 Remark** *([DZ15])* The role of the generators $v_1, \dots, v_n$ is symmetric, so the conclusion of Theorem 2.4.4 holds for

$$A_\sigma := \gamma_{\sigma(1)} \dots \gamma_{\sigma(n)}(D),$$

where $\sigma$ is an arbitrary permutation of $1, \dots, n$. However, different permutations $\sigma$ yield in general different subspaces $A_\sigma$.

The projections $\pi_i$ preserve the degree, hence the maps $\gamma_i$ are also compatible with the grading on $E$:

**2.4.6 Proposition** *([DZ15])*

*(i)* *If $D \subseteq \bigoplus_{k \in I} E_k$ for some $I \subseteq [n]$, then $\gamma_i(D) \subseteq \bigoplus_{k \in I} E_k$.*

*(ii)* *If $D \subseteq \bigoplus_{k \in I} E_k$ and $A \subseteq \bigoplus_{k \in J} E_k$ where $I, J \subseteq [n]$ are disjoint subsets then $\gamma_i(A \oplus D) = \gamma_i(A) \oplus \gamma_i(D)$.*

*(iii)* *If $D = \bigoplus_{k=0}^{n}(D \cap E_k)$ is spanned by its homogeneous components, then we have $\gamma_i(D) = \bigoplus_{k=0}^{n} \gamma_i(D \cap E_k)$ (and $\gamma_i(D \cap E_k) \subseteq E_k$ for all $k$).*

Let $b \in E$ a non-zero element. Write $b^{\min}$ for the homogeneous component of $b$ of minimal degree. To any subspace $A$ of $E$, one can canonically associate a subspace $A^{\min}$ spanned by homogeneous elements as follows:

$$A^{\min} := \mathrm{Span}_{\mathbb{F}}\{b^{\min} \mid b \in A, b \neq 0\}.$$

The following statements are straightforward to prove:

**2.4.7 Proposition** *([DZ15])*

*(i)* $\dim(A^{\min}) = \dim(A)$;

*(ii)* *If $A$ is a subalgebra of $E$, then $A^{\min}$ is a subalgebra of $E$. Moreover, if $A$ is commutative then $A^{\min}$ is commutative. If $A$ is a square zero subspace, then $A^{\min}$ is a square zero subspace.*

Note also that for any graded subalgebra $A$ of $E$, the subalgebra $B := \gamma_1 \dots \gamma_n(A)$ spanned by monomials has the same Hilbert series as $A$: we have $\dim(A \cap E_k) = \dim(B \cap E_k)$ for $k = 0, 1, \dots, n$ by Proposition 2.4.6.

**2.4.8 Remark** *([DZ15])* Not all subalgebras of $E$ are isomorphic to a graded subalgebra of $E$, and not all graded subalgebras of $E$ are isomorphic to a subalgebra generated by monomials.

## 2.5 Odd intersecting families

Theorem 2.4.4 opens the way to reduce certain questions on square zero subspaces of $E_{\overline{1}}$ to questions about odd intersecting families. Recall that a set $\mathcal{F} \subseteq 2^{[n]}$ of subsets of $[n]$ is called an ***intersecting family*** if $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{F}$, and it is an ***odd intersecting family*** if in addition $|A|$ is odd for all $A \in \mathcal{F}$.

**2.5.1 Proposition** *([DZ15]) Let $\mathcal{F} \subseteq 2^{[n]}$ be an odd intersecting family.*

*(i) If $n$ is even, then*

$$|\mathcal{F}| \leq 2^{n-2}.$$

*(ii) If $n$ is odd, $\mathcal{F} \subseteq \binom{[n]}{i} \cup \binom{[n]}{n-i-1}$ for some odd $i$ with $i < n/2 - 1$ and $\mathcal{F}$ is of maximal possible size, then*

$$\mathcal{F} = \binom{[n]}{n-i-1}.$$

*(iii) If $n = 4k+1$ (where $k$ is a non-negative integer) and $|\mathcal{F}|$ is maximal then*

$$\mathcal{F} = \bigcup_{n/2 < i \text{ odd}} \binom{[n]}{i}.$$

*(iv) If $n = 4k+3$ (where $k$ is a non-negative integer) and $|\mathcal{F}|$ is maximal then*

$$\mathcal{F} = \bigcup_{n/2 < i \text{ odd}} \binom{[n]}{i} \cup \left\{ X \in \binom{[n]}{2k+1} \,\middle|\, l \in X \right\},$$

