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DOCTORAL THESIS BOOKLET

Asymptotic properties of mean field coupled maps

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1 Introduction

Systems of coupled maps have become a subject of considerable interest within the theory of dynamical systems in the past few decades. In broad terms, a coupled map system consist of interacting individual units, each governed by a simple discrete time dynamical system perturbed by the effect of the interaction.

The variety in the many models studied essentially stems from the structure of the interaction. Earliest examples include *coupled map lattices*, chiefly motivated by statistical physics, in particular the Ising model. In a coupled map lattice, individual units are placed at the points of a lattice (usually $\mathbf{Z}$ or $\mathbf{Z}^d$) and interaction is defined between sites of various distance depending on the modell.

The *mean field model*, providing the foundation of the system studied in this thesis has also motivation from classical statistical physics (e.g. the Curie-Weiss model). A mean field system (also known as globally coupled system in the literature) can be understood by imagining that the sites generate a so-called *mean field*, acting as a common environment and providing a perturbation for each of the individual dynamics. This produces a compound dynamics where each site interacts with every other, and the role of any individual site is the same.

The main interest in such models is the emergence of *bifurcations*: how do the characteristic features of such a compound system change when the strength of interaction is varied. When there is no coupling, the individual sites behave independently; on the other hand for strong interactions some kind of synchronization can be expected. Such phenomena can be thought of as a deterministic analogue of the phase transitions of Ising models [27, 47]. Mathematically rigorous results use most sophisticated tools of dynamical systems theory, and typically prove the lack of phase transitions (unique SRB measure) for small interaction strength [16, 18, 25, 32, 33, 39, 41].

The results are more scarce for strong coupling. Koiller and Young [42] provided one of the first mathematically rigorous steps by studying some simple, finite models in terms of hyperbolic theory. They proved the emergence of contracting directions once the coupling is above a threshold and showed that this can indeed be interpreted as an example of synchronization.

In the first part of this thesis we consider a system of N globally coupled maps of the circle, as defined by Fernandez [24] (based on [42]). This model is a discrete time, chaotic analogue of the Kuramoto model of coupled oscillators [45, 46]. We first explore certain statistical properties of this system for weak coupling (like the existence of an absolutely continuous invariant measure, ergodicity, mixing etc.), then study different types of synchronization phenomena for strong coupling. We introduce an infinite version of this model in the second part of the thesis. This model enables us to study more general individual dynamics than the doubling map. In this model we are interested in the existence and properties of invariant distributions for weak coupling and the nature of synchronization for strong coupling.

In the remainder of this chapter we will give a short introduction of our basic model and an overview of previous results.

1.1 The basic model and previous results

The main idea behind this model is to formulate a discrete time version of the *Kuramoto model* with chaotic local dynamics. The Kuramoto model of coupled oscillators was originally designed to model chemical instabilities [45, 46] but since then it has become a popular way to represent the time evolution of systems consisting of many simple individuals connected by pairwise interaction [1, 49, 54].

The Kuramoto model considers $N \in \mathbb{N}$ oscillators represented by their phase θ_i, which is an element of the unit circle $\mathbb{T} = \mathbb{R}\backslash 2\pi\mathbb{Z}$. The dynamics is described by the following set of ordinary differential equations:

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \qquad i = 1, \dots, N \tag{1.1}$$

where $K \in \mathbb{R}^+$ is the *coupling strength* and $\omega_i \in \mathbb{R}$ are time-independent randomly drawn frequencies. The sine function of the difference of two phases represents a simple interaction between pairs of oscillators.

For weak interaction, that is for small K, the dynamics is very much like in the $K = 0$ case: it is *unorganized* in the sense that the system state returns to an arbitrarily small neighborhood of the initial state infinitely many times. On the other hand, for sufficiently large K full phase locking occurs: difference of each pair of phases becomes constant asymptotically, meaning *organized* behavior, or in other words, synchronization [7, 22, 31].

The model of Fernandez, providing the starting point for this thesis is the following: let $N \in \mathbb{N}$ and $\mathbb{T}^N = (\mathbb{R}\backslash\mathbb{Z})^N$ be the N-dimensional (flat) torus. We are going to represent $\mathbb{T}^N$ as $[0,1]^N \subset \mathbb{R}^N$ with opposite faces identified. We define $F_{\varepsilon,N} : \mathbb{T}^N \to \mathbb{T}^N$ as

$$F_{\varepsilon,N} = F_{0,N} \circ \Phi_{\varepsilon,N}$$

where

$$\begin{aligned} (F_{0,N}(x))_i &= 2x_i \mod 1 \\ (\Phi_{\varepsilon,N}(x))_i &= x_i + \frac{\varepsilon}{N}\sum_{j=1}^{N} g(x_j - x_i) \qquad \mod 1 \qquad i = 1, \dots, N, \ \ x = (x_i)_{i=1}^N \in \mathbb{T}^N. \end{aligned} \tag{1.2}$$

The function $g : \mathbb{R} \to \mathbb{R}$ is defined as the periodic extension of

$$\hat{g}(u) = \begin{cases} 0 & \text{if } u = \pm\frac{1}{2}, \\ u & \text{if } u \in (-\frac{1}{2}, \frac{1}{2}) \end{cases}$$

See Figure 1 for the graph of this function. We can see that the function g can be

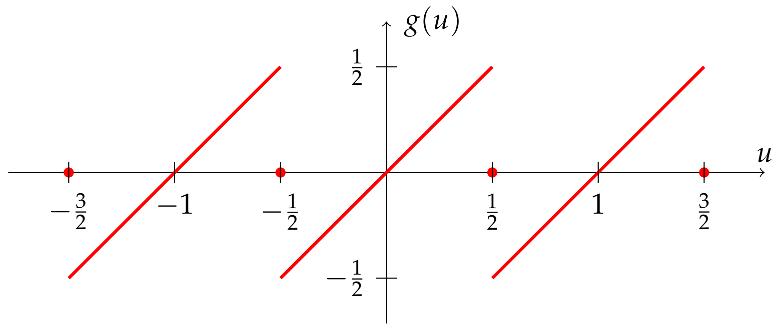


FIGURE 1: The function g.

thought of as a crude simplification of the sine function.

The map $F_{\varepsilon,N}$ is a piecewise affine map of the unit cube (discontinuities arising from the discontinuities of $g(\cdot)$) and the domains of continuity are N dimensional polyhedra. The linear part of the map has two distinct eigenvalues, $\lambda_1 = 2$ is of multiplicity 1 and $\lambda_2 = 2(1-\varepsilon)$ is of multiplicity $N-1$. The eigenvectors associated to these eigenvalues are independent of the domain of continuity. So it makes sense to talk about *contracting* and *expanding* directions: all directions are expanding if $\varepsilon \in [0, 1/2)$, and all but one direction become contracting once ε becomes greater than $1/2$. We are going to call $\varepsilon \in [0, 1/2)$ the *expanding regime* of the parameter and $\varepsilon \in (1/2, 1)$ the *contracting regime*. We are not going to discuss the degenerate cases $\varepsilon = 1/2$ and $\varepsilon = 1$.

