On Commutative Substructures of Special Algebraic Structures

PhD Dissertation

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Introduction

In the dissertation we deal with three topics; all of them require the commutativity of certain substructures of special algebraic structures. The purpose of the examinations on the first topic is to give an upper bound to the dimension of the commutative subalgebras of Grassmann algebras. This is a joint work with Mátyás Domokos, see [DZ15]. Investigations on the second topic are designed to give an upper bound to the size of subsemilattices of finite semilattice indecomposable semigroups, see [Zub16]. The third theme is related to semigroups S whose sets of congruences forms commutative subsemigroups in the semigroup of all binary relations of S; these semigroups are called congruence permutable semigroups. This is a joint work with Attila Nagy, see [NZ16].

The dissertation contains an unnumbered Introduction and further four numbered chapters.

Chapter 1 is the Preliminaries, in which we present those notations, basic notions and results which are used in the dissertation.

In Chapter 2, the maximal dimension of commutative subalgebras of Grassmann algebras is in the focus. The ***Grassmann algebra*** is the associative $\mathbb{F}$-algebra given in terms of generators and relations as

$$E = \mathbb{F}\langle v_1, ..., v_n \mid \quad v_i v_j = -v_j v_i \quad (1 \le i, j \le n)\rangle.$$

The number of generators of E is n. Write k for the lower integer part of $n/4$. For other related definitions and notations see Section 2.1. An $\mathcal{F} \subseteq 2^{[n]}$ set family is an ***odd intersecting family*** if intersecting family which only contains sets of odd size.

We prove the following assertions:

(i) Let A be a commutative subalgebra of E of maximal dimension. Then

$$\dim(A) = \dim(E_{\bar{0}}) + |\mathcal{F}|,$$

where $\mathcal{F} \subseteq 2^{[n]}$ is an odd intersecting family of maximal possible size. Hence

$$\dim(A) = \begin{cases} 3 \cdot 2^{n-2} & \text{if } n \text{ is even;} \\ 2^{n-1} + \sum_{l=k}^{2k} \binom{n}{2l+1} & \text{if } n = 4k+1; \\ 2^{n-1} + \sum_{l=k}^{2k} \binom{n}{2l+3} + \binom{n-1}{2k} & \text{if } n = 4k+3. \end{cases}$$

(ii) If n is even, then all maximal commutative subalgebras of E have the same dimension, but they are not all isomorphic for $n > 2$.

(iii) If $n = 4k+1$, then $E_{\bar{0}} \oplus (\bigoplus_{n/2<i \text{ odd}} E_i)$ is the only maximal dimensional commutative subalgebra of E.

(iv) If $n = 4k+3$, then the maximal dimensional commutative subalgebras of E are exactly the subspaces of the form

$$E_{\overline{0}} \oplus (\bigoplus_{n/2<i \text{ odd}} E_i) \oplus C,$$

where $C \subset E_{2k+1}$ is a square zero subspace of dimension $\binom{n-1}{2k}$.

(v) When n is odd, then exist maximal commutative subalgebras that are not maximal dimensional commutative subalgebras in E.

In Chapter 3, subsemilattices of finite semilattice indecomposable semigroups are in the focus. It is known that every semigroup is a semilattice of semilattice indecomposable (s-indecomposable) semigroups. In the literature of semigroup theory there are many papers about s-indecomposable semigroups (see, for example, the papers [Chr69], [Nag84], [Nag85], [Nag92], [Nag92-2], [Nag93], [Nag98], [NJ04], [Nor88] [PuW71], [Tam82], [TK54], and the books [Gri01], [Nag01]). Some of them deal with s-indecomposable semigroups without idempotents, others investigate s-indecomposable semigroups containing at least one idempotent. In this chapter we deal with finite s-indecomposable semigroups in terms of what can be said about the size of their subsemilattices. The answer is known in special classes of semigroups. As $ef = fe$ implies $e = f$ for all idempotent elements e and f of a completely simple semigroup, in a completely simple semigroup the order of the subsemilattices is one. In the classes of semigroups investigated in [Chr69], [Nag84], [Nag85], [Nag92], [Nag92-2], [Nag93], [Nag98], [NJ04], [Nor88], [TK54], the finite s-indecomposable semigroups are ideal extensions of special completely simple semigroups by nilpotent semigroups. Thus their idempotents are in the completely simple part, and so the order of their subsemilattices is one.

The situation is more interesting in general. In Chapter 3 we show that if Y is a subsemilattice of a finite semilattice indecomposable semigroup S, then

$$|Y| \leq 2\left\lfloor \frac{|S|-1}{4} \right\rfloor + 1.$$

We also show that, for every positive integer n, there are finite semilattice indecomposable semigroups S with n elements which contain a subsemilattice Y such that $|Y| = 2\left\lfloor \frac{|S|-1}{4} \right\rfloor + 1$. These semigroups (called B_2-combinatorial) are characterized here in the special case when $|S| = 4k+1$ (k is a non-negative integer). We prove that a semigroup S is a B_2-combinatorial semigroup if and only if S has a zero and for every non-zero element a of S, the principal factor $J(a)/I(a)$ is isomorphic to the semigroup B_2. B_2 is a specific five element Brandt semigroup (completely 0-simple inverse semigroups).

In Chapter 4, the semigroup algebras of congruence permutable semigroups are in the focus. The examined problem is the following. Let S be a semigroup and $\mathbb{F}$ a field. For an arbitrary congruence α on S, let $\mathbb{F}[\alpha]$ denote the kernel of the extended canonical homomorphism $\mathbb{F}[S] \to \mathbb{F}[S/\alpha]$. By Lemma 5 of Chapter 4 of [Okn91], for every semigroup S and every field $\mathbb{F}$, the mapping

$$\varphi_{\{S;\mathbb{F}\}} : Con(\mathbb{F}[S]) \to Con(S)$$

$$J \mapsto \varrho_J$$

is a surjective $\wedge$-homomorphism such that $\varrho_{\mathbb{F}[\alpha]} = \alpha$ for every congruence α on S. Denote the semigroup of binary relations on S by $(\mathcal{B}_S; \circ)$. Homomorphic image of semigroups are also semigroups, the $\alpha \circ \beta = \alpha \vee \beta$ is satisfied for every congruences α and β of a congruence permutable semigroup, the following assertions are obvious:

(i) If S is a semigroup such that, for a field $\mathbb{F}$,

$$\varphi_{\{S;\mathbb{F}\}} : Con(\mathbb{F}[S]) \to \mathcal{B}_S; J \mapsto \varrho_J$$

is a $\circ$-homomorphism then S is a congruence permutable semigroup.

(ii) If S is a congruence permutable semigroup, then $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism if and only if $\varphi_{\{S;\mathbb{F}\}}$ is a $\vee$-homomorphism, that is, $ker_{\varphi_{\{S;\mathbb{F}\}}}$ is $\vee$-compatible.

We show that the converse of the (i) assertion is not true in general; for a congruence permutable semigroup S, the condition:

(i) $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism

depends on the field $\mathbb{F}$. We show that if $S = C_4$ is the cyclic group of order 4, then $\varphi_{\{C_4;\mathbb{F}_3\}}$ is not a $\circ$-homomorphism, where $\mathbb{F}_3$ is the field 3 element. At the same time, we show that $\varphi_{\{C_4;\mathbb{F}_2\}}$ is a $\circ$-homomorphism, where $\mathbb{F}_2$ is the field of 2 element.

By the above, it is a natural idea to find all pairs $(S, \mathbb{F})$ of congruence permutable semigroups S and fields $\mathbb{F}$, for which the mapping $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism. We prove that if S is an arbitrary congruence permutable semilattice or an arbitrary congruence permutable rectangular band, then $\varphi_{\{S;\mathbb{F}\}}$ satisfies the condition (i) for an arbitrary field $\mathbb{F}$. Thus, in the class of all semilattices and the class of all rectangular bands, the congruence permutability of semigroups is not only necessary but also sufficient condition for the mapping $\varphi_{\{S;\mathbb{F}\}} : Con(\mathbb{F}[S]) \mapsto Con(S)$ to be a $\circ$-homomorphism.

Chapter 1

Preliminaries

In this chapter we present a collection of definitions and basic theorems from various fields of algebra (namely, semigroup theory, symplectic linear algebra, gradings on associative algebras and intersecting families) which will be used in this dissertation.

1.1 Semigroup theory

In this section we introduce some definitions and results from the theory of semigroups related to Chapters 3 and 4.

1.1.1 Definition *A **semigroup** is a non-empty set together with a binary operation in which the operation is associative. A semigroup containing an identity element is called a **monoid**.*

Let S be a semigroup and let 1 be a symbol not representing any element of S. Extend the given binary operation in S to one in $S \cup \{1\}$ by defining $11 = 1$ and $1s = s1 = s$ for every $s \in S$. Then $S \cup \{1\}$ is a monoid (with identity element 1). We say that this monoid is obtained from S by adjunction an identity element to S.

Similarly, one may adjoin an element 0 to S by defining $00 = 0s = s0 = 0$ for every $s \in S$. Then $S \cup \{0\}$ is a semigroup with the zero 0. We shall use the following notations.

Let S be a semigroup. Then S^1 denotes the semigroup S if S has an identity element and the semigroup $S \cup \{1\}$ otherwise. Similarly S^0 denotes the semigroup S if S has a zero element and $|S| > 1$ and the semigroup $S \cup \{0\}$ otherwise.

Relation semigroups, congruences on semigroups

Let X be a non-empty set. For arbitrary binary relations α and β on X, $\alpha \circ \beta$ denotes the binary relation on X defined by $(a, b) \in \alpha \circ \beta$ if and only if there is an element $x \in X$ such that $(a, x) \in \alpha$ and $(x, b) \in \beta$. The set $\mathcal{B}_X$ of all binary relations on X is a semigroup with respect to the operation $\circ$.

1.1.2 Definition *An equivalence relation α on a semigroup S is called a **congruence relation** (or a **congruence**) on S if, for every $a, b, c, d \in S$, the assumption $(a, b) \in \alpha$ and $(c, d) \in \alpha$ implies $(ac, bd) \in \alpha$.*

It is known that an equivalence relation α on a semigroup S is a congruence if and only if, for every $a, b, s \in S$, the assumption $(a, b) \in \alpha$ implies $(as, bs) \in \alpha$ and $(sa, sb) \in \alpha$.

1.1.3 Definition *A semigroup S is said to be a **congruence permutable semigroup** if, for every congruences α and β on S, $\alpha \circ \beta = \beta \circ \alpha$.*

It is well-known that a semigroup S is congruence permutable if and only if the congruences on S form a subsemigroup in the semigroup $\mathcal{B}_S$ of all binary relations on S.

Bands

1.1.4 Definition *An element e of a semigroup S is called an **idempotent element** if $e^2 = e$.*

For arbitrary idempotent elements e and f of a semigroup S, let $e \leq f$ denote the fact that $ef = fe = e$. It is known that $\leq$ is a partial order on the set $E(S)$ of all idempotent elements of a semigroup S. If a semigroup contains a zero element 0 then $0 \leq e$ is satisfied for every $e \in E(S)$.

1.1.5 Definition *An idempotent element $e \neq 0$ of a semigroup S is said to be a **primitive idempotent element** of S if $f \leq e$ implies $f = e$ or $f = 0$ (if S has a zero).*

1.1.6 Definition *A semigroup S is called a **band** if every element of S is an idempotent element. A commutative band is called a **semilattice**.*

1.1.7 Theorem *([Ham75]) A semilattice is congruence permutable if and only if it has at most two elements.*

1.1.8 Definition *A semigroup satisfying the identity $ab = a$ [$ab = b$] is called a **left zero semigroup** [**right zero semigroup**].*

It is clear that every left zero [right zero] semigroup is a band.

1.1.9 Definition *A semigroup satisfying the identity $aba = a$ is called a* ***rectangular band****.*

The identity $aba = a$ implies $a^2 = a$. It means the semigroups with identity $aba = a$ are bands.

1.1.10 Theorem *([Pet77, II.1.5. Lemma]) A semigroup is a rectangular band if and only if it is a direct product $L \times R$ of a left zero semigroup L and a right zero semigroup R.*

1.1.11 Theorem *([BC81]) A rectangular band $L \times R$ is congruence permutable if and only if $|L|, |R| \leq 2$.*

Regular semigroups, inverse semigroups

1.1.12 Definition *An element a of a semigroup S is called a* ***regular element*** *if there is an element $x \in S$ such that $axa = a$ is satisfied.*

It is easy to see that $axa = a$ implies that ax and xa are idempotent elements of S. It is clear that every idempotent element of a semigroup is regular. Thus a semigroup has a regular element if and only if it has an idempotent element.

1.1.13 Definition *A semigroup is called a* ***regular semigroup*** *if every element of S is regular.*

1.1.14 Definition *We say that the element b of a semigroup S is an **inverse** of an element a of S if $aba = a$ and $bab = b$ are satisfied.*

It is easy to see that if a is a regular element of a semigroup S such that $axa = a$ for some $x \in S$, then $b = xax$ is an inverse of a. Thus every element of a regular semigroup has an inverse.

1.1.15 Definition *A regular semigroup in which every element has exactly one inverse is called an **inverse semigroup**.*

1.1.16 Theorem *([CP61, Theorem 1.17]) A regular semigroup is an inverse semigroup if and only if the idempotents of S commute with each other.*

Ideals, simple and 0-simple semigroups

1.1.17 Definition *A non-empty subset I of a semigroup S is called an **ideal** of S if both $as, sa \in I$ for every $a \in I$ and $s \in S$.*

If I is an ideal of a semigroup S then the relation

$$\varrho_I = \{(a, b) \in S \times S \mid a = b \quad \text{or} \quad a, b \in I\}$$

is a congruence on S. This congruence is called the ***Rees congruence*** on S defined by the ideal I. The factor semigroup $Q = S/\varrho_I$ is said to be the ***Rees factor semigroup*** of S defined by the ideal I.

If I is an ideal of a semigroup S and Q denotes the Rees factor semigroup S/ϱ_I, then we also say that S is an ***ideal extension*** (briefly: an extension) of the semigroup I by the semigroup Q.

1.1.18 Definition *An ideal I of a semigroup S is called a **proper ideal** of S if $I \neq S$. A semigroup S is called a **simple semigroup** if it has no proper ideal. The trivial semigroup is considered a simple semigroup.*

1.1.19 Definition *A semigroup S with zero element 0 is called 0**-simple** if $S^2 \neq \{0\}$ and 0 is the only proper ideal of S.*

1.1.20 Theorem *([CP61, Lemma 2.28]) A semigroup S with zero 0 containing at least two elements is 0-simple if and only if $SaS = S$ for every element $a \neq 0$ of S.*

Completely simple, completely 0-simple semigroups

1.1.21 Definition *By a **completely [0-] simple** semigroup we mean a [0-] simple semigroup containing a primitive idempotent. The trivial semigroup is considered as a completely simple semigroup.*

In the characterization of completely simple and completely 0-simple semigroups the following semigroups play an important role.

Let G be a group, and I, J be arbitrary sets. By a ***Rees** $I \times J$ **matrix over** G^0* we mean an $I \times J$ matrix over G^0 having at most one non-zero entry. Denote the zero matrix with $\mathbf{0}$. If $a \in G$, $i \in I$ and and $j \in J$, then $(a)_{ij}$ will denote the Rees $I \times J$ matrix over G^0 having a in the ith row and jth column, its remaining entries being 0.