*for some $l \in [n]$.*

## 2.6 Commutative subalgebras of maximal dimension

**2.6.1 Theorem** *([DZ15]) Write $k$ for the lower integer part of $n/4$.*

*(i) Let $A$ be a commutative subalgebra of $E$ of maximal dimension. Then*

$$\dim(A) = \dim(E_{\overline{0}}) + |\mathcal{F}|,$$

*where $\mathcal{F} \subseteq 2^{[n]}$ is an odd intersecting family of maximal possible size. Hence*

$$\dim(A) = \begin{cases} 3 \cdot 2^{n-2} & \text{if } n \text{ is even;} \\ 2^{n-1} + \sum_{l=k}^{2k} \binom{n}{2l+1} & \text{if } n = 4k+1; \\ 2^{n-1} + \sum_{l=k}^{2k} \binom{n}{2l+3} + \binom{n-1}{2k} & \text{if } n = 4k+3. \end{cases}$$

*(ii) If $n$ is even, then all maximal commutative subalgebras of $E$ have the same dimension, but they are not all isomorphic for $n > 2$.*

*(iii) If $n = 4k+1$, then $E_{\overline{0}} \oplus (\bigoplus_{n/2<i \text{ odd}} E_i)$ is the only maximal dimensional commutative subalgebra of $E$.*

*(iv) If $n = 4k+3$, then the maximal dimensional commutative subalgebras of $E$ are exactly the subspaces of the form*

$$E_{\overline{0}} \oplus (\bigoplus_{n/2<i \text{ odd}} E_i) \oplus C,$$

*where $C \subset E_{2k+1}$ is a square zero subspace of dimension $\binom{n-1}{2k}$.*

*(v) When $n$ is odd, then exist maximal commutative subalgebras that are not maximal dimensional commutative subalgebras in $E$.*

# 3 Semilattice indecomposable finite semigroups with large subsemilattices

In the literature of the theory of semigroups there are many papers about s-indecomposable semigroups (see, for example, the papers, [Chr69], [Nag84], [Nag85], [Nag92], [Nag92-2], [Nag93], [Nag98], [NJ04], [Nor88] [PuW71], [Tam82], [TK54], and the books [Gri01], [Nag01]). Some of them deal with the s-indecomposable semigroups without idempotents, the others investigate the s-indecomposable semigroups containing at least one idempotent. In this chapter we deal with finite s-indecomposable semigroups in terms of what can be said about the size of their subsemilattices. The answer is known in special classes of semigroups. As $ef = fe$ implies $e = f$ for every idempotent elements $e$ and $f$ of a completely simple semigroup, the cardinality of the subsemilattices in a completely simple semigroup is one. In the classes of semigroups investigated in [Chr69], [Nag84], [Nag85], [Nag92], [Nag92-2], [Nag93], [Nag98], [NJ04], [Nor88], [TK54], the finite s-indecomposable semigroups are ideal extensions of special completely simple semigroups by nilpotent semigroups. Thus their idempotents are in the completely simple part, and so the cardinality of their subsemilattices is one.

The situation is more interesting in general. We show that if $Y$ is a subsemilattice of a finite s-indecomposable semigroup $S$ then $|Y| \leq 2 \left\lfloor \frac{|S|-1}{4} \right\rfloor + 1$. We also show that there are finite s-indecomposable semigroups $S$ which contain a subsemilattice $Y$ such that $|Y| = 2\left\lfloor \frac{|S|-1}{4} \right\rfloor + 1$. Moreover, these semigroups are characterized here, when $|S| = 4k + 1$.

## 3.1 Notions used in this chapter

Let $S$ be a semigroup. Let $\mathbb{C}[S]$ denote the semigroup algebra of $S$ over the field $\mathbb{C}$ of all complex numbers. The contracted semigroup algebra of a semigroup $S$ with a zero (over $\mathbb{C}$) will be denoted by $\mathbb{C}_0[S]$ (see [Okn91, p.35]).

For a finite dimensional algebra $A$ over $\mathbb{C}$, the Jacobson radical of $A$ will be denoted by $J(A)$. It is known that $J(A)$ is the set of all properly nilpotent elements of $A$. We will use the following well-known facts: the factor algebra $A/J(A)$ is semisimple (and so, for a finite semigroup $S$, $\mathbb{C}[S]/J(\mathbb{C}[S])$ is semisimple), moreover a finite dimensional algebra $A$ over $\mathbb{C}$ is semisimple if and only if $A$ is isomorphic to $\bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C})$, where $M_n(\mathbb{C})$ denotes the associative algebra of all $n \times n$ matrices over $\mathbb{C}$.