Koiller and Young focused on finding the parameter values where contracting directions emerge, for a variety of coupled map networks. For the system studied in this thesis, they managed to designate the critical parameter value $\varepsilon_2^* = 1/2$ mentioned above and fully characterize synchronization phenomena in the $N = 3$ case. Fernandez studied the case of globally coupled doubling maps in a more careful detail and found further critical parameter values associated to a different phenomenon, *ergodicity breaking*. It is well known, that for no coupling the system $(\mathbb{T}^N, F_{0,N})$ (where $F_{0,N}$ is just a product of N doubling maps) has a unique Lebesgue-absolutely continuous invariant measure (*acim*). It can be derived from classical results of the literature (see for example [41]), that this property prevails for weak coupling. That is, for small values of ε, the system $(\mathbb{T}^N, F_{\varepsilon,N})$ has a unique acim. On the other hand, no acim exists in the contracting regime $\varepsilon > 1/2$. Fernandez showed that when $N = 3$, there exists a critical value of the coupling parameter in the expanding regime, $0 < \varepsilon_1^* < 1/2 = \varepsilon_2^*$, such that for $\varepsilon \geq \varepsilon_1^*$ the system has *multiple acims*. (He also showed that for $N = 2$ no such value exists.) This can also be expressed as the acim of maximal support no longer being ergodic, hence the term ergodicity breaking. He conjectured, based on computer simulations, that ergodicity breaking occurs for all $N \geq 4$.

This conjecture provided the starting point for the research contained in this thesis. In addition to this, many other aspects and an infinite generalization of the model was studied. In the next sections we are going to present the main results and comment on analogies with similar statements regarding the Kuramoto model.

2 A system of N coupled maps

In this section we study the model defined by (1.2). We are going to introduce a change of coordinates to obtain a factor of this system. Let

$$u_1 = \sum_{s=1}^{N} x_s \mod 1,$$
$$u_{i+1} = x_i - x_{i+1} \mod 1, \quad i = 1, \dots, N-1. \tag{2.1}$$

Define $h : \mathbb{R} \to \mathbb{R}$ as

$$h(v) = \begin{cases} 0 & v = (2k+1) \cdot \frac{1}{2}, \quad k \in \mathbb{Z}, \\ -\lfloor v + 1/2 \rfloor & \text{otherwise}, \end{cases}$$

and $c_i : [0,1]^{N-1} \to \mathbb{R}$, $i = 2, \dots, N$ as

$$c_i(u_2, \dots, u_N) = -2h(u_i) + \sum_{k=2}^{i-1} h\left(\sum_{j=k}^{i-1} u_j\right) - \sum_{k=i+1}^{N} h\left(\sum_{j=i}^{k} u_j\right) - \sum_{k=2}^{i-1} h\left(\sum_{j=k}^{i} u_j\right) + \sum_{k=i+1}^{N} h\left(\sum_{j=i+1}^{k} u_j\right) \mod 1. \tag{2.2}$$

By elementary calculations one can show that in the coordinates (2.1) the law $F_{\varepsilon,N}$ becomes

$$\begin{aligned} T(u_1) &= 2u_1 \mod 1, \\ (G_{\varepsilon,N}(u_2, \dots, u_N))_i &= 2(1-\varepsilon)u_i + \frac{2\varepsilon}{N} c_i(u_2, \dots, u_N) \mod 1, \quad i = 2, \dots, N. \end{aligned} \tag{2.3}$$

We can see that the map $G_{\varepsilon,N}$ is a piecewise similarity with expansion/contraction factor $2(1-\varepsilon)$ and translation depending on the domain of continuity. The discontinuities of $G_{\varepsilon,N}$ are the intersections of the planes

$$u_i + u_{i+1} \cdots + u_{i+j} = (2k+1) \cdot \frac{1}{2}, \quad i = 2, \dots, N, j = 0, \dots, N-i, k = 0, 1, \dots, j$$

with $\mathbb{T}^{N-1}$, giving polyhedral domains of continuity.

Observe that the system $(\mathbb{T}^N, T \times G_{\varepsilon,N})$ is a factor of $(\mathbb{T}^N, F_{\varepsilon,N})$, since for any $x = (x_i)_{i=1}^N \in \mathbb{T}^N$ the points

$$(x_i)_{i=1}^N, \left(x_i + \frac{1}{N}\right)_{i=1}^N, \dots, \left(x_i + \frac{N-1}{N}\right)_{i=1}^N \tag{2.4}$$

share the same u-coordinates. (Addition is understood mod 1.) Nonetheless, the following proposition tells us that we can deduce ergodicity and ergodicity breaking of $(\mathbb{T}^N, F_{\varepsilon,N})$ by proving similar properties of $(\mathbb{T}^{N-1}, G_{\varepsilon,N})$.

Proposition 2.1. *The system $(\mathbb{T}^N, F_{\varepsilon,N})$ has a unique acim if and only if $(\mathbb{T}^{N-1}, G_{\varepsilon,N})$ does. Furthermore, the acim of $(\mathbb{T}^N, F_{\varepsilon,N})$ is mixing if and only if the acim of $(\mathbb{T}^{N-1}, G_{\varepsilon,N})$ is mixing.*

In the next two sections we introduce our results on ergodicity and ergodicity breaking for small ε and synchronization for ε sufficiently large. The results regarding $N = 2$ and $N = 3$ are contained in the paper [2] while the results on $N = 4$ in [57].

2.1 Ergodicity versus ergodicity breaking

N=2

Our result regarding the system of two coupled maps is the following:

Theorem 2.2. *Consider the dynamical system $(\mathbb{T}^2, F_{\varepsilon,2})$.*

1. *Let $0 \le \varepsilon < 1 - \frac{\sqrt{2}}{2}$. The system has a unique absolutely continuous invariant measure, which is mixing.*

2. *Let* $1 - \frac{\sqrt{2}}{2} \leq \varepsilon < \frac{1}{2}$*. The system has a unique absolutely continuous invariant measure, which is not mixing. If* $1 - \frac{\sqrt[2^n]{2}}{2} \leq \varepsilon < 1 - \frac{\sqrt[2^{n+1}]{2}}{2}$ *holds, the attractor is the union of* 2^n *mixing components.*

By *mixing component* we mean a subset of the attractor such that some iterate of the map restricted to it gives a mixing system. So we claim that when $1 - \frac{\sqrt[2^n]{2}}{2} \leq \varepsilon < 1 - \frac{\sqrt[2^{n+1}]{2}}{2}$ holds, the attractor is a union of 2^n sets which are mapped onto each other and the 2^nth iterate of the map restricted to any of these sets is mixing.

By attractor, we mean the *Milnor attractor* of the system. This is the smallest closed set which attracts the orbit of every initial condition up to a zero Lebesgue measure set. For further reference see [17] and [48].