Let $P = (p_{ji})$ be an arbitrary but fixed $J \times I$ matrix over G^0. We define a binary operation $\circ$ on the set of Rees $I \times J$ matrices over G^0 as follows: For arbitrary Rees $I \times J$ matrices A and B, let $A \circ B = APB$. Easy to see, that this operation is associative. Hence the set of all Rees $I \times J$ matrices over G^0 is a semigroup with respect to the binary operation $\circ$; we call it the ***Rees*** $I \times J$ ***matrix semigroup over the group with zero*** G^0 ***with sandwich matrix*** P, and denote it by $\mathcal{M}^0(G; I, J; P)$. We call G the structure group of $\mathcal{M}^0$. If P contains no zero entry, then there are no proper divisors of zero in $\mathcal{M}^0(G; I, J; P)$. We call semigroup $\mathcal{M}^0 \setminus \{0\}$ the ***Rees*** $I \times J$ ***matrix semigroup without zero over the group*** G ***with sandwich matrix*** P, and denote by $\mathcal{M}(G; I, J; P)$.

1.1.22 Theorem *([CP61, Lemma 3.1]) A Rees $I \times J$ matrix semigroup $\mathcal{M}^0(G; I, J; P)$ over a group with zero G^0, and with sandwich matrix P, is regular if and only if each row and each column of P contains a non-zero entry.*

1.1.23 Theorem *(Rees; [CP61, Theorem 3.5]). A semigroup is completely 0-simple if and only if it is isomorphic to a regular Rees matrix semigroup over a group with zero. A semigroup is completely simple if and only if it is isomorphic to a Rees matrix semigroup without zero over a group.*

1.1.24 Definition *A* ***Brandt semigroup*** *is a semigroup S with zero satisfying the following conditions: (1) To each element $a \neq 0$ of S there correspond a unique element e of S such that $ea = a$, a unique element $f \in S$ such that $af = a$, and a unique element $a-1$ of S such that $a^{-1}a = f$; (2) If e and f are non-zero idempotents of S then $eSf \neq 0$.*

1.1.25 Theorem *([CP61, Thm. 3.9. p.102]) The following three conditions on a semigroup S with zero are equivalent.*

- *S is a Brandt semigroup.*
- *S is a completely 0-simple inverse semigroup.*
- *S is isomorphic to a (regular) Rees $I \times I$ matrix semigroup*

$$\mathcal{M}^0(G; I, I; E_I)$$

over a group with zero G^0 where E_I is the $I \times I$ identity matrix.

In Chapter 3, a special type of Brandt semigroups is in the focus. This is the semigroup $\mathcal{M}^0(1; 2; 2; E_2)$, where 1 denote the trivial group. We will denote this Brandt semigroup by B_2. The semigroup B_2 has five elements and it has a representation with five 2×2 matrix with matrix multiplication.

$$B_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

Semilattice decomposition of semigroups

1.1.26 Definition *A congruence relation α of a semigroup S is called a* ***semilattice congruence*** *if the factor semigroup $I = S/\alpha$ is a semilattice.*

The α-classes S_i ($i \in I$) are subsemigroups of S such that $S_i S_j \subseteq S_{ij}$, where ij is the product of i and j in the semilattice I. We also say that the semigroup S is a semilattice I of subsemigroups S_i ($i \in I$).

Every semigroup has a semilattice congruence, namely the universal relation.

1.1.27 Definition *A semigroup S is said to be **semilattice indecomposable** (or s-indecomposable) if the universal relation ω_S is the only semilattice congruence on S.*

Let S be a semigroup and σ a relation on S defined by $a\sigma b$ if and only if a divides some power of b, that is, $xay = b^m$ for some $x, y \in S^1$ and some positive integer m. Let ϱ be the transitive closure of σ, and let ϱ' defined by $a\varrho' b$ if and only if $a\varrho b$ and $b\varrho a$.

1.1.28 Theorem *([Tam68, THEOREM]) ϱ' is a smallest semilattice congruence on a semigroup S, and each ϱ'-class is an s-indecomposable semigroup.*

In other words: every semigroup is decomposable into a semilattice of s-indecomposable semigroups. The next result is a consequence of Theorem 1.1.28.

1.1.29 Theorem *([Tam68, COROLLARY]) A semigroup S is s-indecomposable if and only if, for every $a, b \in S$, there is a sequence*

$$a = a_0, a_1, \ldots, a_{k-1}, a_k = b$$

of elements of S for some k such that for all $1 \leq i \leq k$ a_{i-1} divides some power of a_i.

Semigroup Algebra

Let S be a semigroup and $\mathbb{F}$ a field. Let $\mathbb{F}[S]$ denote the set of all mappings

$$a : s \mapsto a(s)$$

of S into $\mathbb{F}$ such that the support of a, that is the set of all $s \in S$ such that $a(s) \neq 0$, is finite. Define the sum $a+b$ of elements a and b of $\mathbb{F}[S]$ such that

$$(a+b)(s) = a(s) + b(s)$$

for every $s \in S$, and the product αa of $a \in \mathbb{F}[S]$ and $\alpha \in \mathbb{F}$ such that

$$(\alpha a)(s) = \alpha(a(s))$$

for every $s \in S$. It is clear that $\mathbb{F}[S]$ is a vector space over $\mathbb{F}$. For arbitrary $a, b \in \mathbb{F}[S]$, define the product ab such that, for every $s \in S$,

$$(ab)(s) = \sum_{rt=s} a(r)b(t).$$

It is easy to see that $\mathbb{F}[S]$ becomes an associative algebra over $\mathbb{F}$. This algebra is called the ***semigroup algebra*** of the semigroup S over the field $\mathbb{F}$. For more details see [CP61, page 159].

For every $s \in S$ define $\overline{s} \in \mathbb{F}[S]$ as

$$\overline{s}(x) = \begin{cases} 1 & \text{if } x = s \\ 0 & \text{if } x \neq s. \end{cases}$$

The $\{\overline{s} | s \in S\}$ forms a basis of $\mathbb{F}[S]$ and it forms a semigroup with the multiplication of the semigroup algebra. This semigroup is isomorphic to S. So this basis of $\mathbb{F}[S]$ can be identified with S by the mapping $\overline{s} \mapsto s$. Since S is a basis of $\mathbb{F}[S]$ every $a \in \mathbb{F}[S]$ can be written in the following form. If the support of a is $\{s_1, \dots, s_n\}$ then

$$a = a(s_1)s_1 + \cdots + a(s_n)s_n.$$

1.2 Definitions and results from various fields

In this section we introduce some definitions and results from the followings: symplectic linear algebra, G-gradings on associative algebras and intersecting families. These concepts and theorems will be used in Chapter 2.

Symplectic Linear Algebra

1.2.1 Definition *A* ***symplectic vector space*** *is a vector space V over a field $\mathbb{F}$ equipped with a* ***symplectic bilinear form*** *ω which is a non-degenerate alternating bilinear form.*

- ***non-degenerate****: $\omega(u, v) = 0$ for all $v \in V$ implies that u is zero.*
- ***alternating****: $\omega(v, v) = 0$ holds for all $v \in V$*

If the characteristic of field $\mathbb{F}$ is not 2, alternation is equivalent to skew-symmetry.

- ***skew-symmetry***: $\omega(u, v) = -\omega(v, u)$ for all $u, v \in V$.

Let $W \subseteq V$ be a subspace. The ***symplectic complement*** of W is

$$W^{\perp} = \{v \in V \mid \omega(v, w) \text{ for all } w \in W\}$$

1.2.2 Theorem *Let (V, ω) be a symplectic vector space and $W \subseteq V$ a linear subspace. Then*

$$\dim(W) + \dim(W^\perp) = \dim(V).$$

1.2.3 Definition *Let (V, ω) be a symplectic vector space and $W \subseteq V$ a linear subspace. If $W = W^\perp$, then W is called a* ***Lagrangian subspace****.*

1.2.4 Corollary *If W is a Lagrangian subspace of V then*

$$\dim(W) = \frac{\dim(V)}{2}.$$

G-Graded Algebras

1.2.5 Definition *Let G be a monoid. A* ***G-graded algebra*** *A is an associative algebra with a direct sum decomposition $A = \bigoplus_{i \in G} A_i$ such that*

$$A_i A_j \subseteq A_{ij}.$$

Note that the Grassmann algebra (see 2.1) has a $\mathbb{Z}$ gradation and a $\mathbb{Z}/2\mathbb{Z}$ gradation.

Intersecting families

1.2.6 Definition *Let $\mathcal{F}$ be a family of subsets of S. If for all $A, B \in \mathcal{F}$ $A \cap B \neq \emptyset$, then $\mathcal{F}$ is called an* ***intersecting family****.*

1.2.7 Theorem (Erdős-Ko-Rado) *[Erd61] Let $\mathcal{F}$ be a family of r-element subsets of an n-element set. If $n \geq 2r$ and $\mathcal{F}$ is an intersecting family, then*

$$|\mathcal{F}| \leq \binom{n-1}{r-1}.$$

Chapter 2

Commutative subalgebras of Grassmann algebras

In this chapter, the maximal dimension of commutative subalgebras of Grassmann algebras is determined. It is shown that for any commutative subalgebra A of a Grassmann algebra E, there exists a commutative subalgebra of E which is spanned by monomials and has the same dimension as A. It follows that the maximal dimension of a commutative subalgebra can be expressed in terms of the maximal size of an intersecting family of subsets of odd size in a finite set.

2.1 Introduction

The ***Grassmann algebra*** (also called ***exterior algebra***) E of an n-dimensional vector space $\mathrm{Span}_{\mathbb{F}}\{v_1, \ldots, v_n\}$ over a field $\mathbb{F}$ (assumed throughout to have characteristic different from 2) is the associative $\mathbb{F}$-algebra given in terms of generators and relations as

$$E = \mathbb{F}\langle v_1, ..., v_n \mid \quad v_i v_j = -v_j v_i \quad (1 \leq i, j \leq n)\rangle.$$

Throughout this chapter the number of generators of E will be n. In this chapter we shall investigate the algebra structure of this fundamental object. The algebra E is not commutative, but it is not far from being commutative: it has a large center, and satisfies the polynomial identity $[[x, y], z] = 0$ (Lie nilpotency of index 2). We are mainly interested in the commutative subalgebras of E. Our main result Theorem 2.6.1 gives in particular the maximal dimension of commutative subalgebras of E, and gives some partial results on their structure. It turns out that in the case when n is even, all maximal (with respect to inclusion) subalgebras have the same dimension. When n is odd, then there are maximal commutative subalgebras of different dimensions.

Along the way we show in Theorem 2.4.3 that to any commutative subalgebra of E one can associate via a simple linear algebra process an equidimensional commutative subalgebra spanned by monomials (products of generators). This result has some independent interest, and also makes it possible to make a tight connection between our question and the Erdős-Ko-Rado theorem on intersecting families.

Our interest in maximal commutative subalgebras was inspired by [Mar15], where the existence of large commutative subalgebras of E is used as an obstruction for embeddability of E into the full matrix algebra $\mathbb{F}^{m\times m}$ for small

m. The study of commutative subalgebras in non-commutative algebras has a considerable literature. We only mention the theorem of Schur [Sch05] determining the maximal dimension of a commutative subalgebra of $\mathbb{F}^{n\times n}$, see [Jac44] and [Gus76] for alternative proofs.

2.2 Square zero subspaces

For a subset $J \subseteq [n] := \{1,\ldots,n\}$ set $v_J := v_{i_1}\cdots v_{i_k}$, where $J = \{i_1,\ldots,i_k\}$ and $i_1 < \cdots < i_k$. Clearly, $\{v_J : \ J \subseteq [n]\}$ is an $\mathbb{F}$-vector space basis of E. We shall refer to the elements $v_J \in E$ as ***monomials***. The Grassmann algebra is graded:

$$E = \bigoplus_{k=0}^{\infty} E_k \text{ where } E_k = \mathrm{Span}_{\mathbb{F}}\{v_J \mid I \subseteq [n], \quad |J| = k\}$$

(of course, for $k > n$ we have $E_k = \{0\}$). Sometimes we pay attention to the $\mathbb{Z}/2\mathbb{Z}$-grading induced by the above $\mathbb{Z}$-grading:

$$E = E_{\overline{0}} \oplus E_{\overline{1}} \text{ where } E_{\overline{0}} := \bigoplus_{k \text{ is even}} E_k, \quad E_{\overline{1}} := \bigoplus_{k \text{ is odd}} E_k.$$

The defining relations of E imply the multiplication rules

$$v_J v_K = (-1)^{|J|\cdot|K|} v_K v_J$$

and when $J \cap K \neq \emptyset$, then $v_J v_K = 0$. It follows that $E_{\overline{0}}$ is contained in the center of E, and the elements of $E_{\overline{1}}$ anticommute: $ab = -ba$ for any pair $a, b \in E_{\overline{1}}$. In particular, $a, b \in E_{\overline{1}}$ commute if and only if $ab = 0$. So the commutative subalgebras of E have a natural connection with square zero subspaces. For subspaces $C, D \subseteq E$ we write CD for the subspace $\mathrm{Span}_{\mathbb{F}}\{cd \mid c \in C, d \in D\}$, and we call a subspace $D \subseteq E$ a ***square zero subspace*** if $D^2 = 0$, that is, if $cd = 0$ for all $c, d \in D$. A commutative subalgebra A of E is called ***maximal*** if there is no commutative subalgebra of E properly containing A. Similarly, a square zero subspace of $E_{\bar{1}}$ is called ***maximal*** if it is not properly contained in a square zero subspace of $E_{\bar{1}}$.

2.2.1 Proposition *([DZ15])*

(i) If $D \subseteq E_{\bar{1}}$ is a square zero subspace, then $K := E_{\bar{0}}D \subseteq E_{\bar{1}}$ is also a square zero subspace and $E_{\bar{0}} \oplus K$ is a commutative subalgebra of E.

(ii) The map $D \mapsto E_{\bar{0}} \oplus D$ gives a bijection between the maximal square zero subspaces in $E_{\bar{1}}$ and the maximal commutative subalgebras of E.

Proof. Statement (i) follows from the centrality of $E_{\bar{0}}$. In order to show (ii), suppose that $A \supseteq E_{\bar{0}}$ is a commutative subalgebra in E. Then as a vector space, $A = E_{\bar{0}} \oplus D$, where $D := A \cap E_{\bar{1}}$. Moreover, D is an $E_{\bar{0}}$-submodule in $E_{\bar{1}}$. Since elements of $E_{\bar{1}}$ anticommute, this forces that $D^2 = 0$. Taking into account (i) we get that $D \mapsto E_{\bar{0}} \oplus D$ gives a bijection between square zero $E_{\bar{0}}$-submodules of $E_{\bar{1}}$ and commutative subalgebras of E that contain $E_{\bar{0}}$. This bijection restricts to the bijection claimed in (ii), since a maximal commutative subalgebra of E necessarily contains the center $E_{\bar{0}}$, and a maximal square zero subspace of $E_{\bar{1}}$ is necessarily an $E_{\bar{0}}$-submodule. □

2.2.2 Remark *([DZ15])* It is interesting to compare the above "structure theorem" with the case of the matrix algebra $\mathbb{F}^{n\times n}$, where by a theorem of Schur [Sch05], a commutative subalgebra of maximal possible dimension is also of the form

$$Z(\mathbb{F}^{n\times n}) \oplus D = \mathbb{F}I_n \oplus D,$$

where D is a subspace with $D^2 = 0$, I_n stands for the identity matrix and $Z(\mathbb{F}^{n\times n})$ denote the center of $\mathbb{F}^{n\times n}$. However, unlike for E, in $\mathbb{F}^{n\times n}$ not all maximal subalgebras are of this form.

Maximal square zero subspaces in $E_{\overline{1}}$ can be characterized in terms of certain bilinear maps on $E_{\overline{1}}$ defined as the composition of the multiplication map $E_{\overline{1}} \times E_{\overline{1}} \to E_{\overline{0}}$ and a projection from $E_{\overline{0}}$ to one of its homogeneous components. Recall that any $x \in E$ can be uniquely written as

$$x = \sum_{J\subseteq[n]} x_J v_J, \tag{2.1}$$

where $x_J \in \mathbb{F}$. For every $x \in E$ and $J \subseteq [n]$ the x_J denote the coefficient of v_J in the equation 2.1.