If a semigroup $S$ has a minimal ideal $K_S$, then $K_S$ is called the ***it kernel*** of $S$. Every finite semigroup evidently has a kernel. If a semigroup $S$ has a kernel, then $K_S$ is a simple subsemigroup of $S$ [CP61, Cor. 2.30. p.69]. Every finite simple [0-simple] semigroup is completely simple [completely 0-simple] by [CP61, Cor. 2.56. p.83].

The notions of the Rees matrix semigroups and the completely 0-simple semigroups were defined in Chapter 1 of the dissertation. It is known that a semigroup is completely 0-simple if and only if it is isomorphic with a regular Rees matrix semigroup over a group with a zero. A completely 0-simple semigroup is an inverse semigroup if and only if it is a Brandt semigroup. In our investigation a special type of Brandt semigroups is in the focus. This is the semigroup $\mathcal{M}^0(1; 2, 2; I)$ where 1 denote the one-element group and $I$ is the $2 \times 2$ identity matrix. We will denote this Brandt semigroup by $B_2$.

## 3.2 Semilattice indecomposable semigroups

A semigroup $S$ is said to be a ***semilattice indecomposable*** (s-indecomposable) semigroup if every semilattice homomorphic image of $S$ is trivial (that is, it contains only one element). An ideal $I$ of a semigroup $S$ is called a ***completely prime ideal*** if $S \setminus I$ is a subsemigroup of $S$. It is known ([Pet77, I.8.3. Prop. p.15]) that a semigroup is s-indecomposable if and only if it does not contain completely prime ideals. Corollary in [Tam72] gives an other characterization of s-indecomposable semigroups. A semigroup $S$ is s-indecomposable if and only if, for every $a, b \in S$, there is a sequence $a = a_0, a_1, \ldots, a_{n-1}, a_n = b$ of elements of $S$ such that $a_{i-1}$ divides some power of $a_i$ $(i = 1, \ldots, n)$.

In Theorem 3.2.1 we give a new characterization of finite s-indecomposable semigroups $S$ by the terms of semigroup algebras $\mathbb{C}[S/K_S]$.

**3.2.1 Theorem** *([Zub16]) A finite semigroup $S$ is s-indecomposable if and only if, $\mathbb{C}[S/K_S]/J(\mathbb{C}[S/K_S])$ has exactly one 1-dimensional ideal.*

**3.2.2 Remark** *([Zub16]) In case of finite semigroups with zero, s-indecomposability can completely be described in terms of the semigroup algebra (Theorem 3.2.1). It is not true for finite semigroups in general. For example if $G$ is a finite Abelian group and $Y$ is a finite semilattice such that $|G| = |Y|$, then $\mathbb{C}[G] \cong \bigoplus_{i\in G} \mathbb{C} \cong \mathbb{C}[Y]$. Thus, if $1 < |G| = |Y|$, then $G$ is s-indecomposable, $Y$ is not, but $\mathbb{C}[G] \cong \mathbb{C}[Y]$.*

## 3.3 Embeddings into semilattice indecomposable semigroups

Let $A, B$ be semigroups with zeros $z_A, z_B$. Then $A \times B$ has an ideal

$$I = (\{z_A\} \times B) \cup (A \times \{z_B\}).$$

Let $A \times_0 B$ denote the Rees factor semigroup $(A \times B)/I$.

**3.3.1 Proposition** *([Zub16]) For arbitrary semigroups $A$ and $B$ with zeros, the semigroup $A \times_0 B$ is s-indecomposable if and only if $A$ or $B$ is s-indecomposable.*

Our next goal is to describe the smallest s-indecomposable semigroup which contains a 2-element subsemilattice. We will see that, it has 5 elements. We also show that, the smallest s-indecomposable semigroup which contains a 3-element subsemilattice is isomorphic to the semigroup $B_2$ (Theorems 3.4.2 and 3.5.5) below. First we show that there are only three nonisomorphic 5-element s-indecomposable semigroups with a 2-element subsemilattice.

**3.3.2 Corollary** *([Zub16]) Let $S$ be an s-indecomposable semigroup such that $|S| \leq 5$ and $S$ has at least two commuting idempotents. Then*

$$S \cong \mathcal{M}^0(1; 2, 2; P),$$

*where $P$ is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. When $P$ is the identity matrix then $S \cong B_2$.*

**3.3.3 Corollary** *([Zub16]) Every finite semigroup $S$ can be embedded into an s-indecomposable semigroup containing $4|S| + 1$ elements.*

**3.3.4 Proposition** *([Zub16]) Every finite s-indecomposable semigroup $S$ with a zero can be embedded into an s-indecomposable semigroup containing $|S| + 1$ elements.*

## 3.4 On order of subsemilattices of semilattice indecomposable finite semigroups

In this section we answer the question: what the order of subsemilattices of s-indecomposable finite semigroups is. First we deal with the case when the semigroup in question has a zero (Proposition 3.4.1). Then we consider the general case (Theorem 3.4.2).