By Proposition 2.1, it is enough to prove the statements of the theorem for $(\mathbb{T}, G_{\varepsilon,2})$:

$$G_{\varepsilon,2}(u) = \begin{cases} 2(1-\varepsilon)u & \text{if } 0 \leq u < \frac{1}{2}, \\ 1 & \text{if } u = \frac{1}{2}, \\ 2(1-\varepsilon)u + 2\varepsilon - 1 & \text{if } \frac{1}{2} < u \leq 1. \end{cases} \tag{2.5}$$

This is a Lorenz-type interval map studied by W. Parry [51]. Thus the vital parts of the proof can be deduced from [51] and some remaining details from Glendinning and Sparrow [28].

N=3

Our result regarding the system of three coupled maps is the following:

Theorem 2.3. *Let us consider the dynamical system* $(\mathbb{T}^3, F_{\varepsilon,3})$ *and define* $\varepsilon^*(3) = \frac{4-\sqrt{10}}{2}$.

1. *Let* $0 \leq \varepsilon < 1 - \frac{\sqrt{2}}{2}$*. The system has a unique absolutely continuous invariant measure.*

2. *If* $\varepsilon^*(3) \leq \varepsilon < \frac{1}{2}$*, at least six ergodic absolutely continuous invariant measures exist.*

By Proposition 2.1, it is enough to prove the statements of the theorem for the system $(\mathbb{T}^2, G_{\varepsilon,3})$.

We prove part 1 by showing that $G_{\varepsilon,3}$ is locally eventually onto for the stated values of ε. We now comment on the proof of part 2. In our context, we define a *symmetry* of a map F as a linear transformation S such that

$$S \circ F = F \circ S.$$

The symmetries of the map $F_{\varepsilon,N}$ which are of interest to us arise from two sources:

- The inversion symmetry of g and that of the doubling map T (namely, $g(1-s) = 1 - g(s)$ and $T(1-s) = 1 - T(s)$) imply the inversion symmetry of $F_{\varepsilon,N}$. More precisely,
$$F_{\varepsilon,N} \circ I = I \circ F_{\varepsilon,N},$$
where
$$I : (x_1, \ldots, x_N) \to (1 - x_1, \ldots, 1 - x_N).$$

- Every permutation of $x_1, \ldots, x_N$ is a symmetry of $F_{\varepsilon,N}$:

$$F_{\varepsilon,N} \circ \pi = \pi \circ F_{\varepsilon,N},$$

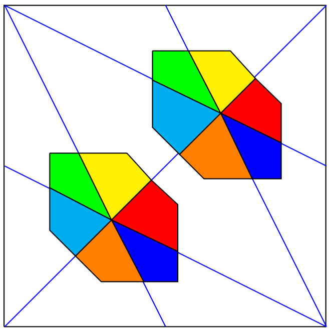


FIGURE 2: The asymmetric invariant set $\mathcal{A}$ (green). The symmetric images of $\mathcal{A}$ giving the six asymmetric invariant sets. Color code: $S_3S_0(\mathcal{A})$ (yellow), $S_1(\mathcal{A})$ (red), $S_0(\mathcal{A})$ (blue), $S_3(\mathcal{A})$ (orange), $S_1S_0(\mathcal{A})$ (cyan).

where π is a permutation of the coordinates.

We are going to call the group of these transformations the *symmetry group* of $F_{\varepsilon,N}$. This symmetry group induces the symmetry group of $G_{\varepsilon,N}$, which we denote by $S_{G,N}$. In the $N = 3$ case we are going to use the notation S_0 for the symmetry induced by I, S_1 for the symmetry induced by π_{12}, S_2 for the symmetry induced by π_{13} and S_3 for the symmetry induced by π_{23}.

Now we define what we mean by *asymmetric* and *invariant* set.

Definition 2.4. A set $\mathcal{B} \subset \mathbb{T}^{N-1}$ is symmetric with respect to $S \in S_{G,N}$ if $\mathcal{B} = S(\mathcal{B})$, and asymmetric with respect to $S \in S_{G,N}$ if $\mathcal{B}$ and its symmetric image $S\mathcal{B}$ are disjoint.

A set $\mathcal{B}$ is symmetric if it is symmetric with respect to every element of $S_{G,N}$, and asymmetric if there exists a symmetry in $S_{G,N}$ for which $\mathcal{B}$ is asymmetric.

Note that asymmetric means more than not symmetric. To keep terminology brief, we are going to say that $\mathcal{B}$ *breaks* S, when $\mathcal{B}$ is asymmetric with respect to some symmetry S.

Definition 2.5. A set $\mathcal{B} \subset \mathbb{T}^{N-1}$ is (forward) invariant if $G_{\varepsilon,N}(\mathcal{B}) \subseteq \mathcal{B}$.

Notice that if $\mathcal{B}$ is invariant under the dynamics, $S\mathcal{B}$ is also invariant since

$$G_{\varepsilon,N}(\mathcal{B}) \subseteq \mathcal{B} \Rightarrow G_{\varepsilon,N}(S(\mathcal{B})) = SG_{\varepsilon,N}(\mathcal{B}) \subseteq S(\mathcal{B}).$$

Suppose this set $\mathcal{B}$ is a (not necessarily connected, but positive Lebesgue measure) polyhedral region of $\mathbb{T}^{N-1}$. Suppose further that it is asymmetric with respect to some $S \in S_{G,N}$. Of course in this case the symmetric image of $\mathcal{B}$ is also a polyhedral domain and disjoint from $\mathcal{B}$. Since the map is completely expanding (as $\varepsilon < 1/2$), we can use [58, Theorem 1.7] for these sets independently to obtain that on both of these sets an absolutely continuous invariant measure is supported. In conclusion, the existence of such an asymmetric invariant set implies that multiple acims exist. Or from another point of view, the acim of maximal support cannot be ergodic.

So by proving that an asymmetric invariant set exists we obtain ergodicity breaking. So we construct a set which breaks six symmetries and see that the symmetric images of $\mathcal{A}$ with respect to these six symmetries are pairwise disjoint. We then show that the set $\mathcal{A}$ is dynamically invariant if $\varepsilon \geq \frac{4-\sqrt{10}}{2}$. To construct this set, we

	$\mathbf{H_1}$	$\mathbf{H_2}$
u	$\varepsilon/3 < u < 1-\sigma$	$\sigma < u < 1-\varepsilon/3$
v	$\varepsilon/3 < v < 1-\sigma$	$\sigma < v < 1-\varepsilon/3$
$\mathbf{u+v}$	$\sigma < u+v < 1-\varepsilon/3$	$1+\varepsilon/3 < u+v < 2-\sigma$

TABLE 1: The hexagons containing the attractor. We used the notation u, v for the coordinates on $[0,1]^2$ and $\frac{2\varepsilon}{3}(2-\varepsilon) = \sigma$.

first notice that the attractor is contained in the union the two hexagons described in Table 1. What we can observe by tuning the value of ε, that for coupling strengths weaker than the critical value, the two hexagons contain a symmetric invariant set. But when the threshold is surpassed, this set is decomposed into six invariant sets along the symmetry axes pictured in blue on Figure 2. See $\mathcal{A}$ explicitely on Figure 2.