Define a bilinear map ϕ as follows:

- if n is even then $\Phi : E_{\overline{1}} \times E_{\overline{1}} \to E_n$, $\Phi(a, b) = (ab)_{[n]}v_{[n]}$;
- if n is odd then $\Phi : E_{\overline{1}} \times E_{\overline{1}} \to E_{n-1}$, $\Phi(a, b) = \sum_{J\in\binom{[n]}{n-1}}(ab)_J v_J$.

Here $\binom{[n]}{k}$ stands for the set of k-element subsets of $[n]$. Since the multiplication map on $E_{\overline{1}}$ is skew-symmetric, the bilinear map Φ is also skew-symmetric. Since

$$\dim(\mathrm{im}(\Phi)) = \begin{cases} 1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd,} \end{cases}$$

when n is even, the $\mathrm{im}(\Phi) = E_n$ can be identified with $\mathbb{F}$, so Φ is a skew-symmetric bilinear form. It is a non-degenerate form, since if $x_J \neq 0$ for some $x \in E_{\bar{1}}$ and $J \subseteq [n]$, then $v_{[n]\setminus J} \in E_{\bar{1}}$ and $\Phi(x, v_{[n]\setminus J}) = x_J \neq 0$. It means if n is even then $(E_{\bar{1}}, \Phi)$ is a symplectic vector space.

Given a subspace $D \subseteq E_{\bar{1}}$ we write

$$D^{\perp} := \{x \in E_{\bar{1}} : \Phi(x, w) = 0 \ \forall w \in D\}.$$

2.2.3 Proposition *([DZ15]) A subspace $D \subseteq E_{\bar{1}}$ is a maximal square zero subspace in $E_{\bar{1}}$ if and only if D is an $E_{\bar{0}}$-submodule and $D = D^{\perp}$.*

Proof. Let D be an $E_{\bar{0}}$-submodule of $E_{\bar{1}}$ with $D = D^{\perp}$. Suppose that there exist $x, y \in D$ with $xy \neq 0$. Then there exists a (homogeneous) $z \in E_{\bar{0}}$ such that $0 \neq xyz = \Phi(x, yz)$, hence $yz \notin D^{\perp}$. This is a contradiction, since $yz \in DE_{\bar{0}} = D = D^{\perp}$. Thus $D^2 = 0$.

To show the reverse implication suppose that $D \subseteq E_{\bar{1}}$ is a maximal square zero subspace in $E_{\bar{1}}$. By Proposition 2.2.1 (i) (and since $1 \in E_{\bar{0}}$) we have $D = DE_{\bar{0}}$. Clearly, $D^2 = 0$ implies $\Phi(x, y) = 0$ for all $x, y \in D$, so $D \subseteq D^{\perp}$. Moreover, if $x \in E_{\bar{1}} \setminus D$ then by maximality of D there exists $y \in D$ such that $0 \neq xy \in E_{\bar{0}}$. Thus there exists a $z \in E_{\bar{0}}$ such that $0 \neq xyz = \Phi(x, yz)$. Here $yz \in DE_{\bar{0}} = D$, showing $x \notin D^{\perp}$. So we proved $D^{\perp} \subseteq D$. Hence we get $D = D^{\perp}$, and the proof is finished. □

2.2.4 Corollary *([DZ15]) If $n \geq 2$ is even, then any maximal commutative subalgebra of E has dimension $3 \cdot 2^{n-2}$.*

Proof. Let A be a maximal commutative subalgebra of E. Then by Proposition 2.2.1 (ii) $A = E_{\bar{0}} \oplus D$, where $D \subseteq E_{\bar{1}}$ is a maximal square zero subspace. By Proposition 2.2.3 we get $D = D^{\perp}$ is a Lagrangian subspace in the symplectic vector space $(E_{\bar{1}}, \Phi)$, hence $\dim(D) = \dim(E_{\bar{1}})/2 = 2^{n-2}$. So $\dim(A) = \dim(E_{\bar{0}}) + 2^{n-2} = 3 \cdot 2^{n-2}$. □

2.3 Commuting projections

Let W be a vector space over a field $\mathbb{F}$, and $\pi_1, \dots, \pi_r \in \mathrm{End}_{\mathbb{F}}(W)$ pairwise commuting projections for some positive r. So $\pi_i^2 = \pi_i$ and $\pi_i \pi_j = \pi_j \pi_i$ for all $1 \leq i, j \leq r$. Recall the corresponding direct sum decompositions $W = \ker(\pi_j) \oplus \mathrm{im}(\pi_j)$. Given a subset $J \subseteq [r]$ we set

$$W_J := \bigcap_{j \in J} \ker(\pi_j) \cap \bigcap_{j \notin J} \mathrm{im}(\pi_j).$$

Let $\mathrm{Gras}(W)$ stand for the set of subspaces of W, and for $j = 1, \dots, r$ define a map

$$\gamma_j : \mathrm{Gras}(W) \to \mathrm{Gras}(W), \qquad D \mapsto \ker(\pi_j|_D) \oplus \mathrm{im}(\pi_j|_D) \tag{2.2}$$

where $\pi_j|_D : D \to W$ stands for the restriction of π_j to the subspace D. Note that for $A, D \in \mathrm{Gras}(W)$ we have $\gamma_j(A) + \gamma_j(D) \subseteq \gamma_j(A + D)$ (this inclusion is proper in general). It is also obvious that if $A \subseteq D$, then $\gamma_j(A) \subseteq \gamma_j(D)$.

2.3.1 Lemma *([DZ15]) Take* $D \in \mathrm{Gras}(W)$ *and denote* $A := \gamma_1 \dots \gamma_r(D)$*. Then we have the equalities*

(i) $\dim(A) = \dim(D)$*;*

(ii) $A = \bigoplus_{J \subseteq [r]} (A \cap W_J)$.

Proof. The case $r = 1$ is a basic fact of linear algebra. For $r = 1$ we have

(i) $\dim(D) = \dim(\ker(\pi_1|_D) \oplus \mathrm{im}(\pi_1|_D))$;

(ii) $A = (A \cap \ker(\pi_1)) \oplus (A \cap \mathrm{im}(\pi_1))$.

Statement (i) follows by a repeated application of the special case $r = 1$ of (i). To prove statement (ii) we apply induction on r. Suppose $r > 1$. Set $W' := \ker(\pi_r)$, $W'' := \mathrm{im}(\pi_r)$. Since $\pi_j \pi_r = \pi_r \pi_j$ for any $j = 1, \dots, r-1$, it follows that $\pi_j(W') \subseteq W'$, $\pi_j(W'') \subseteq W''$, and if $C = C' + C''$ with $C' \subseteq W'$, $C'' \subseteq W''$, then

$$\gamma_j(C') \subseteq W', \quad \gamma_j(C'') \subseteq W'', \text{ and } \gamma_j(C) = \gamma_j(C') \oplus \gamma_j(C''). \tag{2.3}$$

By definition of γ_r we have $\gamma_r(D) = (\gamma_r(D) \cap W') \oplus (\gamma_r(D) \cap W'')$, hence by a repeated application of (2.3) we get

$$A = (\gamma_1 \dots \gamma_{r-1})(\gamma_r(D)) = (A \cap W') \oplus (A \cap W''), \tag{2.4}$$

$$A \cap W' = \gamma_1 \dots \gamma_{r-1}(\gamma_r(D) \cap W') \tag{2.5}$$

and

$$A \cap W'' = \gamma_1 \dots \gamma_{r-1}(\gamma_r(D) \cap W''). \tag{2.6}$$

For a subset $J \subseteq [r-1]$ set

$$W'_J := \bigcap_{j \in J} \ker(\pi_j|_{W'}) \cap \bigcap_{j \in [r-1] \setminus J} \operatorname{im}(\pi_j|_{W'})$$

$$W''_J := \bigcap_{j \in J} \ker(\pi_j|_{W''}) \cap \bigcap_{j \in [r-1] \setminus J} \operatorname{im}(\pi_j|_{W''}).$$

Applying the induction hypothesis to $r-1$ (2.5) and (2.6) yield

$$A \cap W' = \bigoplus_{j \subseteq [r-1]} A \cap W'_J \quad \text{and} \quad A \cap W'' = \bigoplus_{j \subseteq [r-1]} A \cap W''_J. \tag{2.7}$$

Observe that

$$W'_J = W_{J \cup \{r\}} \quad \text{and} \quad W''_J = W_J. \tag{2.8}$$

Now (2.4), (2.7) and (2.8) yield statement (ii). □

2.3.2 Remark *([DZ15])* Note that though $\pi_1, \ldots, \pi_r$ commute, the maps $\gamma_1, \ldots, \gamma_r$ do not necessary commute, i.e. $\gamma_i \gamma_j(D)$ may be different from $\gamma_j \gamma_i(D)$ (see Example 2.7.1 and 2.7.4).

2.4 Projections on the Grassmann algebra

We shall apply Lemma 2.3.1 for the case when $W = E$ is the Grassmann algebra. For $i = 1, \ldots n$, define the linear map $\pi_i : E \to E$ by

$$\pi_i(x) := \sum_{i \notin J \subseteq [n]} x_J v_J$$

(see (2.1) for the notation). Then $\pi_i^2 = \pi_i$ and $\pi_i \pi_j = \pi_j \pi_i$, so the considerations of Section 2.3 apply for these projections. Keeping the notation of Section 2.3 define $\gamma_i : \operatorname{Gras}(E) \to \operatorname{Gras}(E)$ as in equation (2.2).

Observe that

$$\ker(\pi_i) = v_i E = E v_i,$$

$$\operatorname{im}(\pi_i) = \operatorname{Span}_{\mathbb{F}}\{v_J | J \subseteq [n], i \notin J\}.$$

Clearly, $W_J = E_J$ is the 1-dimensional subspace spanned by v_J. An extra feature now is that the maps π_i are algebra homomorphisms, moreover from

$$\ker(\pi_i) = v_i E = E v_i \text{ and } v_i^2 = 0$$

we get

$$\ker(\pi_i)^2 = \{0\}. \tag{2.9}$$

2.4.1 Proposition *([DZ15]) Let $D, A \in \operatorname{Gras}(E)$ be subspaces and $i \in [n]$. The following hold for γ_i:*

(i) $\gamma_i(A)\gamma_i(D) \subseteq \gamma_i(AD)$.

(ii) If D is a subalgebra of E, then $\gamma_i(D)$ is also a subalgebra.

(iii) If $D^2 = \{0\}$, then $\gamma_i(D)^2 = \{0\}$.

(iv) If D is a right ideal [left ideal] in E, then $\gamma_i(D)$ is also a right ideal [left ideal] in E.

(v) If D is a commutative subalgebra of E, then $\gamma_i(D)$ is a commutative subalgebra of E.

Proof. To simplify notation set $\gamma := \gamma_i$ and $\pi := \pi_i$.

(i) Observe that $x - \pi(x) \in \ker(\pi)$ for all $x \in E$. Hence that $\ker(\pi)^2 = \{0\}$ implies that for any $x \in E$ and $y \in \ker(\pi)$,

$$xy = \pi(x)y \text{ and } yx = y\pi(x). \tag{2.10}$$

It follows that

$$\pi(A)\ker(\pi|_D) = A\ker(\pi|_D) \subseteq \ker(\pi|_{AD}),$$

and similarly,

$$\ker(\pi|_A)\pi(D) \subseteq \ker(\pi|_{AD}).$$

Taking into account that $\pi(A)\pi(D) = \pi(AD)$ we conclude

$$\gamma(A)\gamma(D) = \big(\ker(\pi|_A) + \operatorname{im}(\pi|_A)\big)\big(\ker(\pi|_D) + \operatorname{im}(\pi|_D)\big) \subseteq$$

$$\subseteq \ker(\pi)^2 + \ker(\pi|_{AD}) + \pi(A)\pi(D) = \gamma(AD).$$

Statements (ii), (iii), (iv) are immediate corollaries of (i).

(v) A general pair of elements in $\gamma(D)$ can be written as $\pi(a) + a'$ and $\pi(b) + b'$ where $a, b \in D$ and $a', b' \in \ker(\pi|_D)$. Since D is commutative and π is an algebra homomorphism, (2.10) implies

$$(\pi(a) + a')(\pi(b) + b') = \pi(ab) + a'b + ab' + a'b' =$$

$$= \pi(ba) + ba' + b'a + b'a' = (\pi(b) + b')(\pi(a) + a').$$

□

2.4.2 Remark *([DZ15])* Note that in the situation of Proposition 2.4.1 (ii), the algebra $\gamma_i(D)$ is not necessarily isomorphic to the algebra D (see Examples 2.7.2 and 2.7.5).

Combining Lemma 2.3.1, Proposition 2.4.1 and the fact that $E_J = \mathbb{F}v_J$ we obtain the following:

2.4.3 Theorem *([DZ15]) Let $D \subseteq E$ be a subalgebra (not necessarily unitary), and set $A := \gamma_1 \dots \gamma_n(D)$. Then the following hold:*

(i) A is a subalgebra of E,

(ii) as an $\mathbb{F}$ vector space it is spanned by elements in the form v_J, $J \subseteq [n]$,

(iii) $\dim(A) = \dim(D)$,

(iv) if D is commutative then so is A,

(v) if $D^2 = \{0\}$ then $A^2 = \{0\}$.

2.4.4 Remark *([DZ15])* The role of the generators $v_1, \dots, v_n$ is symmetric, so the conclusion of Theorem 2.4.3 holds for

$$A_\sigma := \gamma_{\sigma(1)} \dots \gamma_{\sigma(n)}(D),$$

where σ is an arbitrary permutation of $1, \dots, n$. However, different permutations σ yield in general different subspaces A_σ, see Example 2.7.4.

The projections π_i preserve the degree, hence the maps γ_i are also compatible with the grading on E:

2.4.5 Proposition *([DZ15])*

(i) If $D \subseteq \bigoplus_{k \in I} E_k$ for some $I \subseteq [n]$, then $\gamma_i(D) \subseteq \bigoplus_{k \in I} E_k$.

(ii) *If* $D \subseteq \bigoplus_{k\in I} E_k$ *and* $A \subseteq \bigoplus_{k\in J} E_k$ *where* $I, J \subseteq [n]$ *are disjoint subsets then* $\gamma_i(A \oplus D) = \gamma_i(A) \oplus \gamma_i(D)$*.*

(iii) *If* $D = \bigoplus_{k=0}^{n}(D \cap E_k)$ *is spanned by its homogeneous components, then we have* $\gamma_i(D) = \bigoplus_{k=0}^{n} \gamma_i(D \cap E_k)$ *(and* $\gamma_i(D \cap E_k) \subseteq E_k$ *for all* k*).*

Let $b \in E$ a non-zero element. Write $b^{\min}$ for the homogeneous component of b of minimal degree. To any subspace A of E, one can canonically associate a subspace $A^{\min}$ spanned by homogeneous elements as follows:

$$A^{\min} := \mathrm{Span}_{\mathbb{F}}\{b^{\min} \mid b \in A, b \neq 0\}.$$

The following statements are straightforward to prove:

2.4.6 Proposition *([DZ15])*

(i) $\dim(A^{\min}) = \dim(A)$*;*

(ii) *If* A *is a subalgebra of* E*, then* $A^{\min}$ *is a subalgebra of* E*. Moreover, if* A *is commutative then* $A^{\min}$ *is commutative. If* A *is a square zero subspace, then* $A^{\min}$ *is a square zero subspace.*

Note also that for any graded subalgebra A of E, the subalgebra $B := \gamma_1 \dots \gamma_n(A)$ spanned by monomials has the same Hilbert series as A: we have $\dim(A \cap E_k) = \dim(B \cap E_k)$ for $k = 0, 1, \dots, n$ by Proposition 2.4.5.