**3.4.1 Proposition** *([Zub16]) Let $S$ be an s-indecomposable finite semigroup with a zero. If $Y$ is a subsemilattice of $S$ then $|Y| \leq 2\left\lfloor \frac{|S|-1}{4} \right\rfloor + 1$.*

**3.4.2 Theorem** *([Zub16]) Let $S$ be an s-indecomposable finite semigroup.*

*(i) If $Y$ is a subsemilattice of $S$, then $|Y| \leq 2\left\lfloor \frac{|S|-1}{4} \right\rfloor + 1$.*

*(ii) For every positive integer $n$, there is a semigroup $S$ such that $|S| = n$ and $S$ has a subsemilattice $Y$ of $S$ such that $|Y| = 2\left\lfloor \frac{|S|-1}{4} \right\rfloor + 1$.*

## 3.5 $B_2$-combinatorial semigroups

In this section we only deal with s-indecomposable semigroups $S$ with $4k + 1$ elements which containing a subsemilattice $Y$ with $2\left\lfloor \frac{|S|-1}{4} \right\rfloor + 1$ elements. In the this section we describe the structure of these ones.

**3.5.1 Definition** *([Zub16]) A semigroup $S$ is said to be $B_2$**-combinatorial** if $S$ is s-indecomposable, $|S| = 4k + 1$ ($k$ is a non-negative integer) and $S$ has a subsemilattice $Y$ with $|Y| = 2\left\lfloor \frac{|S|-1}{4} \right\rfloor + 1 = \frac{|S|+1}{2} = 2k + 1$.*

The name $B_2$-combinatorial will be clear in Theorem 3.5.5. First of all we note that the semigroup $B_2$ is $B_2$-combinatorial.

**3.5.2 Proposition** *([Zub16]) Let $S$ be a $B_2$-combinatorial semigroup. Then all of the following assertions hold:*

*(i) $S$ has a zero.*

*(ii) The semigroup algebra $\mathbb{C}[S]$ is isomorphic to $\mathbb{C} \oplus \bigoplus_{i=1}^{k} M_2(\mathbb{C})$.*

*(iii) Every ideal of $S$ is $B_2$-combinatorial.*

*(iv) Every homomorphic image of $S$ is $B_2$-combinatorial.*

**3.5.3 Lemma** *([Zub16]) Let $S$ be a completely 0-simple semigroup and $Y$ a subsemilattice of $S$. Then*

$$|Y| \leq \sqrt{|S| - 1} + 1.$$

*If $|Y| = \sqrt{|S| - 1} + 1$, then $S \cong \mathcal{M}^0(1; n, n; I)$, where $n = \sqrt{|S| - 1}$.*

**3.5.4 Proposition** *([Zub16]) If $S$ is a $B_2$-combinatorial 0-simple semigroup, then $S \cong B_2$.*

On ***principal ideal*** of a semigroup we mean an ideal generated by a single element. Let $S$ be a semigroup. Let $J(a)$ denote the principal ideal of $S$ generated by an element $a \in S$. Then

$$I(a) := \{b \mid b \in J(a); J(a) \neq J(b)\}$$

is either empty or an ideal of $S$. The factor semigroup $J(a)/I(a)$ is called a ***principal factor*** of $S$. It is known that every principal factor of semigroup is a 0-simple, a simple or a null semigroup ([CP61, Lemma 2.39. p.73]).

**3.5.5 Theorem** *([Zub16]) Let $S$ be a finite semigroup. Then (i) and (ii) are equivalent:*

*(i) $S$ is a $B_2$-combinatorial semigroup,*

*(ii) $S$ has a zero and for every non-zero element $a$ of $S$, the principal factor $J(a)/I(a)$ is isomorphic to the semigroup $B_2$.*

# 4 Congruence permutable semigroups

In this chapter we consider a semigroup algebraic problem in which the congruence permutable semigroups are in the focus. For an ideal $J$ of a semigroup algebra $\mathbb{F}[S]$, let $\varrho_J$ denote the congruence on the semigroup $S$ which is the restriction of the congruence on $\mathbb{F}[S]$ defined by $J$. We show that if $S$ is a semilattice or a rectangular band, then the mapping $\varphi_{\{S;\mathbb{F}\}} \mid J \mapsto \varrho_J$ is a $\circ$-homomorphism ($\circ$ is the relation composition) if and only if $S$ is congruence permutable.