N=4

Our result regarding the system of four coupled maps is the following:

Theorem 2.6. *Let us consider the system $(\mathbb{T}^4, F_{\varepsilon,4})$. There exists an $\varepsilon^*(4) < 1/2$, such that for $\varepsilon^*(4) \leq \varepsilon < 1/2$ at least six absolutely continuous invariant measures exist.*

As seen in the section discussing the $N = 3$ case, it is enough to show that the system $(\mathbb{T}^3, G_{\varepsilon,4})$ has an asymmetric invariant set of positive Lebesgue measure that breaks at least six symmetries.

To construct this set, consider first

$$L_{\varepsilon/2}(s) = \begin{cases} 2(1-\varepsilon)s + \frac{\varepsilon}{2} & \text{if } \frac{\varepsilon}{2} < s < \frac{1}{2}, \\ 2(1-\varepsilon)s + \frac{3\varepsilon}{2} - 1 & \text{if } \frac{1}{2} < s < 1 - \frac{\varepsilon}{2}. \end{cases} \tag{2.6}$$

This is a centrally symmetric Lorenz map with a period two point

$$p^* = \frac{(1-\varepsilon)\varepsilon + 3\varepsilon/2 - 1}{1 - 4(1-\varepsilon)^2} = \frac{\varepsilon - 2}{4\varepsilon - 6}. \tag{2.7}$$

Denote the coordinates on $[0,1]^3$ by u, v and w. Define the polytopes P_1 and P_2 by the inequalities of Table 2 (the missing constraints can be deduced from the ones given).

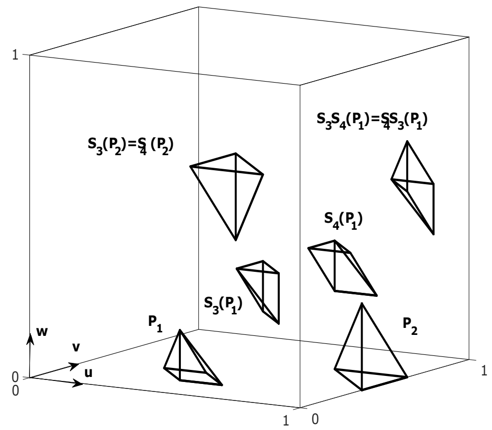
	$\mathbf{P_1}$	$\mathbf{P_2}$
u		$u < 1$
v	$v > \varepsilon/2$	
w	$w > 0$	$w > 0$
$\mathbf{u+v}$	$u+v > 1-p^*$	$u+v > 1+p^*$
$\mathbf{v+w}$	$v+w < p^*$	$v+w < 1-p^*$
$\mathbf{u+v+w}$	$u+v+w < 1-\varepsilon/2$	

TABLE 2

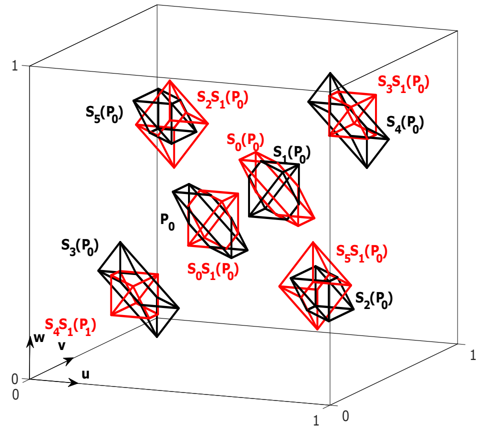
We use the following notation for the symmetries: S_0 is induced by I, S_1 is induced by π_{12}, S_2 is induced by π_{13}, S_3 is induced by π_{14}, S_4 is induced by π_{23}, S_5 is induced by π_{24} and S_6 is induced by π_{34}.

To prove our theorem, we show that if $\varepsilon^*(4)$ is the unique real solution of

$$p^* = (1-\varepsilon)^2, \quad \text{or more explicitly} \quad 4\varepsilon^3 - 14\varepsilon^2 + 15\varepsilon - 4 = 0,$$



(a) The asymmetric set $\mathcal{A}$.



(b) The symmetric set $\mathcal{S}$ for $1 - \frac{\sqrt{2}}{2} \leq \varepsilon$.

then the set

$$\mathcal{A} = P_1 \cup P_2 \cup S_3(P_1) \cup S_4(P_1) \cup S_3S_4(P_1) \cup S_3(P_2)$$

is symmetric with respect to S_3 and S_4, but asymmetric with respect to

$$S_0,\ S_1,\ S_2,\ S_5,\ \text{and}\ S_6.$$

The sets $\mathcal{A}$, $S_0(\mathcal{A})$, $S_1(\mathcal{A})$, $S_2(\mathcal{A})$, $S_5(\mathcal{A})$ and $S_6(\mathcal{A})$ are pairwise disjoint. Furthermore, $\mathcal{A}$ is invariant with respect to $G_{\varepsilon,4}$ if and only if

$$0.397 \approx \varepsilon^*(4) \leq \varepsilon.$$

As an addition to Theorem 2.6, we have one more result. We construct a symmetric set $\mathcal{S}$, such that by simulation of trajectories it is clear that this set contains the attractor when ε is sufficiently large but still smaller than $\varepsilon^*(4)$. When ε becomes larger than $\varepsilon^*(4)$ the asymmetric invariant sets arise outside of this set. This is in contrast to the case of $N = 3$, where the asymmetric invariant sets appeared as a decomposition of a symmetric invariant set.

Let P_0 be defined as follows:

	$\mathbf{P_0}$
u	$L_{\varepsilon/2}(\varepsilon/2) < u < L_{\varepsilon/2}(1-\varepsilon/2)$
v	$\varepsilon/2 < v < L^2_{\varepsilon/2}(1-\varepsilon/2)$
w	$L_{\varepsilon/2}(\varepsilon/2) < w < L_{\varepsilon/2}(1-\varepsilon/2)$
$\mathbf{u+v+w}$	$1+\varepsilon/2 < u+v+w < 1+L^2_{\varepsilon/2}(1-\varepsilon/2)$

We claim the following:

Proposition 2.7. *The set*

$$\begin{aligned}\mathcal{S} = & P_0 \cup S_0(P_0) \cup S_1(P_0) \cup S_2(P_0) \cup S_3(P_0) \cup S_4(P_0) \cup S_5(P_0) \\ & S_0S_1(P_0) \cup S_2S_1(P_0) \cup S_3S_1(P_0) \cup S_4S_1(P_0) \cup S_5S_1(P_0)\end{aligned}$$

is symmetric and it is invariant with respect to $G_{\varepsilon,4}$ if and only if

$$1 - \frac{\sqrt{2}}{2} \leq \varepsilon.$$

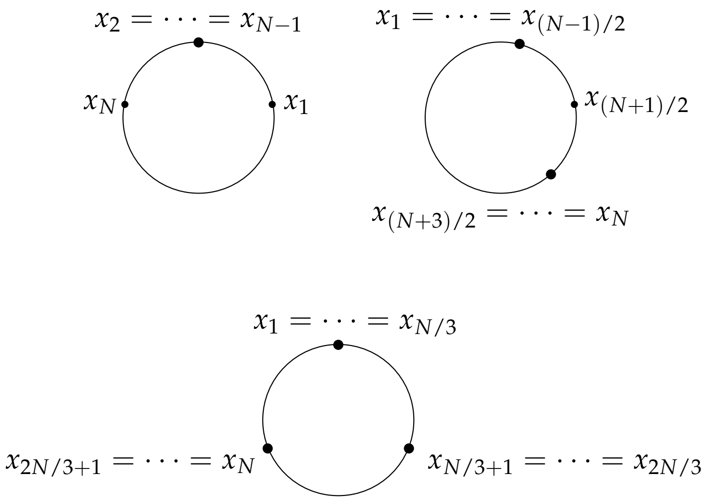


FIGURE 4: Types of clusters: one cluster, two clusters and clusters of even size.

2.2 Synchronization

In this section we introduce our results on the system $(\mathbb{T}^N, F_{\varepsilon,N})$ in the contracting regime, that is when $\varepsilon \in (1/2, 1)$. Now the second component of the factor system, $(\mathbb{T}^{N-1}, G_{\varepsilon,N})$, is a piecewise contracting similarity, hence the existence of an absolutely continuous invariant measure is not possible. Nevertheless, we can characterize some features of the attractor which can be interpreted as synchronization phenomena in terms of the N-particle system. Full characterization of the types of synchronization is beyond the scope of this work, we only list some interesting special cases for various values of N.

These results are unpublished, except for the special cases of $N = 2$ and $N = 3$.

Complete synchronization

Let

$$\Delta_N = \{x \in \mathbb{T}^N : x_1 = \cdots = x_N\}.$$

This corresponds to a state in the N particle system where all the particles form a single cluster. This is what we are going to call a *completely synchronized* state. We claim that such a state is asymptotically stable in the contracting regime.

Proposition 2.8. *Let $N > 1$ and consider the system $(\mathbb{T}^N, F_{\varepsilon,N})$ for $\varepsilon \in (1/2, 1)$. The diagonal Δ_N attracts the orbit of an open set. In the $N = 2$ case, Δ_2 attracts all orbits.*

Partial synchronization

Evenly placed formation. Let $N \geq 3$ be odd. Consider the set

$$\Sigma_N = \{(x_i)_{i=1}^N \in \mathbb{T}^N : x_i - x_j = \frac{k}{N}, \quad k \in \mathbb{Z} \setminus \ell N, \ell \in \mathbb{Z}, \quad \forall i \neq j = 1, \ldots, N\}.$$

Observe that the elements of Σ_N corresponds to a formation of evenly placed particles on the circle. We claim the following:

Proposition 2.9. *Let $N \geq 3$ odd and consider the system $(\mathbb{T}^N, F_{\varepsilon,N})$ for $\varepsilon \in (1/2, 1)$. The set Σ_N is invariant and attracts the orbit of an open set of initial conditions. In the $N = 3$ case, $\Delta_3 \cup \Sigma_3$ attracts almost all orbits.*

One cluster. Suppose that $N \geq 4$. Consider a formation where we have a cluster of $N-2$ particles and the other two particles are of equal distance on opposite sides of this cluster. We claim that such a formation is invariant and asymptotically stable in the sense as the completely synchronized and the evenly placed formation.

Proposition 2.10. *Let $\varepsilon \in (1/2, 1)$ and $N < \frac{4\varepsilon}{2\varepsilon-1}$. Consider a formation where x_i are equal for all $i \in \{1, \dots, N\} \backslash \{j, k\}$ and $g(x_j - x_i) = g(x_i - x_k) = d_1$ $(i \neq j, k)$. If*

$$d_1 = \pm \frac{2\varepsilon - N}{2\varepsilon N - 3N},$$

a formation like this is invariant and asymptotically stable.

Two clusters. Suppose that $N \geq 5$. Consider a formation where we have two clusters of equal size with equal distance from a single particle, see the right hand side of Figure 4. We claim that a formation like this is invariant and asymptotically stable.

Proposition 2.11. *Let $\varepsilon \in (1/2, 1)$ and $N < \frac{4\varepsilon}{2\varepsilon-1}$. Consider a formation with $x_1 = \dots = x_{(N-1)/2}$, $x_{(N+3)/2} = \dots = x_N$ and $g(x_{(N+1)/2} - x_1) = g(x_N - x_{(N+1)/2}) = d_2$. If*

$$d_2 = \pm \frac{2\varepsilon(N-1) - N}{2\varepsilon N - 3N},$$

a formation like this is invariant and asymptotically stable.

Clusters of same size. Let $N \geq 6$ and suppose that it is not a power of two. This means that N is a multiple of an odd number, say $N = kM$, M odd. Consider a formation of M evenly placed clusters of k particles, see the bottom of Figure 4 for the $M = 3$ case. We claim that a formation like this is invariant and asymptotically stable.

Proposition 2.12. *Let $\varepsilon \in (1/2, 1)$ and $N = kM$, $k \in \mathbb{N}$, M odd. Then an evenly placed formation of M clusters of size k is invariant and asymptotically stable.*

These phenomena resemble synchronization in the Kuramoto model [7, 22, 31]: full phase locking is observable if ω_i are independent of i. This means that the phases asymptotically become equal, analogously to our case of *complete synchronization*. For more general assumptions on the initial conditions of the Kuramoto oscillators, one can prove that limit

$$\lim_{t \to \infty} |\theta_i - \theta_j|$$

exists for each pair (i, j), provided that the coupling is strong enough. This is analogous to our case of *partial synchronization*.

3 An infinite system

In this section we discuss the generalization of our model with N particles to a continuum of interacting units.

Consider a Borel probability measure μ_0 defined on $\mathbb{T}$. This measure is going to represent the distribution of a continuum of particles on the circle. We define the dynamics

$$F_{\varepsilon,\mu_0} : \mathbb{T} \to \mathbb{T}, \quad F_{\varepsilon,\mu_0} = T \circ \Phi_{\varepsilon,\mu_0},$$

where T is an expanding circle map (to be specified later in this chapter) and Φ_{ε,μ_0} is the coupling function

$$\Phi_{\varepsilon,\mu_0} : \mathbb{T} \to \mathbb{T}, \quad \Phi_{\varepsilon,\mu_0}(x) = x + \varepsilon \int_{\mathbb{T}} g(y-x) \mathrm{d}\mu_0(y).$$

Here $g(\cdot)$ is the same distance function defined in Section 1. For the coupling strength we have the standing assumption $0 \leq \varepsilon < 1$ throughout this section.