2.4.7 Remark *([DZ15])* Not all subalgebras of E are isomorphic to a graded subalgebra of E, and not all graded subalgebras of E are isomorphic to a subalgebra generated by monomials, see Example 2.7.7 and 2.7.6.

2.5 Odd intersecting families

Theorem 2.4.3 opens the way to reduce certain questions on square zero subspaces of $E_{\bar{1}}^{-}$ to questions about odd intersecting families. Recall that a set $\mathcal{F} \subseteq 2^{[n]}$ of subsets of $[n]$ is called an ***intersecting family*** if $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{F}$, and it is an ***odd intersecting family*** if in addition $|A|$ is odd for all $A \in \mathcal{F}$.

2.5.1 Proposition *([DZ15]) Let $\mathcal{F} \subseteq 2^{[n]}$ be an odd intersecting family.*

(i) If n is even, then

$$|\mathcal{F}| \leq 2^{n-2}.$$

(ii) If n is odd, $\mathcal{F} \subseteq \binom{[n]}{i} \cup \binom{[n]}{n-i-1}$ for some odd i with $i < n/2 - 1$ and $\mathcal{F}$ is of maximal possible size, then

$$\mathcal{F} = \binom{[n]}{n-i-1}.$$

(iii) If $n = 4k + 1$ (where k is a non-negative integer) and $|\mathcal{F}|$ is maximal then

$$\mathcal{F} = \bigcup_{n/2<i \text{ odd}} \binom{[n]}{i}.$$

(iv) If $n = 4k + 3$ (where k is a non-negative integer) and $|\mathcal{F}|$ is maximal then

$$\mathcal{F} = \bigcup_{n/2<i \text{ odd}} \binom{[n]}{i} \cup \left\{ X \in \binom{[n]}{2k+1} \,\middle|\, l \in X \right\},$$

for some $l \in [n]$.

Proof. (i) If n is even, $X \subseteq [n]$ such that $|X|$ is odd, then $|[n] \setminus X|$ is also odd. Since $\mathcal{F}$ cannot simultaneously contain X and its complement.

(ii) Write $\mathcal{F}^c$ for the complement of $\mathcal{F}$ in $2^{[n]}$. We have the inclusion

$$\left\{(A,B) \,\middle|\, A \in \mathcal{F} \cap \binom{[n]}{i}, B \in \binom{[n]}{n-i-1}, A \cap B = \emptyset\right\}$$

$$\subseteq \left\{(A,B) \,\middle|\, B \in \mathcal{F}^c \cap \binom{[n]}{n-i-1}, A \in \binom{[n]}{i}, A \cap B = \emptyset\right\}.$$

It follows that

$$\left|\mathcal{F} \cap \binom{[n]}{i}\right| \cdot (n-i) \leq \left|\mathcal{F}^c \cap \binom{[n]}{n-i-1}\right| \cdot (i+1)$$

and hence

$$\left|\mathcal{F} \cap \binom{[n]}{i}\right| \cdot \frac{n-i}{i+1} + \left|\mathcal{F} \cap \binom{[n]}{n-i-1}\right| \leq \binom{n}{n-i-1}.$$

Since $1 < \frac{n-i}{i+1}$, we get $|\mathcal{F}| \leq \binom{n}{n-i-1}$ with equality only if $\mathcal{F} \subseteq \binom{[n]}{n-i-1}$. Note that since $n/2 < n-i-1$, $\binom{[n]}{n-i-1}$ is an intersecting family.

(iii) follows from (ii).

(iv) follows from (ii) and the Erdős-Ko-Rado Theorem for $r = \frac{n-1}{2}$ 1.2.7 Theorem, [Erd61]. $\square$

2.6 Commutative subalgebras of maximal dimension

2.6.1 Theorem *([DZ15]) Write k for the lower integer part of $n/4$.*

(i) Let A be a commutative subalgebra of E of maximal dimension. Then

$$\dim(A) = \dim(E_{\bar{0}}) + |\mathcal{F}|,$$

where $\mathcal{F} \subseteq 2^{[n]}$ is an odd intersecting family of maximal possible size. Hence

$$\dim(A) = \begin{cases} 3 \cdot 2^{n-2} & \text{if } n \text{ is even;} \\ 2^{n-1} + \sum_{l=k}^{2k} \binom{n}{2l+1} & \text{if } n = 4k+1; \\ 2^{n-1} + \sum_{l=k}^{2k} \binom{n}{2l+3} + \binom{n-1}{2k} & \text{if } n = 4k+3. \end{cases}$$

(ii) If n is even, then all maximal commutative subalgebras of E have the same dimension, but they are not all isomorphic for $n > 2$.

(iii) If $n = 4k+1$, then $E_{\bar{0}} \oplus (\bigoplus_{n/2<i \text{ odd}} E_i)$ is the only maximal dimensional commutative subalgebra of E.

(iv) If $n = 4k+3$, then the maximal dimensional commutative subalgebras of E are exactly the subspaces of the form

$$E_{\bar{0}} \oplus (\bigoplus_{n/2<i \text{ odd}} E_i) \oplus C,$$

where $C \subset E_{2k+1}$ is a square zero subspace of dimension $\binom{n-1}{2k}$.

(v) When n is odd, then exist maximal commutative subalgebras that are not maximal dimensional commutative subalgebras in E.

Proof. (i) Note that a set of monomials $\{v_J : J \in \mathcal{F}\}$ spans a square zero subspace in $E_{\bar{1}}$ if and only if $\mathcal{F} \subseteq 2^{[n]}$ is an odd intersecting family. Now let A be a maximal dimensional commutative subalgebra in E. Then, in particular, A is a maximal commutative subalgebra of E, hence $A = E_{\bar{0}} \oplus D$ by Proposition 2.2.1, where $D \subseteq E_{\bar{1}}$ is a maximal dimensional square zero subspace. By Theorem 2.4.3, $\gamma_1 \dots \gamma_n(D)$ is a maximal dimensional square zero subspace of $E_{\bar{1}}$, hence $\mathcal{F}(D) := \{J \subseteq [n] \mid v_J \in \gamma_1 \dots \gamma_n(D)\}$ is an odd intersecting family of maximal size. Since $|\mathcal{F}(D)| = \dim(D)$, statement (i) follows from Proposition 2.5.1.

(ii) It was shown in Corollary 2.2.4 that for even n all maximal commutative subalgebras have the same dimension. Set

$$A := E_{\bar{0}} + \mathrm{Span}_{\mathbb{F}}\{v_J \mid 1 \in J\}$$

and

$$B := E_{\bar{0}} \oplus (\oplus_{n/2<l \text{ odd}} E_l) \oplus C,$$

where

$$C = \begin{cases} 0 & \text{if n=4k} \\ \mathrm{Span}_{\mathbb{F}}\{v_J \mid 1 \in J, |J| = 2k+1\} & \text{if n=4k+2} \end{cases}$$

These algebras are local, their radical is their unique maximal ideal spanned by their homogeneous components of positive degree. By the definition of A

$$\mathrm{rad}(A) = \mathrm{rad}(A)^2 \oplus E_2 \oplus \mathbb{F}v_1,$$

showing

$$\dim(\mathrm{rad}(A)/\mathrm{rad}(A)^2) = \binom{n}{2} + 1.$$

On the other hand

$$\mathrm{rad}(B) = \begin{cases} \mathrm{rad}(B)^2 \oplus E_2 \oplus E_{2k+1} & \text{if } n = 4k \\ \mathrm{rad}(B)^2 \oplus E_2 \oplus C \oplus \mathrm{Span}_{\mathbb{F}}\{v_J \mid |J| = 2k+3, 1 \notin J\} & \text{if } n = 4k+2. \end{cases}$$

It follows that

$$\dim(\mathrm{rad}(B)/\mathrm{rad}(B)^2) = \begin{cases} \binom{n}{2} + \binom{n}{2k+1} & \text{if } n = 4k \\ \binom{n}{2} + \binom{n-1}{2k} + \binom{n-1}{2k+3} & \text{if } n = 4k+2. \end{cases}$$

This shows that $\dim(\mathrm{rad}(A)/\mathrm{rad}(A)^2) \neq \dim(\mathrm{rad}(B)/\mathrm{rad}(B)^2)$ for $n > 2$, hence $A \ncong B$ are nonisomorphic algebras.

(iii) Let $n = 4k+1$. Assume $A \neq E_{\overline{0}} \oplus (\bigoplus_{n/2<i \text{ odd}} E_i)$ is another commutative subalgebra of maximal dimension. Since A is maximal, using Proposition 2.2.1 we get $A = E_{\overline{0}} \oplus D$, where $D \subseteq E_{\overline{1}}$ is a square zero subspace. Moreover $D \neq \bigoplus_{n/2<i \text{ odd}} E_i$. So D has an element x such that $x_J \neq 0$ for some $J \subseteq n$ with $|J| < n/2$. Choose a permutation $\sigma \in S_n$ such that $\sigma(\{1, \dots, |J|\}) = J$. Observe that

$$J \in \mathcal{F}_\sigma(D) := \{I \subseteq [n] \mid v_I \in \gamma_{\sigma(1)} \dots \gamma_{\sigma(n)}(D)\}.$$

As explained above (and by Remark 2.4.4), $\mathcal{F}_\sigma(D)$ is an odd intersecting family of maximal size. Thus $J \in \mathcal{F}_\sigma(D)$ contradicting Proposition 2.5.1.

(iv) It is obvious that the subspaces in the statement are commutative subalgebras, and their dimensions agree with the value given in (i). A similar argument as in (iii) shows that by Proposition 2.2.1, Theorem 2.4.3, Remark 2.4.4, and Proposition 2.5.1 a maximal dimensional commutative subalgebra A is of the form $A = E_{\overline{0}} \oplus D$ where $D \subseteq E_{\overline{1}}$, $D^2 = \{0\}$, and $D \subset \bigoplus_{i \geq 2k+1 \text{ odd}} E_i$. In addition, Propositions 2.4.5 and 2.4.6 imply $\dim(D^{\min} \cap E_{2k+1}) = \binom{n-1}{2k}$. It follows that

$$\dim\left(D \cap \bigoplus_{i>2k+1 \text{ odd}} E_i\right) = \dim\left(\bigoplus_{i>2k+1 \text{ odd}} E_i\right).$$

Consequently, $D = C \oplus \bigoplus_{i>2k+1 \text{ odd}} E_i$ where $C \subset E_{2k+1}$, and so $C^2 = \{0\}$, $\dim(C) = \binom{n-1}{2k}$.

(v) Clearly, $D := \mathrm{Span}_{\mathbb{F}}\{v_J \mid 1 \in J, |J| \text{ is odd}\}$ is a maximal square zero subspace in $E_{\overline{1}}$, hence $E_{\overline{0}} \oplus D$ is a maximal commutative subalgebra. Its dimension is $3 \cdot 2^{n-2}$, so when n is odd this is strictly smaller than the maximal possible dimension of a commutative subalgebra of E, which is given in (i). □

2.6.2 Conjecture *([DZ15]) If $n = 4k+3$ and $A_1, A_2 \subseteq E$ are maximal dimensional commutative subalgebras, then $A_1 \cong A_2$.*

2.6.3 Conjecture *([DZ15]) If $n = 4k+1$ and A is a maximal commutative subalgebra of E, then $\dim(A) \geq 3 \cdot 2^{n-2}$.*

2.6.4 Conjecture *([DZ15]) If $n = 4k+3$ and A is a maximal commutative subalgebra of E, then $\dim(A) \geq 2^{n-1} + \sum_{l=k}^{2k} \binom{n}{2l+3} + |\mathcal{F}|$, where $\mathcal{F} \subseteq \binom{[n]}{\lfloor n/2 \rfloor}$ is a maximal intersecting family of minimal possible size.*

Conjecture 2.6.3 holds for $n = 5$ and Conjecture 2.6.4 holds for $n = 7$ by Proposition 2.6.5 below. We finish this section with a result that classifies maximal commutative subalgebras in E of a special form:

2.6.5 Proposition *([DZ15]) The maximal commutative subalgebras of E that contain an element whose degree 1 component is non-zero are exactly the subalgebras of the form $\alpha(A)$, where α is an $\mathbb{F}$-algebra automorphism of E,* and $A = E_{\bar{0}} + \mathrm{Span}_{\mathbb{F}}\{v_J \mid 1 \in J \subseteq [n]\}$ *(in particular, these subalgebras have dimension $3 \cdot 2^{n-2}$).*

Proof. Let B be a maximal commutative subalgebra of E containing an element a whose degree 1 component is non-zero. Then by Proposition 2.2.1 B contains an element b with $b \in E_{\bar{1}}$ and the degree 1 component b_1 of b is non-zero. A linear automorphism of V sending b_1 to v_1 extends to an $\mathbb{F}$-algebra automorphism β of E, and $\beta(B)$ contains $\beta(b) = v_1 + c$ with $c \in E_3 + E_5 + E_7 + \cdots$. It is also known (and easy to see) that the map $v_1 + c \mapsto v_1$, $v_2 \mapsto v_2$, $\ldots$, $v_n \mapsto v_n$ extends to an $\mathbb{F}$-algebra automorphism ρ of E (see [Bav10] for more details on the automorphism group of E). Then $\rho\beta(B)$ is a maximal commutative subalgebra of E containing v_1. If D is a square zero subspace of $E_{\bar{1}}$ containing v_1, then necessarily $D \subseteq v_1 E \cap E_{\bar{1}}$. It follows by Proposition 2.2.1 that $\rho\beta(B) = A$, where A is the subalgebra of E in the statement. □

2.7 Examples

2.7.1 Example For $n = 2$ and the subspace $W = \mathbb{F}(v_1 + v_2)$ we have

$$\gamma_1(\gamma_2(W)) = \mathbb{F}v_1 \neq \mathbb{F}v_2 = \gamma_2(\gamma_1(W)).$$

2.7.2 Example Let $n = 4$. Consider the 6-dimensional ideal

$$A := \mathbb{F}(v_1v_2 + v_3v_4) \oplus E_3 \oplus E_4$$

of E. Then

$$\gamma_1(A) = \mathbb{F}v_3v_4 \oplus E_3 \oplus E_4.$$

Since

$$A^2 = \mathbb{F}v_1v_2v_3v_4 \quad \text{ and } \quad \gamma_1(A)^2 = 0,$$

we obtain $A \ncong \gamma_1(A)$ and $\gamma_1(A)^2 \subsetneq \gamma_1(A^2)$.

2.7.3 Example Let $n = 6$ and let

$$D := \mathrm{Span}_{\mathbb{F}}\{v_1v_2v_3 + v_4v_5v_6, v_1v_2v_4 + v_3v_5v_6\}.$$

Then D is a square zero subspace and there are permutations $\sigma, \rho \in S_6$ such that $\mathcal{F}_\sigma(D)$ and $\mathcal{F}_\rho(D)$ are not isomorphic as intersecting families (see the proof of Theorem 2.6.1 (iii) for this notation). Indeed, for $\sigma = \mathrm{id}$, $\rho = (36)$ (transposition) we have

$$\mathcal{F}_\sigma(D) = \{\{1,2,3\},\{1,2,4\}\} \quad \text{ and } \quad \mathcal{F}_\rho(D) = \{\{4,5,6\},\{1,2,4\}\}.$$

The intersection of the elements of $\mathcal{F}_\sigma(D)$ contains 2 elements, whereas thats of the elements of $\mathcal{F}_\rho(D)$ have only 1 element.