## 4.1 The general case

Let $S$ be a semigroup and $\mathbb{F}$ a field. For an arbitrary congruence $\alpha$ on $S$, denote the kernel of the extended canonical homomorphism $\mathbb{F}[S] \to \mathbb{F}[S/\alpha]$ by $\mathbb{F}[\alpha]$. By Lemma 5 of Chapter 4 of [Okn91], for every semigroup $S$ and every field $\mathbb{F}$, the mapping

$$\varphi_{\{S;\mathbb{F}\}} : Con(\mathbb{F}[S]) \to Con(S)$$

$$J \mapsto \varrho_J$$

is a surjective $\wedge$-homomorphism such that $\varrho_{\mathbb{F}[\alpha]} = \alpha$ for every congruence $\alpha$ on $S$. Int his chapter we examine that, the mapping $\mathbb{F}[S]$ preserves $\circ$ or $\vee$ or not, where $\circ$ is composition of relations and $\vee$ is the join operation of the congruence lattice. As a homomorphic image of a semigroup is also a semigroup, and $\alpha \circ \beta = \alpha \vee \beta$ is satisfied for every congruence $\alpha$ and $\beta$ of a congruence permutable semigroup, the assertions of the following lemma are obvious.

**4.1.1 Lemma** *([NZ16]) Let $S$ be a semigroup and $\mathbb{F}$ a field. Assume that the mapping $\varphi_{\{S;\mathbb{F}\}} : Con(\mathbb{F}[S]) \to \mathcal{B}_S;\ J \mapsto \varrho_J$ is a $\circ$-homomorphism. Then $S$ is a* *congruence permutable semigroup. Moreover, if $S$ is a congruence permutable semigroup, then $\varphi_{\{S;\mathbb{F}\}} : Con(\mathbb{F}[S]) \to Con(S);\ J \mapsto \varrho_J$ is a $\circ$-homomorphism if and only if $\vee$-homomorphism, that is, $ker_{\varphi_{\{S;\mathbb{F}\}}}$ is $\vee$-compatible.*

The next example shows that the converse of the first assertion of Lemma 5.0.1 is not true, in general; for a congruence permutable semigroup $S$, the condition "$\varphi_{\{S;\mathbb{F}\}}$ is a homomorphism of $(Con(\mathbb{F}[S]);\circ)$ onto the semigroup $(Con(S);\circ)$" depends on the field $\mathbb{F}$.

**4.1.2 Example** *Let $C_4$, $\mathbb{F}_3$ and $\mathbb{F}_2$ denote the cyclic group of order 4, the fields of 3 and 2 elements, respectively. It is known that every group is a congruence permutable semigroup. The $ker_{\varphi_{\{C_4;\mathbb{F}_3\}}}$ is not $\vee$-compatible however $ker_{\varphi_{\{C_4;\mathbb{F}_2\}}}$ is $\vee$-compatible.*

By Lemma 5.0.1 and the Example 5.0.2, it is a natural idea to find all pairs $(S,\mathbb{F})$ of congruence permutable semigroups $S$ and fields $\mathbb{F}$, for which the mapping $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism. Next we show that if $S$ is an arbitrary congruence permutable semilattice or an arbitrary congruence permutable rectangular band, then $\varphi_{\{S;\mathbb{F}\}}$ is $\vee$-compatible for an arbitrary field $\mathbb{F}$.

## 4.2 Semilattices

**4.2.1 Theorem** *([NZ16]) Let $S$ be a congruence permutable semilattice. Then, for an arbitrary field $\mathbb{F}$, $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism.*

**4.2.2 Corollary** *([NZ16]) Let $S$ be a semilattice. Then, for a field $\mathbb{F}$, $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism if and only if $S$ is congruence permutable.*

# 5 Rectangular bands

**5.0.1 Theorem** *([NZ16]) Let $S = L \times R$ be a congruence permutable rectangular band where $L$ is a left zero semigroup, $R$ is a right zero semigroup. Then, for an arbitrary field $\mathbb{F}$, $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism.*

**5.0.2 Corollary** *([NZ16]) Let $S = L \times R$ be a rectangular band. Then, for a field $\mathbb{F}$, $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism if and only if $S$ is congruence permutable.*

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