From now on we assume that μ_0 is absolutely continuous with respect to the Lebesgue measure λ. Denote

$$\frac{\mathrm{d}\mu_0}{\mathrm{d}\lambda} = f_0 \in L^1(\mathbb{T}).$$

By pushing μ_0 forward with F_{ε,μ_0}, we obtain the distribution of the particles after one iteration of the coupled dynamics. Iterating this further, we get a sequence of measures

$$\mu_{n+1} = (F_{\varepsilon,\mu_n})_* \mu_n, \quad n = 0, 1, \ldots$$

describing the distribution of particles in each step. Our (monotonicity) assumptions on T to be introduced later are going to ensure that all the pushforward measures are absolutely continuous. Let us use the notation $F_{\varepsilon,f_0} = T \circ \Phi_{\varepsilon,f_0}$ where

$$\Phi_{\varepsilon,f_0}(x) = x + \varepsilon \int_{\mathbb{T}} g(y-x) f_0(y) \mathrm{d}y. \tag{3.1}$$

Instead of the sequence of measures we can equivalently consider the sequence of density functions by the rule

$$f_{n+1} = P_{F_{\varepsilon,f_n}} f_n, \quad n = 0, 1, \ldots \tag{3.2}$$

where $P_{F_{\varepsilon,f_n}}$ is the Perron-Frobenius (or transfer) operator associated to the dynamics F_{ε,f_n}. That is, for $h \in L^1(\mathbb{T})$, $P_{F_{\varepsilon,f}} h$ is defined as the unique function in $L^1(\mathbb{T})$ such that

$$\int_A P_{F_{\varepsilon,f}} h \, \mathrm{d}\lambda = \int_{F_{\varepsilon,f}^{-1}(A)} h \, \mathrm{d}\lambda$$

for all Borel sets A. However, a more explicit formula can be given if we assume that both h and the dynamics $F_{\varepsilon,f}$ are sufficiently regular (as they will be for us in what follows):

$$(P_{F_{\varepsilon,f}} h)(y) = \sum_{x \in F_{\varepsilon,f}^{-1}(y)} \frac{h(x)}{|F'_{\varepsilon,f}(x)|} \quad y \in \mathbb{T}.$$

Since the dynamics in each step depends on the current distribution of the particles, we see from (3.2) that the transfer operator is *self-consistent* in the sense that it depends on the same function f_n to which it is applied. Note that since $F_{\varepsilon,f} = T \circ \Phi_{\varepsilon,f}$ we have $P_{F_{\varepsilon,f}} = P_T P_{\Phi_{\varepsilon,f}}$. To introduce more convenient notation let

$$\mathcal{F}_\varepsilon(f) = P_{F_{\varepsilon,f}} f \qquad f \in L^1(\mathbb{T}).$$

We claim that the time n measure μ_n depends Lipschitz continuously on the initial measure μ_0 in the weak topology.

We recall that the weak topology is metrizable with respect to the *bounded Lipschitz distance* (see for example [11, Theorem 8.3.2]), which is defined as

$$d_{BL}(\mu,\nu) = \sup\left\{ \left| \int_{\mathbb{T}} \varphi \, \mathrm{d}\mu - \int_{\mathbb{T}} \varphi \, \mathrm{d}\nu \right|, \ \mathrm{Lip}(\varphi) \leq 1 \right\},$$

for μ,ν probability measures on $\mathbb{T}$, where $\mathrm{Lip}(\varphi)$ is the Lipschitz constant of φ. We remind the reader that we denoted the density of the time n measure by f_n.

Proposition 3.1. *Suppose $T \in C^1(\mathbb{T})$. Suppose further that $f_n, \tilde{f}_n \in C(\mathbb{T})$ and $|f_n|, |\tilde{f}_n| \leq M$ for all $n \in \mathbb{N}$. Then there exists a constant $K = K(T', M) > 0$*

$$d_{BL}(\mu_n, \tilde{\mu}_n) \leq K^n \cdot d_{BL}(\mu_0, \tilde{\mu}_0).$$

The continuity and uniform bound on the densities might seem restrictive, but it is going to hold in the specific settings we are going to work subsequently. We note that this is an unpublished result.

We note that in the continuum version of the Kuramoto model, one can also obtain Lipschitz continuity of the time t measure as a function of the initial one [19]. As one expects, the proof of this fact relies heavily on the Lipschitz continuity of the sine function in the coupling map. In our case $g(\cdot)$ is not continuous, but it turned out that this fact does not hinder us from proving Lipschitz continuity of the time n measure.

In the next sections we are going to introduce our results on the asymptotic behavior of the coupled map system. In Section 3.1 we regard the case of small ε in detail, while in Section 3.2 the case of $\varepsilon \gg 0$ is discussed. In each of these sections we first study the toy model where T is the doubling map. These results were published in [2]. We then discuss more general cases, when T is a more general uniformly expanding map. These results were published in [3].

3.1 Stability of the invariant distribution

In this section we are going to introduce our results regarding the case of weak coupling, that is when ε is close to zero. We are going to assume throughout this section that T is at least C^2 smooth on $\mathbb{T}$ and expanding (further assumptions will be introduced later). By standard results this implies that T has a unique invariant density $h = P_T h$ which is continuously differentiable [43] (for some more recent results see [4, Section 2.2]). Furthermore,

$$P_T^n f \underset{n\to\infty}{\rightarrow} h$$

exponentially for any $f \in L^1(\mathbb{T})$. This can be thought of as the uncoupled ($\varepsilon = 0$) case of our coupled system, since $T = F_{0,f}$. When we introduce coupling, the dynamics (hence the associated transfer operator) changes from step to step as the distribution of the particles changes. This makes matters more complicated, but we expect to see similar behavior to the uncoupled case, provided that ε is small enough.

In the first part of this section we are going to discuss the case when T is the doubling map. This case is the direct generalization of the model in Section 2, and enables us to introduce most of the interesting features in a simple setting. We then show that our statements in a slightly weaker form carry over to a wider class of expanding circle maps T.

Coupled doubling maps.

Consider the coupled map system

$$F_{\varepsilon,f} : \mathbb{T} \to \mathbb{T}, \quad F_{\varepsilon,f}(x) = 2\left(x + \varepsilon \int_0^1 g(y-x) f(y) dy\right) \mod 1, \tag{3.3}$$

where $f \in C^1(\mathbb{T})$ is a density function, which means that $\int_{\mathbb{T}} f \,\mathrm{d}\lambda = 1$ and $f \geq 0$.

We are going to denote by $\mathrm{var}(f)$ the total variation of f and let $\|f\|_{BV} = \mathrm{var}(f) + \int_{\mathbb{T}} |f| \,\mathrm{d}\lambda$ be the bounded variation norm.

It is easy to see that the uniform density is an invariant density of this coupled map system for any value of ε. We claim that for weak enough coupling, sufficiently smooth initial densities approach this invariant distribution with exponential speed.