2.7.4 Example Let the elements

$$s_k := \sum_{I \in \binom{[n]}{k}} v_I$$

of E, for $1 \leq k \leq n$. For the n-dimensional subspace

$$D := \mathrm{Span}_{\mathbb{F}}\{s_k \mid 1 \leq k \leq n\}$$

and a permutation $\sigma \in S_n$ we have

$$\gamma_{\sigma(1)}\gamma_{\sigma(2)} \dots \gamma_{\sigma(n)}(D) = \mathrm{Span}_{\mathbb{F}}\{v_{\{\sigma(1)\}}, v_{\{\sigma(1),\sigma(2)\}}, v_{\{\sigma(1),\sigma(2),\sigma(3)\}}, \dots, v_{[n]}\}.$$

So if $\sigma \neq \rho \in S_n$ then $\gamma_{\sigma(1)}\gamma_{\sigma(2)} \dots \gamma_{\sigma(n)}(D) \neq \gamma_{\rho(1)}\gamma_{\rho(2)} \dots \gamma_{\rho(n)}(D)$.

2.7.5 Example Let $n = 3$, consider the 4-dimensional subalgebra

$$D := \mathrm{Span}_{\mathbb{F}}\{v_1v_2 + v_3, v_1, v_1v_3, v_1v_2v_3\}$$

of E. We have

$$\gamma_1\gamma_2\gamma_3(D) = \mathrm{Span}_{\mathbb{F}}\{v_1v_2, v_1, v_1v_3, v_1v_2v_3\}$$

and

$$\gamma_3\gamma_2\gamma_1(D) = \mathrm{Span}_{\mathbb{F}}\{v_3, v_1, v_1v_3, v_1v_2v_3\}.$$

Easy to calculate that

$$D^2 = \mathrm{Span}_{\mathbb{F}}\{v_1v_3, v_1v_2v_3\},$$

$$(\gamma_1\gamma_2\gamma_3(D))^2 = 0$$

and

$$(\gamma_3\gamma_2\gamma_1(D))^2 = \mathbb{F}v_1v_3.$$

Then D, $\gamma_1\gamma_2\gamma_3(D)$ and $\gamma_3\gamma_2\gamma_1(D)$ are pairwise non-isomorphic subalgebras, since the dimensions of their squares are 2, 0, and 1 respectively.

2.7.6 Example For $x = \sum_{J\in\binom{[n]}{2}} x_J v_J \in E_2$ we have $x^2 = 0$ if and only if

$$x_{\{i,j\}}x_{\{k,l\}} - x_{\{i,k\}}x_{\{j,l\}} + x_{\{i,l\}}x_{\{j,k\}} = 0 \text{ holds for all } \{i,j,k,l\} \in \binom{[n]}{4}.$$

These are the well-known *Grassmann-Plücker relations* (see for example [Stu93]), so $x^2 = 0$ if and only if $x \in E_2 \cong \bigwedge^2 V$ is *decomposable*, i.e. $x \in D := \{yz \mid y, z \in E_1\}$. The subset D is Zariski closed in the $\binom{n}{2}$-dimensional affine space E_2, and the dimension of this affine algebraic variety is $2n-3$. Therefore, if $n \geq 4$, and $L \subset E_2$ is a subspace with $\dim(L) \leq \binom{n}{2} - 2n + 3$,then $L \cap D = \{0\}$. Now let A be the subalgebra of E generated by such a non-zero L. Then $A \setminus A^2 = L + A^2$ contains no non-zero element a with $a^2 = 0$, so A can not be isomorphic to a subalgebra B of E generated by monomials, since a minimal set of monomials generating B consists of square zero elements in $B \setminus B^2$. For example, in $E^{(4)}$ take $L := \mathbb{F}(v_1v_2 + v_3v_4)$; the subalgebra generated by L is $\mathrm{Span}_{\mathbb{F}}\{v_1v_2 + v_3v_4, v_1v_2v_3v_4\}$. The square zero elements span a proper subspace in it, so it is not isomorphic to any subalgebra of E generated by monomials.

2.7.7 Example Let $n = 3$ and Let

$$A := \mathrm{Span}_{\mathbb{F}}\{v_1 + v_2v_3, v_1v_2v_3\}.$$

We have

$$(v_1 + v_2v_3)^2 = 2v_1v_2v_3, \quad (v_1v_2v_3)^2 = 0 \quad \text{and} \quad (v_1 + v_2v_3)v_1v_2v_3 = 0.$$

So A is a 2-dimensional nilpotent subalgebra of $E^{(3)}$, containing an element whose square is not zero. It is easy to see that a 2-dimensional nilpotent graded subalgebra of E must be a square zero subspace, so A is not isomorphic to a graded subalgebra of E generated by 3 elements. On the other hand, A is isomorphic as an $\mathbb{F}$-algebra to the graded subalgebra

$$\mathrm{Span}_{\mathbb{F}}\{v_1v_2 + v_3v_4, v_1v_2v_3v_4\}$$

of E generated by 4 elements.

Chapter 3

Semilattice indecomposable finite semigroups with large subsemilattices

By Theorem 1.1.28, every semigroup is a semilattice of semilattice indecomposable (s-indecomposable) semigroups. In the literature of the semigroup theory there are many papers about s-indecomposable semigroups (see, for example, the papers, [Chr69], [Nag84], [Nag85], [Nag92], [Nag92-2], [Nag93], [Nag98], [NJ04], [Nor88] [PuW71], [Tam82], [TK54], and the books [Gri01], [Nag01]). Some of them deal with s-indecomposable semigroups without idempotents, the others investigate s-indecomposable semigroups containing at least one idempotent. In this chapter we deal with finite s-indecomposable semigroups in terms of what can be said about the size of their subsemilattices. The answer is known in special classes of semigroups. As $ef = fe$

implies $e = f$ for all idempotents e and f of a completely simple semigroup (see Definition 1.1.21), each subsemilattice in a completely simple semigroup has order 1. In the classes of semigroups investigated in [Chr69], [Nag84], [Nag85], [Nag92], [Nag92-2], [Nag93], [Nag98], [NJ04], [Nor88], [TK54], the finite s-indecomposable semigroups are ideal extensions of special completely simple semigroups by nilpotent semigroups. Thus their idempotents are in the completely simple part, and hence their subsemilattices has order 1.

The situation is more interesting in general. We show that if Y is a subsemilattice of a finite s-indecomposable semigroup S, then $|Y| \leq 2\left\lfloor \frac{|S|-1}{4} \right\rfloor + 1$. We prove that, this inequality cannot be sharpened, that is, there are finite s-indecomposable semigroups S with a subsemilattice Y with $2\left\lfloor \frac{|S|-1}{4} \right\rfloor + 1$ elements. Moreover, these semigroups are characterized here, when $|S| = 4k+1$.

3.1 Notions used in this chapter

Let S be a semigroup. Let $\mathbb{C}[S]$ denote the semigroup algebra of S over the field $\mathbb{C}$ of all complex numbers.

Let S a semigroup with zero z. Notice that the $z \neq 0$ in the $\mathbb{C}[S]$ algebra. This led us to the construction of the contracted semigroup algebra which factor of the semigroup algebra. The $\mathbb{C}z$ is a 1-dimensional ideal of $\mathbb{C}[S]$. The contracted semigroup algebra of a semigroup S with a zero (over $\mathbb{C}$) is

$$\mathbb{C}_0[S] = \mathbb{C}[S]/\mathbb{C}z.$$

For further properties see [Okn91, p. 35].

For a finite dimensional algebra A over $\mathbb{C}$, the Jacobson radical of A is the intersection of all maximal right ideals of A. We will denote it by $J(A)$. We will use the following well-known facts: the factor algebra $A/J(A)$ is semisimple (and so, for a finite semigroup S, $\mathbb{C}[S]/J(\mathbb{C}[S])$ is semisimple). Moreover, a finite dimensional algebra A over $\mathbb{C}$ is semisimple if and only if A is isomorphic to $\bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C})$, where $M_n(\mathbb{C})$ denotes the associative algebra of all $n \times n$ matrices over $\mathbb{C}$.

If a semigroup S has a minimal ideal K_S, then K_S is called the kernel of S. Every finite semigroup evidently has a kernel. If a semigroup S has a kernel, then K_S is a simple subsemigroup of S [CP61, Cor. 2.30. p.69]. Every finite simple [0-simple] semigroup is completely simple [completely 0-simple] by [CP61, Cor. 2.56. p.83].

The notions of the Rees matrix semigroups and the completely 0-simple semigroups were defined in Chapter 1. By Theorem 1.1.23, a semigroup is completely
0-simple if and only if it is isomorphic with a regular Rees matrix semigroup over a group with a zero. By Theorem 1.1.25, a completely 0-simple semigroup is an inverse semigroup (see Definition 1.1.15) if and only if it is a Brandt semigroup (see Definition 1.1.24). In our investigation a special type of Brandt semigroups is in the focus. This is the semigroup $\mathcal{M}^0(1; 2, 2; I)$ where 1 denotes the trivial group and I is the 2×2 identity matrix. We will denote this Brandt semigroup by B_2. The semigroup B_2 has five elements and it has a representation with five 2×2 matrix with matrix multiplication.

$$B_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

3.2 Semilattice indecomposable semigroups

By Definition 1.1.27, a semigroup S is ***semilattice indecomposable*** (s-indecomposable) if every semilattice homomorphic image of S is trivial (that is, it contains only one element). An ideal I of a semigroup S is called a ***completely prime ideal*** if $S \setminus I$ is a subsemigroup of S. It is known ([Pet77, I.8.3. Prop. p.15]) that a semigroup is s-indecomposable if and only if it does not contain completely prime ideals. Corollary in [Tam72] gives another characterization of s-indecomposable semigroups. A semigroup S is s-indecomposable if and only if, for every $a, b \in S$, there is a sequence $a = a_0, a_1, \ldots, a_{n-1}, a_n = b$ of elements of S such that a_{i-1} divides some power of a_i $(i = 1, \ldots, n)$.

In Theorem 3.2.2 we give a new characterization of finite s-indecomposable semigroups S in terms of the semigroup algebra $\mathbb{C}[S/K_S]$. In our investigation we shall use the next lemma, which is a special case of Theorem 4.1 of [Col68].

3.2.1 Lemma *([Zub16]) If Y is a finite semilattice, then the algebra $\mathbb{C}[Y]$ is semisimple and hence isomorphic to the direct sum $\bigoplus_{i \in Y} \mathbb{C}$.*

3.2.2 Theorem *([Zub16]) A finite semigroup S is s-indecomposable if and only if $\mathbb{C}[S/K_S]/J(\mathbb{C}[S/K_S])$ has exactly one 1-dimensional ideal.*

Proof. Let S be a finite semigroup. We first consider the case when S has a zero z. In this case $K_S = \{z\}$ and so $S/K_S \cong S$.

Let α be a semilattice congruence on S. Then there is a surjective homomorphism

$$\varphi : \mathbb{C}[S] \to \mathbb{C}[S/\alpha].$$

The algebra $\mathbb{C}[S/\alpha]$ is semisimple by Lemma 3.2.1, and so $J(\mathbb{C}[S]) \subseteq \ker(\varphi)$. Hence there is a surjective homomorphism $\phi : \mathbb{C}[S]/J(\mathbb{C}[S]) \to \mathbb{C}[S/\alpha]$. Since every ideal of $\mathbb{C}[S]/J(\mathbb{C}[S]) \cong \bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C})$ is a direct summand,

$$\mathbb{C}[S]/J(\mathbb{C}[S]) \cong \ker(\phi) \oplus \mathbb{C}[S/\alpha].$$

By Lemma 3.2.1 we get

$$\mathbb{C}[S]/J(\mathbb{C}[S]) \cong \ker(\phi) \oplus \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{|S/\alpha| \text{ times}}.$$

Therefore, if S is not s-indecomposable, then $\mathbb{C}[S/K_S]/J(\mathbb{C}[S/K_S])$ has more than one 1-dimensional ideal.

Now we show that if S is s-indecomposable, then the semigroup algebra $\mathbb{C}[S]/J(\mathbb{C}[S])$ has exactly one 1-dimensional ideal. The factor algebra $\mathbb{C}[S]/J(\mathbb{C}[S])$ is semisimple in which $\mathbb{C}(z + J(\mathbb{C}[S]))$ is a 1-dimensional ideal. To show that this is the only 1-dimensional ideal of $\mathbb{C}[S]/J(\mathbb{C}[S])$, it is sufficient to show that

$$A := \mathbb{C}[S]/(J(\mathbb{C}[S]) + \mathbb{C}[z])$$

does not contain 1-dimensional ideals. We will show that

$$A \cong \mathbb{C}_0[S]/J(\mathbb{C}_0[S]).$$

It is easy to see that

$$J(\mathbb{C}[S]) \cap \mathbb{C}[z] = 0.$$

By ([Okn91, Cor. 9 p.38])

$$\mathbb{C}[S] \cong \mathbb{C}_0[S] \oplus \mathbb{C}[z]$$

So

$$A \cong (\mathbb{C}_0[S] \oplus \mathbb{C}[z])/(J(\mathbb{C}[S]) \oplus \mathbb{C}[z]) \cong \mathbb{C}_0[S]/J(\mathbb{C}_0[S]).$$

Since A is semisimple, we get

$$A \cong \bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C}).$$

Suppose that $n_j = 1$ for some j with $1 \leq j \leq k$. Denote the composition of the canonical homomorphisms $\mathbb{C}[S] \to \mathbb{C}_0[S]$ and $\mathbb{C}_0[S] \to A$ by ϕ. Let $\pi : A \to M_{n_j}(\mathbb{C}) \cong \mathbb{C}$ be the canonical projection. Let

$$I := \ker(\pi \circ \phi) \cap S = \{s \in S \mid \pi(\phi(s)) = 0\}.$$

Clearly $\phi(S)$ generates A but $\phi(I)$ does not. Hence $S \neq I$. $\phi(z) = 0$ so $z \in I$ which means I is a nonempty proper subset of S. By construction I is a completely prime ideal of S, which contradicts the assumption that S is s-indecomposable. Thus our assertion is proved for semigroups with zero.

As a finite semigroup S is s-indecomposable if and only if S/K_S is s-indecomposable. Hence the general case is an easy corollary of the case when S has a zero. □

3.2.3 Remark *([Zub16]) In case of finite semigroups with zero, s-indecomposability can completely be described in terms of the semigroup algebra*

(Theorem 3.2.2). It is not true for finite semigroups in general. For example if G is a finite Abelian group and Y is a finite semilattice such that $|G| = |Y|$, then $\mathbb{C}[G] \cong \bigoplus_{i \in G} \mathbb{C} \cong \mathbb{C}[Y]$. Thus, if $1 < |G| = |Y|$, then G is s-indecomposable, Y is not, but $\mathbb{C}[G] \cong \mathbb{C}[Y]$.

3.3 Embeddings into semilattice indecomposable semigroups

Let A, B be semigroups with zeros z_A, z_B. Then $A \times B$ has an ideal

$$I = (\{z_A\} \times B) \cup (A \times \{z_B\}).$$

Let $A \times_0 B$ denote the Rees factor semigroup $(A \times B)/I$.