Theorem 3.2. *Let $f \in C^1(\mathbb{T})$ be such that $\mathrm{var}(f) \leq \delta$ for some $\delta > 0$. Assume that $\varepsilon > 0$ is such that*

$$\varepsilon < \frac{1}{1+4\delta}.$$

Then there exists a constant $q = q(\varepsilon, \delta) < 1$ such that

$$\mathrm{var}(\mathcal{F}_\varepsilon^{n+1}(f)) \leq q \cdot \mathrm{var}(\mathcal{F}_\varepsilon^n(f)) \qquad n = 0, 1, \ldots.$$

implying that

$$\lim_{n\to\infty} \|\mathcal{F}_\varepsilon^n(f) - \mathbf{1}\|_{BV} = 0.$$

Some coupled expanding circle maps.

In this section we make the following assumptions on our densities f and our dynamics T:

- (F) $f \in C^1(\mathbb{T})$, f' is Lipschitz continuous, f is a density function,
- (T) $T \in C^2(\mathbb{T})$, T'' is Lipschitz continuous and T is strictly expanding: that is, $\min |T'| = \omega > 1$. We further suppose that T is a degree N covering map of $\mathbb{T}$ such that
$$N < \omega^2.$$

Consider the set

$$\mathcal{C}_{R,S}^c = \{f \text{ is of property (F)}, \mathrm{var}(f) \leq R, |f'| \leq S, \mathrm{Lip}(f') \leq c\},$$

where $R, S, c > 0$ and we denoted by $\mathrm{Lip}(f)$ the Lipschitz constant of f.

We are now state our results.

Theorem 3.3. *Suppose that T is of property (T). There exist R^*, S^* and $c^* > 0$ such that for all $R > R^*$, $S > S^*$ and $c > c^*$ there exists an $\varepsilon^* = \varepsilon^*(R, S, c) > 0$, for which the following holds: For all $0 \leq \varepsilon < \varepsilon^*$, there exists a density $f_*^\varepsilon \in \mathcal{C}_{R^*,S^*}^{c^*}$ for which $\mathcal{F}_\varepsilon(f_*^\varepsilon) = f_*^\varepsilon$. Furthermore,*

$$\lim_{n\to\infty} \mathcal{F}_\varepsilon^n(f) = f_*^\varepsilon \quad \textit{exponentially for all } f \in \mathcal{C}_{R,S}^c$$

in the sense that there exist $C = C(\varepsilon, R, S, c) > 0$ and $\gamma = \gamma(\varepsilon) \in (0,1)$ such that

$$\|\mathcal{F}_\varepsilon^n(f) - f_*^\varepsilon\|_{BV} \leq C\gamma^n \|f - f_*^\varepsilon\|_{BV} \quad \textit{for all } n \in \mathbb{N} \textit{ and } 0 \leq \varepsilon < \varepsilon^*.$$

Now as we can see, we cannot claim that the fixed density of $\mathcal{F}_\varepsilon$ is the same for all ε, as in the case of the doubling map. But we claim that it is Lipschitz continuous in the variable ε.

Theorem 3.4. *Suppose that T is of property (T). Let R, S, c and ε^* be chosen as in Theorem 3.3. Then there exists a $K(R, S, c) = K > 0$ such that for any $0 \le \varepsilon, \varepsilon' < \varepsilon^*$*

$$\|f_*^\varepsilon - f_*^{\varepsilon'}\|_{BV} \le K|\varepsilon - \varepsilon'|$$

holds for the densities $f_*^\varepsilon, f_*^{\varepsilon'} \in \mathcal{C}_{R,S}^c$ for which $\mathcal{F}_\varepsilon(f_*^\varepsilon) = f_*^\varepsilon$ and $\mathcal{F}_{\varepsilon'}(f_*^{\varepsilon'}) = f_*^{\varepsilon'}$.

3.2 Synchronization

In Section 2.2 we have observed various synchronization phenomena in the case of N interacting particles provided that the coupling is sufficiently strong. We expect to see similar behavior in this currently studied infinite model. In this section we discuss an analogue of the complete synchronization of the finite model.

We say that a density is *localized* if it satisfies the following property (F'):

(F') f is a density and there exists an interval $I \subset \mathbb{T}$, $\lambda(I) \ge \frac{1}{2}$ such that $\operatorname{supp}(f) \cap I = \emptyset$.

We now introduce a notation. Let $\operatorname{supp}^*(f)$ be the smallest closed interval on $\mathbb{T}$ containing the support of f. Note that this interval is well defined if f is localized.

This section is organized very much like Section 3.1. We first introduce our results in the case of coupled doubling maps then show how general our statements actually are by considering a wider class of circle maps.

Synchronization of coupled doubling maps.

Consider the coupled map system defined by the law (3.3), that is

$$F_{\varepsilon,f} : \mathbb{T} \to \mathbb{T}, \quad F_{\varepsilon,f}(x) = 2\left(x + \varepsilon \int_0^1 g(y - x) f(y)\, dy\right) \mod 1,$$

such that f is of property (F'). Now $\operatorname{supp}^*(f)$ can be represented either by an interval $[a_1, a_2] \subset [0, 1]$ where we have $a_2 - a_1 \le \frac{1}{2}$ or a union of intervals $[0, b_2] \cup [b_1, 1] \subset [0, 1]$ with $1 - (b_1 - b_2) \le \frac{1}{2}$.

If $\operatorname{supp}(f) \subseteq [a_1, a_2]$ we define the center of mass as

$$M(f) = \int_{a_1}^{a_2} y f(y)\, dy, \tag{3.4}$$

while if $\operatorname{supp}(f) \subseteq [0, b_2] \cup [b_1, 1]$, let

$$M(f) = \int_0^{b_2} (y + 1) f(y)\, dy + \int_{b_1}^1 y f(y)\, dy \qquad \mod 1. \tag{3.5}$$

We claim that for a localized density, $\mathcal{F}_\varepsilon(f)$ is just a linear rescaling of f by the factor $2(1 - \varepsilon)$ (this is a contraction if $\varepsilon > 1/2$) and a shift such that the center of mass moves according to the doubling map. This implies convergence to a point mass with support moving on the doubling map trajectory of the initial center of mass.

Theorem 3.5. *Let $\varepsilon \in (1/2, 1)$. Then if f is of property (F′), $\mathcal{F}_\varepsilon(f)$ is also of property (F′). Furthermore,*

- $|\text{supp}^*(\mathcal{F}_\varepsilon(f))| = 2(1-\varepsilon)|\text{supp}^*(f)|$,
- $\sup \mathcal{F}_\varepsilon(f) = \frac{\sup f}{2(1-\varepsilon)}$,
- $M(\mathcal{F}_\varepsilon(f)) = 2M(f) \mod 1$.