3.3.1 Proposition *([Zub16]) For arbitrary semigroups A and B with zeros, the semigroup $A \times_0 B$ is s-indecomposable if and only if A or B is s-indecomposable.*

Proof. Let A and B be arbitrary semigroups with zeros z_A and z_B. Assume that A is s-indecomposable. Consider the semigroup $A \times_0 B$. We show that $A \times_0 B$ is s-indecomposable. Let φ denote the canonical homomorphism of $A \times B$ onto $A \times_0 B$. Let $x, y \in A \times_0 B$ be arbitrary elements. Let (a_x, b_x) and (a_y, b_y) be elements of $A \times B$ such that $\varphi((a_x, b_x)) = x$ and $\varphi((a_y, b_y)) = y$. As A is s-indecomposable, there are elements $a_x = a_1, \ldots, a_t = z_A$ and $z_A = a_t, a_{t+1}, \cdots, a_k = a_y$ such that a_i divides some power of a_{i+1} for every

$i = 1, \ldots, k-1$ (see Theorem 1.1.29). By this it follows that

$$(a_1; b_x), \ldots, (a_t; b_x) = (z_A; b_x)$$

and

$$(z_A; b_y) = (a_t; b_y), \ldots, (a_k; b_y)$$

are sequences in $A \times B$ such that each element of the sequences (except for the last) divides some power of the next. Then

$$x = \varphi((a_1; b_x)), \ldots, \varphi((a_t; b_x)) = \varphi((z_A; b_x))$$

and

$$\varphi((z_A; b_y)) = \varphi((a_t; b_y)), \ldots, \varphi((a_k; b_y)) = y$$

are sequences in $A \times_0 B$ such that each element of the sequences (except for the last) divides some power of the next. As $\varphi((z_A; b_x)) = \varphi((z_A; b_y))$, obtain

$$x = \varphi((a_1, b_x)), \ldots \varphi((z_A, b_x)) = \varphi((z_A; b_y)), \ldots, \varphi((a_k, b_y)) = y$$

is a sequence in $A \times_0 B$ such that each element of the sequence (except for the last) divides some power of the next. Then, by Theorem 1.1.29, $A \times_0 B$ is s-indecomposable. The proof is similar to the case when the semigroup B is s-indecomposable.

Conversely, assume that $A \times_0 B$ is s-indecomposable. If A and B are not s-indecomposable then there are completely prime ideals $P_A \subset A$ and $P_B \subset B$. It is easy to see that $\varphi((P_A \times B) \cup (A \times P_B))$ is a proper completely prime ideal of $A \times_0 B$ and so $A \times_0 B$ is not s-indecomposable. This is a contradiction, hence A or B must be s-indecomposable. □

3.3.2 Remark *([Zub16])*We have a different proof of Proposition 3.3.1 in the finite case. Suppose A and B are finite. By [Okn91, Cor. 9 p.39 and Lemma 10 p.40] we get:

$$\mathbb{C}[A \times_0 B] \cong \mathbb{C} \oplus \mathbb{C}_0[A \times_0 B] \cong \mathbb{C} \oplus (\mathbb{C}_0[A] \otimes \mathbb{C}_0[B]),$$

so

$$\mathbb{C}[A \times_0 B]/J(\mathbb{C}[A \times_0 B]) \cong \mathbb{C} \oplus ((\mathbb{C}_0[A]/J(\mathbb{C}_0[A])) \otimes \mathbb{C}_0[B]/J(\mathbb{C}_0[B])) .$$

So $\mathbb{C}[A \times_0 B]/J(\mathbb{C}[A \times_0 B])$ has exactly one 1-dimensional ideal if and only if $\mathbb{C}_0[A]/J(\mathbb{C}_0[A])$ or $\mathbb{C}_0[B]/J(\mathbb{C}_0[B])$ has no 1-dimensional ideal. However, $\mathbb{C}_0[A]/J(\mathbb{C}_0[A])$ has no 1-dimensional ideal if and only if $\mathbb{C}[A]/J(\mathbb{C}[A])$ has exactly one. By Theorem 3.2.2 we get the statement.

Our next goal is to describe the smallest s-indecomposable semigroup which contains a 2-element subsemilattice. We will see that, it has 5 elements. We also show that, the smallest s-indecomposable semigroup which contains a 3-element subsemilattice is isomorphic to the semigroup B_2 (Theorems 3.4.2 and 3.5.5) below. First we show that there are only three nonisomorphic 5-element s-indecomposable semigroups with a 2-element subsemilattice.

3.3.3 Corollary *([Zub16]) Let S be an s-indecomposable semigroup such that $|S| \leq 5$ and S has at least two commuting idempotents. Then*

$$S \cong \mathcal{M}^0(1; 2, 2; P),$$

where P is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. When P is the identity matrix then $S \cong B_2$.

Proof. By Theorem 3.2.2 we get $\mathbb{C}[S/K_S]/J(\mathbb{C}[S/K_S])$ is isomorphic to $\mathbb{C}$ or $\mathbb{C} \oplus M_2(\mathbb{C})$. If $\mathbb{C}[S/K_S]/J(\mathbb{C}[S/K_S]) \cong \mathbb{C}$, then $\mathbb{C}_0[S/K_S]$ is nilpotent. If $\mathbb{C}_0[S/K_S]$ is nilpotent, then all idempotents of S are contained by K_S. So there are two commuting idempotents in K_S which contradicts the fact that K_S is completely simple. Hence $\mathbb{C}[S/K_S]/J(\mathbb{C}[S/K_S]) \cong \mathbb{C} \oplus M_2(\mathbb{C})$. Moreover, $\dim(\mathbb{C}[S/K_S]/J(\mathbb{C}[S/K_S])) = \dim(\mathbb{C}[S])$ and so $J(\mathbb{C}[S/K_S]) = 0$ and $|K_S| = 1$. Thus S has a zero z and $\mathbb{C}[S] \cong \mathbb{C} \oplus M_2(\mathbb{C})$. If I is an ideal of S, then $\mathbb{C}[I]$ is an ideal of $\mathbb{C}[S]$. $\mathbb{C}[S]$ has exactly two proper ideals: one of them is the augmentation ideal (see [Okn91, p.35]) and the other is spanned by z. Consequently, S is a (finite) 0-simple semigroup, so it is completely 0-simple. Using the Rees-theorem, it is a matter of checking to see that S is isomorphic to one of the three semigroups listed in the corollary. □

3.3.4 Corollary *([Zub16]) Every finite semigroup S can be embedded into an s-indecomposable semigroup containing $4|S|+1$ elements.*

Proof. Let S be an arbitrary finite semigroup. Denote S^0 the semigroup S with a zero adjoined (also in that case when S has a zero). Let T be an s-indecomposable semigroup of 5 elements containing a zero and another idempotent e (these semigroups are described in Corollary 3.3.3). Since T is an s-indecomposable semigroup with a zero, $S^0 \times_0 T$ is s-indecomposable by Proposition 3.3.1. Moreover, $|S^0 \times_0 T| = 4|S|+1$. Let φ denote the canonical homomorphism of $S^0 \times T$ onto $S^0 \times_0 T$. Define the homomorphism $\pi : S \to S^0 \times T$ by $\pi(s) := (s, e)$. It is obvious that $\varphi \circ \pi$ is an embedding of S into the s-indecomposable semigroup $S^0 \times_0 T$. □

3.3.5 Proposition *([Zub16]) Every finite s-indecomposable semigroup S with a zero can be embedded into an s-indecomposable semigroup containing $|S|+1$ elements.*

Proof. Let $z \in S$ be the zero. Define S' the semigroup which can be obtained from S by adjunction of an element z' such that $(z')^2 := z$ and $z'x := z$, $xz' := z$ where x is an arbitrary element of S. Then $z - z' \in J(\mathbb{C}[S])$, so $\mathbb{C}[S]/J(\mathbb{C}[S]) \cong \mathbb{C}[S']/J(\mathbb{C}[S'])$. Since S is s-indecomposable by Theorem 3.2.2, we get that S' is also s-indecomposable. □

3.4 On order of subsemilattices of semilattice indecomposable finite semigroups

In this section we answer the question: what the order of subsemilattices of s-indecomposable finite semigroups is. First we deal with the case when the semigroup in question has a zero (Proposition 3.4.1). Then we consider the general case (Theorem 3.4.2).

3.4.1 Proposition *([Zub16]) Let S be an s-indecomposable finite semigroup with a zero. If Y is a subsemilattice of S then $|Y| \leq 2\left\lfloor \frac{|S|-1}{4} \right\rfloor + 1$.*

Proof. By Theorem 3.2.2 $\mathbb{C}[S]/J(\mathbb{C}[S]) \cong \mathbb{C} \oplus \bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C})$ such that $n_i \neq 1$ $(i = 1 \dots k)$. By Lemma 3.2.1, $J(\mathbb{C}[S]) \cap \mathbb{C}[Y] = 0$ and so $\mathbb{C}[S]/J(\mathbb{C}[S])$ has a subalgebra isomorphic to $\mathbb{C}[Y]$. So it contains $|Y|$ commuting linearly independent projections, which are simultaneously diagonalizable. Thus

$$|Y| \leq 1 + \sum_{i=1}^{k} n_i.$$

Since $1 + \sum_{i=1}^{k} n_i^2 \leq |S|$ and $n_i \neq 1$ (n_i positive integer),

$$1 + \sum_{i=1}^{k} n_i \leq 1 + \sum_{i=1}^{\left\lfloor \frac{|S|-1}{4} \right\rfloor} 2.$$

Hence

$$|Y| \leq 1 + \sum_{i=1}^{k} n_i \leq 1 + \sum_{i=1}^{\left\lfloor \frac{|S|-1}{4} \right\rfloor} 2 = 2 \left\lfloor \frac{|S|-1}{4} \right\rfloor + 1. \tag{3.1}$$

□

3.4.2 Theorem *([Zub16]) Let S be an s-indecomposable finite semigroup.*

(i) If Y is a subsemilattice of S, then $|Y| \leq 2 \left\lfloor \frac{|S|-1}{4} \right\rfloor + 1$.

(ii) For every positive integer n, there is a semigroup S such that $|S| = n$ and S has a subsemilattice Y of S such that $|Y| = 2 \left\lfloor \frac{|S|-1}{4} \right\rfloor + 1$.

Proof.

(i) K_S is a finite completely simple semigroup. So if $e, f \in K_S$ are commuting idempotents, then $e = f$. Thus

$$|Y \cap K_S| \leq 1,$$

and so $Y \cong Y/(Y \cap K_S)$. Obviously S/K_S is an s-indecomposable semigroup with a zero and $Y/(Y \cap K_S)$ is a subsemilattice of S/K_S. By Proposition 3.4.1, we get:

$$|Y/(Y \cap K_S)| \leq 2 \left\lfloor \frac{|S/K_S| - 1}{4} \right\rfloor + 1.$$

Thus

$$|Y| = |Y/(Y \cap K_S)| \leq 2\left\lfloor \frac{|S/K_S| - 1}{4} \right\rfloor + 1 \leq 2\left\lfloor \frac{|S| - 1}{4} \right\rfloor + 1. \quad (3.2)$$

(ii) Let n be a positive integer. We can consider n in the form $n = 4k+1+l$ where $0 \leq l < 4$ and $0 \leq k$ (l, k integers). Let Y be a semilattice such that $|Y| = k+1$. By Proposition 3.3.1, $Y \times_0 B_2$ is s-indecomposable, because B_2 is s-indecomposable (Corollary 3.3.3). B_2 has a 3-element subsemilattice V. So $Y \times_0 B_2$ has a subsemilattice $Y \times_0 V$, it has $2k+1$ elements. Applying the embedding described in Proposition 3.3.5 l times to $Y \times_0 B_2$, we get an n-element s-indecomposable semigroup in which $Y \times_0 V$ is a subsemilattice containing $2k+1 = 2\left\lfloor \frac{|S|-1}{4} \right\rfloor + 1$ elements.

□

3.5 B_2-combinatorial semigroups

In this section we only deal with s-indecomposable semigroups S with $4k+1$ elements which containing a subsemilattice Y with $2\left\lfloor \frac{|S|-1}{4} \right\rfloor + 1$ elements. In the this section we describe the structure of these ones.

3.5.1 Definition *([Zub16]) A semigroup S is said to be B_2**-combinatorial** if S is s-indecomposable, $|S| = 4k+1$ (k is a non-negative integer) and S has a subsemilattice Y with $|Y| = 2\left\lfloor \frac{|S|-1}{4} \right\rfloor + 1 = \frac{|S|+1}{2} = 2k+1$.*

The name B_2-combinatorial will be clear in Theorem 3.5.5. First of all we note that the semigroup B_2 is B_2-combinatorial.

3.5.2 Proposition *([Zub16]) Let S be a B_2-combinatorial semigroup. Then all of the following assertions hold:*

(i) S has a zero.

(ii) The semigroup algebra $\mathbb{C}[S]$ is isomorphic to $\mathbb{C} \oplus \bigoplus_{i=1}^{k} M_2(\mathbb{C})$.

(iii) Every ideal of S is B_2-combinatorial.

(iv) Every homomorphic image of S is B_2-combinatorial.

Proof. Let S be a B_2-combinatorial semigroup and let Y denote a subsemilattice of S with $|Y| = 2\left\lfloor \frac{|S|-1}{4} \right\rfloor + 1$.

(i) By (3.2) in proof of Theorem 3.4.2 we have $\left\lfloor \frac{|S/K_S|-1}{4} \right\rfloor = \left\lfloor \frac{|S|-1}{4} \right\rfloor$ and so $|S/K_S| = |S|$ thus $|K_S| = 1$. Hence S has a zero.

(ii) If in the proof of Proposition 3.4.1 inequality (3.1) is an equality then

$$\mathbb{C}[S]/J(\mathbb{C}[S]) \cong \mathbb{C} \oplus \bigoplus_{i=1}^{\left\lfloor \frac{|S|-1}{4} \right\rfloor} M_2(\mathbb{C}).$$

Hence $\dim(\mathbb{C}[S]/J(\mathbb{C}[S])) = \dim(\mathbb{C}[S])$ which means $J(\mathbb{C}[S]) = 0$. For $k = \left\lfloor \frac{|S|-1}{4} \right\rfloor$ we get

$$\mathbb{C}[S] \cong \mathbb{C} \oplus \bigoplus_{i=1}^{k} M_2(\mathbb{C}).$$

(iii) Let I be an ideal of S. Every ideal of an s-indecomposable semigroup is also an s-indecomposable ([Tam82, Lemma 4]), so I is s-indecomposable. As Y is a subsemilattice of S of maximal size, the zero of S is in Y. Hence $Y \cap I \neq \emptyset$. By construction $Y \cap I$ and $Y/(Y \cap I)$ are subsemilattices of I and S/I, respectively. Since I and S/I are s-indecomposable semigroups with zeros, we can use (i) of Theorem 3.4.2. Hence

$$|Y \cap I| \leq 2 \left\lfloor \frac{|I| - 1}{4} \right\rfloor + 1$$

and

$$|Y/(Y \cap I)| \leq 2 \left\lfloor \frac{|S/I| - 1}{4} \right\rfloor + 1.$$

Moreover

$$|Y| = |Y \cap I| + |Y/(Y \cap I)| - 1,$$

$$|Y| = \frac{|S| + 1}{2}$$

and

$$|S| = |I| + |S/I| - 1 = 4k + 1.$$

By the previous equalities and inequalities we get

$$\frac{|S| + 1}{2} + 1 = |Y| + 1 = |Y \cap I| + |Y/(Y \cap I)|,$$

$$\frac{|S| + 1}{2} + 1 \leq 2 \left\lfloor \frac{|I| - 1}{4} \right\rfloor + 1 + 2 \left\lfloor \frac{|S/I| - 1}{4} \right\rfloor + 1,$$

$$2k + 2 \leq 2 \left(\left\lfloor \frac{|I| - 1}{4} \right\rfloor + \left\lfloor \frac{4k + 1 - |I|}{4} \right\rfloor \right) + 2,$$

$$0 \leq \left\lfloor \frac{|I|-1}{4} \right\rfloor + \left\lfloor \frac{1-|I|}{4} \right\rfloor .$$

From this it follows that $|I| = 4l+1$. So $|Y \cap I| = 2\left\lfloor \frac{|I|-1}{4} \right\rfloor + 1$. Indeed, if we suppose indirectly $|Y \cap I| < 2\left\lfloor \frac{|I|-1}{4} \right\rfloor + 1$, then we can get that

$$0 < \left\lfloor \frac{|I|-1}{4} \right\rfloor + \left\lfloor \frac{1-|I|}{4} \right\rfloor ,$$

which is a contradiction. Hence $|Y \cap I| = 2\left\lfloor \frac{|I|-1}{4} \right\rfloor + 1 = 2l+1$. I is an s-indecomposable semigroup with $|I| = 4l+1$, where l is an integer with $0 \leq l \leq k$, and $Y \cap I$ is a subsemilattice of I with $|Y \cap I| = 2l+1$. Thus I is a B_2-combinatorial semigroup.

(iv) Let $\phi : S \to T$ be a surjective homomorphism.

Since every homomorphic image of an s-indecomposable semigroup is also an s-indecomposable semigroup ([Tam82, Lemma 3]), T is s-indecomposable.

Extend ϕ to an algebra homomorphism $\hat{\phi} : \mathbb{C}[S] \to \mathbb{C}[T]$. By (i) S is a semigroup with a zero z so $\phi(z)$ is a zero of T. By Corollary 9 of [Okn91, p. 38], we get

$$\mathbb{C}[T] \cong \mathbb{C} \oplus \mathbb{C}_0[T].$$

Since $\mathbb{C}_0[S] \cong \bigoplus_{i=1}^{k} M_2(\mathbb{C})$ for $k = \frac{|S|-1}{4}$ (see (ii)) and $\mathbb{C}_0[T]$ is a homomorphic image of $\mathbb{C}_0[S]$ we get

$$\mathbb{C}_0[T] \cong \bigoplus_{i=1}^{l} M_2(\mathbb{C})$$

for some $1 \leq l \leq k$. Since $|T| = \dim(\mathbb{C}[T]) = 4l + 1$, we get $l = \frac{|T|-1}{4}$. So $\ker(\hat{\phi}) \cong \bigoplus_{i=1}^{k-l} M_2(\mathbb{C})$.

In $\mathbb{C}[S] \cong \mathbb{C} \oplus \bigoplus_{i=1}^{k} M_2(\mathbb{C})$ the subalgebra $\mathbb{C}[Y]$ consists of all the diagonal matrices of $\mathbb{C} \oplus \bigoplus_{i=1}^{k} M_2(\mathbb{C})$. It is easy to see that

$$\dim(\mathbb{C}[Y] \cap \ker(\hat{\phi})) = 2(k - l).$$

Thus

$$|Y/\ker(\phi|_Y)| = \dim(\mathbb{C}[Y]/(\mathbb{C}[Y] \cap \ker(\hat{\phi}))) = 2l + 1.$$

Hence T is an s-indecomposable semigroup with $|T| = 4l + 1$ and $Y/\ker(\phi|_Y)$ is a subsemilattice of T with $|Y/\ker(\phi|_Y)| = 2l + 1$. It means that T is a B_2-combinatorial semigroup.

□

3.5.3 Lemma *([Zub16]) Let S be a completely 0-simple semigroup and Y a subsemilattice of S. Then*

$$|Y| \leq \sqrt{|S| - 1} + 1.$$

If $|Y| = \sqrt{|S| - 1} + 1$, then $S \cong \mathcal{M}^0(1; n, n; I)$, where $n = \sqrt{|S| - 1}$.

Proof. Let S be a completely 0-simple semigroup and let Y be a subsemilattice of S. Let $k := |Y| - 1$. By Rees-theorem, S is isomorphic to a Rees matrix semigroup $\mathcal{M}^0(G; n, m; P)$. The non-zero idempotents of S are of the form $((P_{j,i})^{-1}; i, j)$ with $P_{j,i} \neq 0$. If $(g; i, j), (h; k, l)$ are different commutable non-zero idempotents of S then $P_{j,k} = 0$ and $P_{l,i} = 0$. It means that there is a $k \times k$ permutation matrix R and a $k \times k$ diagonal matrix D over G^0 such that RD is submatrix of P. From $k \leq \min\{n, m\}$ and $|S| = |G|nm + 1$ we get

$$k \leq \sqrt{|S| - 1}.$$

This inequality is an equality if and only if $|G| = 1$, $n = m$ and every row and every column of P has exactly one non-zero element. Using Lemma 3.6. of [CP61, p.94], we get that the inequality is an equality if and only if

$$S \cong \mathcal{M}^0(1; n, n; I).$$

□

3.5.4 Proposition *([Zub16]) If S is a B_2-combinatorial 0-simple semigroup, then $S \cong B_2$.*

Proof. Since S is B_2-combinatorial, it has a subsemilattice Y such $|Y| = \frac{|S|+1}{2}$. By Lemma 3.5.3, $|Y| \leq \sqrt{|S| - 1} + 1$. From

$$\frac{|S| + 1}{2} \leq \sqrt{|S| - 1} + 1$$

we get $1 \leq |S| \leq 5$. Since S is B_2-combinatorial, $|S| = 4k + 1$ for a non-negative integer k. The trivial semigroup is not 0-simple and so $|S| = 5$. Since S is B_2-combinatorial, it has a subsemilattice Y with $|Y| = 3$. By Lemma 3.5.3, we get $S \cong \mathcal{M}^0(1; 2, 2; I) = B_2$. □

On ***principal ideal*** of a semigroup we mean an ideal generated by a single element. Let S be a semigroup. Let $J(a)$ denote the principal ideal of S

generated by an element $a \in S$. Then

$$I(a) := \{b \mid b \in J(a); J(a) \neq J(b)\}$$

is either empty or an ideal of S. The factor semigroup $J(a)/I(a)$ is called a ***principal factor*** of S. It is known that every principal factor of semigroup is a 0-simple, a simple or a null semigroup ([CP61, Lemma 2.39. p.73]).

3.5.5 Theorem *([Zub16]) Let S be a finite semigroup. Then (i) and (ii) are equivalent:*

(i) S is a B_2-combinatorial semigroup,

(ii) S has a zero and for every non-zero element a of S, the principal factor $J(a)/I(a)$ is isomorphic to the semigroup B_2.

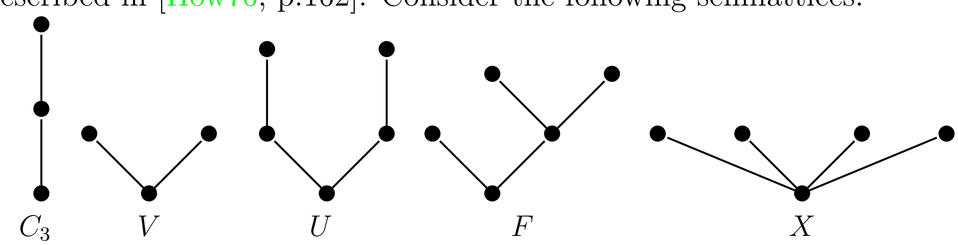
Proof. (i)⇒(ii) Let S be a B_2-combinatorial semigroup. By (i) of Proposition 3.5.2, S has a zero. Let a be a non-zero element of S. By (iii) and (iv) of Proposition 3.5.2, $J(a)/I(a)$ is B_2-combinatorial. It is easy to see that a simple semigroup (simple semigroups has no zero) or a null semigroup is B_2-combinatorial if and only if it contains exactly one element. Hence $J(a)/I(a)$ is 0-simple. By Proposition 3.5.4, $J(a)/I(a) \cong B_2$.

(ii)⇒(i): Every principal factor of S is an inverse semigroup, hence S is also an inverse semigroup. It is well-known that the idempotents of an inverse semigroup forms a subsemilattice. The number of idempotents is $\frac{|S|+1}{2}$. Since every principal factor is s-indecomposable and has a zero divisor, we get that S is s-indecomposable. □

The previous theorem shows why we choose the expression "B_2-combinatorial" for semigroups in Definition 3.5.1: These semigroups are combinatorial inverse semigroups and the principal factors defined by non-zero elements are isomorphic to the semigroup B_2. A semigroup is called ***combinatorial*** if it does not contain a non-trivial group as a subsemigroup.

If Y is a semilattice, then $Y \times_0 B_2$ is a B_2-combinatorial semigroup. The next example shows that not all B_2-combinatorial semigroups can be constructed in this way.

3.5.6 Example For a semilattice E let T_E denote the Munn semigroup of E described in [How76, p.162]. Consider the following semilattices:



We show that there are only 3 non-isomorphic B_2-combinatorial semigroups containing 9 elements:

$C_3 \times_0 B_2 \cong T_U$, $V \times_0 B_2 \subset T_X$ and T_F. There is no semilattice Y such that $Y \times_0 B_2 \cong T_F$.

Proof. If a semigroup is a combinatorial inverse semigroup then it is a fundamental inverse semigroup (an inverse semigroup in which the maximal idempotent separating congruence is identical) [How76, Prop. 5.3.7 p.161]. Let S be a B_2-combinatorial semigroup containing 9 elements. Let $E(S)$ denote the set of idempotents of S. Then $|E(S)| = 5$. So S is isomorphic to a full inverse subsemigroup of the Munn semigroup of $E(S)$ ([How76, Thm.5.4.5

p.165]). It is a matter of checking to see that there are only three Munn semigroups containing a B_2-combinatorial full inverse subsemigroup with 9 elements: T_U, T_F and T_X. The semigroups T_U and T_F are B_2-combinatorial 9-element semigroups. T_X has three 9-element B_2-combinatorial subsemigroups, these are isomorphic to $V \times_0 B_2$.

Suppose that there is a semilattice Y such that $T_F \cong Y \times_0 B_2$. Since $|T_F| = 9$, we get $|Y| = 3$. The non-isomorphic semilattices of 3-element are C_3 and V. It is a matter of checking to see that

$$E(C_3 \times_0 B_2) \cong U, \quad E(V \times_0 B_2) \cong X \quad \text{and} \quad E(T_F) \cong F.$$

Consequently, there is no semilattice Y such that $Y \times_0 B_2 \cong T_F$. $\square$

Chapter 4

Congruence permutable semigroups

In this chapter we consider a semigroup algebraic problem in which the congruence permutable semigroups are in the focus. For an ideal J of a semigroup algebra $\mathbb{F}[S]$, let ϱ_J denote the congruence on the semigroup S which is the restriction of the congruence on $\mathbb{F}[S]$ defined by J. We show that if S is a semilattice or a rectangular band, then the mapping $\varphi_{\{S;\mathbb{F}\}} \mid J \mapsto \varrho_J$ is a $\circ$-homomorphism ($\circ$ is the relation composition) if and only if S is congruence permutable.

4.1 The general case

Let S be a semigroup and $\mathbb{F}$ a field. For an arbitrary congruence α on S, denote the kernel of the extended canonical homomorphism $\mathbb{F}[S] \to \mathbb{F}[S/\alpha]$

by $\mathbb{F}[\alpha]$. By Lemma 5 of Chapter 4 of [Okn91], for every semigroup S and every field $\mathbb{F}$, the mapping

$$\varphi_{\{S;\mathbb{F}\}} : Con(\mathbb{F}[S]) \to Con(S)$$

$$J \mapsto \varrho_J$$

is a surjective $\wedge$-homomorphism such that $\varrho_{\mathbb{F}[\alpha]} = \alpha$ for every congruence α on S. Int his chapter we examine that, the mapping $\mathbb{F}[S]$ preserves $\circ$ or $\vee$ or not, where $\circ$ is composition of relations and $\vee$ is the join operation of the congruence lattice. As a homomorphic image of a semigroup is also a semigroup, and $\alpha \circ \beta = \alpha \vee \beta$ is satisfied for every congruence α and β of a congruence permutable semigroup, the assertions of the following lemma are obvious.

4.1.1 Lemma *([NZ16]) Let S be a semigroup and $\mathbb{F}$ a field. Assume that the mapping $\varphi_{\{S;\mathbb{F}\}} : Con(\mathbb{F}[S]) \to \mathcal{B}_S;\ J \mapsto \varrho_J$ is a $\circ$-homomorphism. Then S is a congruence permutable semigroup. Moreover, if S is a congruence permutable semigroup, then $\varphi_{\{S;\mathbb{F}\}} : Con(\mathbb{F}[S]) \to Con(S);\ J \mapsto \varrho_J$ is a $\circ$-homomorphism if and only if $\vee$-homomorphism, that is, $ker_{\varphi_{\{S;\mathbb{F}\}}}$ is $\vee$-compatible.*

In the Bibliography of the dissertation we can find articles about congruence permutable semigroups (especially, Δ-semigroups; that is, semigroups whose congruences form a chain with respect to inclusion).

The following example shows that the converse of the first assertion of Lemma 4.1.1 is not true in general: let a congruence permutable semigroup

S, the mapping $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism or not depends on the field $\mathbb{F}$.

4.1.2 Example *Let C_4, $\mathbb{F}_3$ and $\mathbb{F}_2$ denote the cyclic group of order 4, the fields of 3 and 2 elements, respectively. It is known that every group is a congruence permutable semigroup. The $ker_{\varphi_{\{C_4;\mathbb{F}_3\}}}$ is not $\vee$-compatible however $ker_{\varphi_{\{C_4;\mathbb{F}_2\}}}$ is $\vee$-compatible.*

Proof. Denote the elements of C_4 by $1, a, a^2, a^3$. It is easy to see that

$$I = \text{Span}\{1 + a^2, a + a^3\} \quad \text{and} \quad J = \text{Span}\{1 + a, a + a^2, a^2 + a^3\}$$

are ideals of $\mathbb{F}_3[C_4]$. Moreover,

$$\varphi_{\{C_4;\mathbb{F}_3\}}(I) = \varrho_I = \iota_{C_4} \quad \text{and} \quad \varphi_{\{C_4;\mathbb{F}_3\}}(J) = \varrho_J = \alpha_{C_2},$$

where α_{C_2} denotes the congruence on C_4 defined by $C_2 = \{1, a^2\}$. From this it follows that

$$\varphi_{\{C_4;\mathbb{F}_3\}}(I) \vee \varphi_{\{C_4;\mathbb{F}_3\}}(J) = \varrho_I \vee \varrho_J$$

$$\varphi_{\{C_4;\mathbb{F}_3\}}(I \vee J) = \varrho_{(I \vee J)} = \varrho_{(I+J)} = \omega_{C_4}.$$

Since $\varrho_I \vee \varrho_J \neq \omega_{C_4}$ the $ker_{\varphi_{\{C_4;\mathbb{F}_3\}}}$ is not $\vee$-compatible and so $\varphi_{\{C_4;\mathbb{F}_3\}}$ is not a $\circ$-homomorphism.

It is a matter of checking to see that the ideals of $\mathbb{F}_2[C_4]$ are preciously $\{0\}$, $\mathbb{F}_2[C_4]$,

$$\mathbb{F}_2[\omega_{C_4}] = \{0, 1 + a + a^2 + a^3, 1 + a^2, a + a^3, 1 + a, a + a^2, a^2 + a^3, 1 + a^3\},$$

$$\mathbb{F}_2[\alpha_{C_2}] = \{0, 1 + a + a^2 + a^3, 1 + a^2, a + a^3\},$$

and

$$\text{Span}\{1 + a + a^2 + a^3\} = \{0, 1 + a + a^2 + a^3\}.$$

Thus $Con(\mathbb{F}_2[C_4])$ is the next:

$$\begin{array}{c} \mathbb{F}_2[C_4] \\ | \\ \mathbb{F}_2[\omega_{C_4}] \\ | \\ \mathbb{F}_2[\alpha_{C_2}] \\ | \\ \text{Span}\{1 + a + a^2 + a^3\} \\ | \\ \{0\} \end{array}$$

It is easy to see that $ker_{\varphi_{\{C_4;\mathbb{F}_2\}}}$ is $\vee$-compatible so $\varphi_{\{C_4;\mathbb{F}_2\}}$ is a $\circ$-homomorphism. □

By Lemma 4.1.1 and the Example 4.1.2, it is a natural idea to find all pairs $(S, \mathbb{F})$ of congruence permutable semigroups S and fields $\mathbb{F}$, for which the mapping $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism. Next we show that if S is an arbitrary congruence permutable semilattice or an arbitrary congruence permutable rectangular band, then $\varphi_{\{S;\mathbb{F}\}}$ is $\vee$-compatible for an arbitrary field $\mathbb{F}$.

4.2 Semilattices

4.2.1 Theorem *([NZ16]) Let S be a congruence permutable semilattice. Then, for an arbitrary field $\mathbb{F}$, $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism.*

Proof. Assume that S is a congruence permutable semilattice. Then, by [Ham75, Lemma 2], $|S| \leq 2$. We consider the case when $|S| = 2$. Let $S = \{z, e\}$, where z and e are the zero and the identity element respectively. It is clear that S has two congruences: ι_S and ω_S.

Let $\mathbb{F}$ be an arbitrary field. It is easy to see that

$$J_z = \{\alpha z \mid \alpha \in \mathbb{F}\} \quad \text{and} \quad J_{z-e} = \{\alpha(z-e) \mid \alpha \in \mathbb{F}\}$$

are proper ideals of $\mathbb{F}[S]$. As

$$dim(J_z) = dim(J_{z-e}) = 1,$$

the ideals J_z and J_{z-e} are minimal ideals of $\mathbb{F}[S]$. We show that the ideals of $\mathbb{F}[S]$ are preciously

$$\{0\}, J_z, J_{z-e} \quad \text{and} \quad \mathbb{F}[S].$$

Let $J \neq \{0\}$ be a proper ideal of $\mathbb{F}[S]$. Clearly $dim(J) = 1$. Let

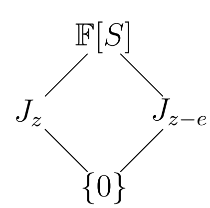
$$A = \alpha z + \beta e$$

be a non-zero element of J. Then

$$zA = z(\alpha z + \beta e) = (\alpha + \beta)z.$$

If $\alpha + \beta \neq 0$, then $J = J_z$. If $\alpha + \beta = 0$, then $J = J_{z-e}$.

Thus the ideals of $\mathbb{F}[S]$ are $\{0\}$, J_z, J_{z-e}, $\mathbb{F}[S]$. So $Con(\mathbb{F}[S])$ is the next:



It is straight forward to see that the $ker_{\varphi_{\{S;\mathbb{F}\}}}$-classes of $Con(\mathbb{F}[S])$ are $\{\{0\}, J_z\}$ and $\{J_{z-e}, \mathbb{F}[S]\}$. It is easy to see that $ker_{\varphi_{\{S;\mathbb{F}\}}}$ is $\vee$-compatible and so, by Lemma 4.1.1, $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism. □

4.2.2 Corollary *([NZ16]) Let S be a semilattice. Then, for a field $\mathbb{F}$, $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism if and only if S is congruence permutable.*

Proof. This is obvious by Lemma 4.1.1 and Theorem 4.2.1. □

4.3 Rectangular bands

4.3.1 Theorem *([NZ16]) Let $S = L \times R$ be a congruence permutable rectangular band where L is a left zero semigroup, R is a right zero semigroup. Then, for an arbitrary field $\mathbb{F}$, $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism.*

Proof. As a rectangular band satisfies the identity $axyb = ayxb$, that is, every rectangular band is a medial semigroup Hence [BC81, Corollary 1.2] implies $|L| \leq 2$ and $|R| \leq 2$.

First consider the case when $|L| = 1$. Then S is isomorphic to the right zero semigroup R, and $|S| \leq 2$. We can suppose that $|S| = 2$. Let $S = \{e, f\}$, where e, f are idempotents, $ef = f$ and $fe = e$. The congruences of S are ι_S and ω_S. We show that the ideals of $\mathbb{F}[S]$ are

$$\{0\}, J_{e-f} = \mathbb{F}[\omega_S] = \{\alpha(e-f) \mid \alpha \in \mathbb{F}\} \quad \text{and} \quad \mathbb{F}[S].$$

Let $J \neq \{0\}$ be an arbitrary ideal. Assume that there is an element

$$0 \neq \alpha e + \beta f \in J$$

for which

$$\alpha + \beta \neq 0$$

is satisfied. Then

$$(\alpha e + \beta f)e = (\alpha + \beta)e$$

and so

$$\frac{1}{\alpha + \beta}(\alpha + \beta)e = e$$

from which we get $ef = f \in J$. Consequently,

$$J = \mathbb{F}[S].$$

Next, consider the case when

$$\alpha + \beta = 0$$

is satisfied for every

$$A = \alpha e + \beta f \in J.$$

Then $\beta = -\alpha$ and so

$$A = \alpha e + \beta f = \alpha e - \alpha f = \alpha(e - f) \in J_{e-f}.$$

Consequently,

$$J \subseteq J_{e-f}.$$

As $dim(J_{e-f}) = 1$, the ideal J_{e-f} is minimal. Hence

$$J = J_{e-f}.$$

Thus the ideals of $\mathbb{F}[S]$ are preciously $\{0\}$, J_{e-f} and $\mathbb{F}[S]$, indeed.

So $Con(\mathbb{F}[S])$ is

$$\begin{array}{c}\mathbb{F}[S]\\ |\\ J_{e-f}\\ |\\ \{0\}\end{array}$$

It is a matter of checking to see that the $ker_{\varphi_{\{S;\mathbb{F}\}}}$-classes of $Con(\mathbb{F}[S])$ are $\{\{0\}\}$ and $\{J_{e-f}, \mathbb{F}[S]\}$. It is easy to see that $ker_{\varphi_{\{S;\mathbb{F}\}}}$ is $\vee$-compatible and so, by Lemma 4.1.1, $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism.

If $|R| = 1$, then S is a left zero semigroup, and $|S| \leq 2$. We can prove that $ker_{\varphi_{\{S;\mathbb{F}\}}}$ is $\vee$-compatible in a similar way, as int he case when $|L| = 1$.

Next, consider the case when $|L| = |R| = 2$. Let

$$L = \{a_1, a_2\}, \quad R = \{b_1, b_2\}.$$

Let α_L and α_R denote the kernels of the projection homomorphisms $S \mapsto L$ and $S \mapsto R$, respectively. The α_L-classes of S are

$$\{(a_1, b_1); (a_1, b_2)\} \quad \text{and} \quad \{(a_2, b_1); (a_2, b_2)\}.$$

The α_R-classes of S are

$$\{(a_1, b_1); (a_2, b_1)\} \quad \text{and} \quad \{(a_1, b_2); (a_2, b_2)\}.$$

It is easy to see that S has no congruences other than ι_S, α_L, α_R and ω_S. We show that the ideals of $\mathbb{F}[S]$ are

$$\mathbb{F}[S],\ \mathbb{F}[\omega_S] = \left\{ \sum_{i,j=1}^{2} \alpha_{i,j}(a_i, b_j) \middle| \sum_{i,j=1}^{2} \alpha_{i,j} = 0 \right\},$$

$$J_L = \mathbb{F}[\alpha_L],\ J_R = \mathbb{F}[\alpha_R],\ J_L \cap J_R, \{0\}.$$

We note that

$$\dim(\mathbb{F}[\omega_S]) = 3,$$

$$\dim(J_L) = \dim(J_R) = 2.$$

First we show that either $J \subseteq \mathbb{F}[\omega_S]$ or $J = \mathbb{F}[S]$ for every ideal J of $\mathbb{F}[S]$. Let J be an arbitrary ideal of $\mathbb{F}[S]$. Assume that there is an element

$$A = \alpha_{1,1}(a_1, b_1) + \alpha_{1,2}(a_1, b_2) + \alpha_{2,1}(a_2, b_1) + \alpha_{2,2}(a_2, b_2) \in J$$

such that $A \notin \mathbb{F}[\omega_S]$, that is $\sum_{i,j=1}^{2} \alpha_{i,j} \neq 0$. Let $i, j \in \{1, 2\}$ be arbitrary elements. Then

$$(a_i, b_1)A(a_1, b_j) = \left(\sum_{i,j=1}^{2} \alpha_{i,j} \right) (a_i, b_j).$$

As $\sum_{i,j=1}^{2} \alpha_{i,j} \neq 0$, we get $(a_i, b_j) \in J$ from which it follows that $S \subseteq J$. Consequently, $J = \mathbb{F}[S]$. Thus $\mathbb{F}[\omega_S]$ is the only maximal ideal of $\mathbb{F}[S]$.

Next we show that $J_L \cap J_R$ is the only ideal of $\mathbb{F}[S]$ of dimension 1. Let

$$A = \alpha_{1,1}(a_1, b_1) + \alpha_{1,2}(a_1, b_2) + \alpha_{2,1}(a_2, b_1) + \alpha_{2,2}(a_2, b_2) \in J_L \cap J_R$$

be a non-zero element. Similarly,

$$(a_1, b_1)\ \alpha_L\ (a_1, b_2) \quad \text{and} \quad (a_2, b_1)\ \alpha_L\ (a_2, b_2),$$

hence

$$\alpha_{1,2} = -\alpha_{1,1} \quad \text{and} \quad \alpha_{2,2} = -\alpha_{2,1}.$$

As

$$(a_1, b_1)\ \alpha_R\ (a_2, b_1) \quad \text{and} \quad (a_1, b_2)\ \alpha_R\ (a_2, b_2),$$

we have

$$\alpha_{2,1} = -\alpha_{1,1} \quad \text{and} \quad \alpha_{2,2} = -\alpha_{1,2}.$$

Thus

$$A = \alpha((a_1, b_1) - (a_1, b_2) - (a_2, b_1) + (a_2, b_2))$$

for some α non-zero element of $\mathbb{F}$. Consequently, the ideal $J_L \cap J_R$ is generated by

$$(a_1, b_1) - (a_1, b_2) - (a_2, b_1) + (a_2, b_2).$$

Hence the dimension of $J_L \cap J_R$ is 1.

To show that $J_L \cap J_R$ is the only ideal of $\mathbb{F}[S]$ whose dimension is 1, consider a one-dimensional ideal J of $\mathbb{F}[S]$ generated by an element

$$0 \neq B = \alpha_{1,1}(a_1, b_1) + \alpha_{1,2}(a_1, b_2) + \alpha_{2,1}(a_2, b_1) + \alpha_{2,2}(a_2, b_2).$$

Since $J \subset \mathbb{F}[\omega_S]$ and

$$(a_1, b_1)B = (\alpha_{1,1} + \alpha_{2,1})(a_1, b_1) + (\alpha_{1,2} + \alpha_{2,2})(a_1, b_2) \in J.$$

There is a coefficient $\xi \in \mathbb{F}$ such that

$$(a_1, b_1)B = \xi B.$$

Assume $\xi \neq 0$. Then $\alpha_{2,1} = \alpha_{2,2} = 0$ and so

$$B = \alpha_{1,1}(a_1, b_1) + \alpha_{1,2}(a_1, b_2).$$

From

$$(a_2, b_2)B = \alpha_{1,1}(a_2, b_1) + \alpha_{1,2}(a_2, b_2) \in J$$

we can conclude that $\alpha_{1,1} = \alpha_{1,2} = 0$ and so $B = 0$. This is a contradiction. Hence $\xi = 0$. Thus

$$B = \alpha_{1,1}(a_1, b_1) + \alpha_{1,2}(a_1, b_2) - \alpha_{1,1}(a_2, b_1) - \alpha_{1,2}(a_2, b_2).$$

As

$$B(a_1, b_1) = (\alpha_{1,1} + \alpha_{1,2}((a_1, b_1) - (a_2, b_1)) \in J,$$

we get $B(a_1, b_1) = \tau B$ for some $\tau \in \mathbb{F}$. Assume $\tau \neq 0$. Then $\alpha_{1,2} = 0$ and so

$$B = \alpha_{1,1}(a_1, b_1) - \alpha_{1,1}(a_2, b_1).$$

From

$$B(a_2, b_2) = \alpha_{1,1}((a_1, b_2) - (a, 2, b_2)) \in J$$

we can conclude that $\alpha_{1,1} = 0$ and so $B = 0$. This is a contradiction. Hence $\tau = 0$. Thus $\alpha_{1,2} = -\alpha_{1,1}$ and so

$$B = \alpha_{1,1}((a_1, b_1) - (a_1, b_2) - (a_2, b_1) + (a_2, b_2)) \in J_L \cap J_R.$$

As $J \neq \{0\}$ and $J_L \cap J_R$ is a minimal ideal of $\mathbb{F}[S]$, we get

$$J = J_L \cap J_R,$$

that is, $J_L \cap J_R$ is the only ideal of $\mathbb{F}[S]$ whose dimension is 1.

As $dim(J_R + J_L) > dim J_R$ and $\mathbb{F}[\omega_S] \supseteq J_R + J_L$, we have

$$J_R + J_L = \mathbb{F}[\omega_S].$$

Let J be an arbitrary ideal of $\mathbb{F}[S]$ which differs from all of the ideals $\mathbb{F}[S], \mathbb{F}[\omega_S], J_L, J_R, J_L \cap J_R, \{0\}$. Then $J \subset \mathbb{F}[\omega_S]$ and $dim(J) = 2$.

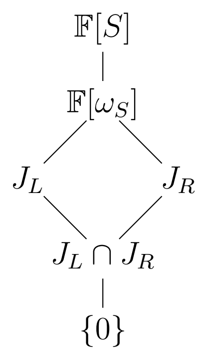
If $J \cap J_L = \{0\}$, then $dim(J + J_L) = 4$ which contradicts $J + J_L \subseteq \mathbb{F}[\omega_S]$. Hence $dim(J \cap J_L) = 1$ and so $J_L \cap J_R = J \cap J_L$. From this we also get $J \cap J_R = J_L \cap J_R$. Recall that $C = (a_1, b_1) - (a_1, b_2) - (a_2, b_1) + (a_2, b_2)$ generates the ideal $J_L \cap J_R$. Let A be an arbitrary element of $J - (J_L \cap J_R)$. Then A and C are linearly independent. So

$$A = \alpha(a_1, b_1) + \beta(a_1, b_2) + \gamma(a_2, b_1) + (-\alpha - \beta - \gamma)(a_2, b_2),$$

where $\alpha \neq -\gamma$. Then

$$(a_1, b_1)A = (\alpha + \gamma)((a_1, b_1) - (a_1, b_2)) \in J_L \cap J = J_L \cap J_R.$$

It means $\alpha = -\gamma$ which is a contradiction. Thus $Con(\mathbb{F}[S])$ is the next:



It is a matter of checking to see that the $ker_{\varphi_{\{S;\mathbb{F}\}}}$-classes of $Con(\mathbb{F}[S])$ are $\{\{0\}, J_L \cap J_R\}, \{J_L\}, \{J_R\}$ and $\{\mathbb{F}[\omega_S], \mathbb{F}[S]\}$. It is easy to see that $ker_{\varphi_{\{S;\mathbb{F}\}}}$ is $\vee$-compatible and so, by Lemma 4.1.1, $\varphi_{\{S;\mathbb{F}\}}$ is a homomorphism of the semigroup $(\mathrm{Con}(\mathbb{F}[S]), \circ)$ onto the semigroup $(Con(S); \circ)$. $\square$

4.3.2 Corollary *([NZ16]) Let $S = L \times R$ be a rectangular band. Then, for a field $\mathbb{F}$, $\varphi_{\{S;\mathbb{F}\}}$ is a $\circ$-homomorphism if and only if S is congruence permutable.*

Proof. It is obvious by Lemma 4.1.1 and Theorem 4.3.1. $\square$

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