This implies that by using the notation $\frac{d\mu_n}{d\lambda} = \mathcal{F}_\varepsilon^n(f)$ and T for the doubling map that

$$d_{BL}(\mu_n, \delta_{T^n(M(f))}) \underset{n\to\infty}{\to} 0 \quad \textit{exponentially}$$

in the sense that

$$d_{BL}(\mu_n, \delta_{T^n(M(f))}) \leq [2(1-\varepsilon)]^n \quad \textit{for all} \quad n \in \mathbb{N}.$$

Note that in the regime $\varepsilon > 1/2$, for sufficiently even initial distributions Theorem 3.2 applies, that is, asymptotically the sites are uniformly distributed on $\mathbb{T}$. On the other hand, for sufficiently concentrated initial distributions, Theorem 3.5 applies, that is, the sites are fully synchronized asymptotically. This is analogous to the phenomena observed in the contracting regime for the case of finitely many odd particles: sufficiently concentrated initial distributions will suffer complete synchronization (Proposition 2.8) while sufficiently spread out initial distributions will be asymptotically evenly distributed (Proposition 2.9). This phenomenon is remarkable as it has no analogue in the Kuramoto model.

Synchronization of continuous circle maps.

We would like to generalize the previous results on the doubling map to a more general class of circle maps and detect similar type of synchronization. Let us assume the following:

(T′) $T \in C^1(\mathbb{T})$, $|T'| > \omega > 0$.

Denote $\Omega = \max |T'|$. Notice that we do not assume expansion, as it has no role in the emergence of complete synchronization. In fact as we shall see, that larger Ω means slower convergence to a synchronized state.

Theorem 3.6. *Let f be of property (F') and T of property (T'). Suppose $1 - \frac{1}{\Omega} < \varepsilon < 1$. Then*

$$|\text{supp}^*(\mathcal{F}_\varepsilon^n(f))| \underset{n\to\infty}{\to} 0 \quad \textit{exponentially}$$

in the sense that

$$|\text{supp}^*(\mathcal{F}_\varepsilon^n(f))| \leq [\Omega(1-\varepsilon)]^n |\text{supp}^*(f)| \quad \textit{for all} \quad n \in \mathbb{N}.$$

Furthermore, by using the notation $\frac{d\mu_n}{d\lambda} = \mathcal{F}_\varepsilon^n(f)$, we claim that there exists an $x^* \in \text{supp}^*(f)$ such that

$$d_{BL}(\mu_n, \delta_{T^n(x^*)}) \underset{n\to\infty}{\to} 0 \quad \textit{exponentially}$$

in the sense that

$$d_{BL}(\mu_n, \delta_{T^n(x^*)}) \leq [\Omega(1-\varepsilon)]^n \quad \textit{for all} \quad n \in \mathbb{N}.$$

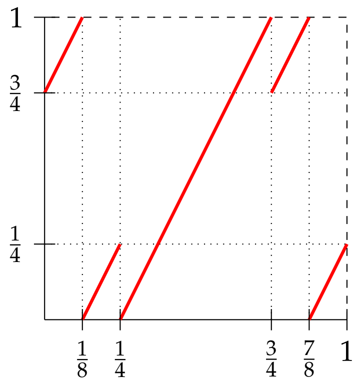


FIGURE 5: The discontinuous map T.

Note that in the regime $\varepsilon > 1 - 1/\max|T'|$, some distributions will converge to an invariant absolutely continuous distribution according to Theorem 3.3, while other, localized distributions will converge to a point mass according to Theorem 3.6.

4 Summary and outlook

In this thesis we studied various asymptotic properties of systems of globally coupled identical circle maps.

In the first main part of the thesis we studied the case of finitely many coupled doubling maps, modeling a system of N interacting particles on the circle. We showed in Section 2.1, that in the case of $N = 3$ and $N = 4$ that the system has an invariant set which breaks the inversion symmetry of the dynamics, provided that the coupling is strong enough. This implied that the ergodicity of the system, well known for $\varepsilon \approx 0$, does not prevail in the whole expanding regime. In Section 2.2 we showed that in the contracting regime, that is when the coupling is relatively strong, certain synchronization phenomena can be observed, reminiscent of the synchronization of the coupled oscillators of the Kuramoto-model.

In the second main part of this thesis we defined an infinite version of our coupled map system. Here the distribution of the particles was represented by a density function, and the evolution of this density was studied with respect to a self-consistent Perron-Frobenius operator. We showed in Section 3.1 that if the individual dynamics is *smooth enough* a unique invariant density exists (which is a Lipschitz continuous function of the coupling strength) and the convergence to this is exponential from any initial element of an appropriate set of densities. In Section 3.2 we showed that for sufficiently strong coupling, localized distributions converge to a point mass which can be interpreted as complete synchronization of the coupled map system.

In the remaining part of this section we are going to provide outlook on some further directions of research. The results to be stated are unpublished.

4.1 Discontinuous dynamics

In this section we are going to introduce an example of a coupled map system with discontinuous individual dynamics T for which the invariant density f_*^ε does not depend continuously on ε, in the sense that $\|f_*^\varepsilon - f_*^0\|_{BV}$ does not converge to zero as $\varepsilon \to 0$. This shows the limits of Theorem 3.4.

Let $T : [0,1] \to [0,1]$ be defined in the following way:

$$T(x) = \begin{cases} 2x + \frac{3}{4} & \text{if } \ 0 \le x < \frac{1}{8}, \\ 2x - \frac{1}{4} & \text{if } \ \frac{1}{8} \le x \le \frac{1}{4}, \\ 2x - \frac{1}{2} & \text{if } \ \frac{1}{4} < x < \frac{3}{4}, \\ 2x - \frac{3}{4} & \text{if } \ \frac{3}{4} \le x < \frac{7}{8}, \\ 2x - \frac{7}{4} & \text{if } \ \frac{7}{8} \le x \le 1. \end{cases}$$

See Figure 5. Let $I = [0,1/4] \cup [3/4,1]$. Throughout this section we restrict to densities of the following type:

(F") f is a density function such that $\text{supp}(f) \subseteq I$ and $f(x) = f(1-x)$ for all $x \in [0,1]$.

Proposition 4.1. *For all $0 \le \varepsilon < \frac{1}{2}$, the operator $\mathcal{F}_\varepsilon$ has a unique fixed element f_*^ε. Furthermore,*

$$\|f_*^\varepsilon - f_*^0\|_{BV} \ge 1 - \frac{1}{4(1-\varepsilon)-1}$$

holds for all $0 < \varepsilon < \frac{1}{2}$.

4.2 A related self-consistent dynamical system

Let X be a compact metric space and $\mathcal{M}(X)$ be the set of Borel probability measures on X. By a *self-consistent dynamical system* we mean the system defined by a law

$$\mathcal{T} : X \times \mathcal{M}(X) \to X \times \mathcal{M}(X), \quad \mathcal{T}(x,\mu) = (T_\mu(x), (T_\mu)_*\mu),$$

as defined in [8]. This abstract concept is a natural generalization of coupled map systems considered in this thesis: take $T_\mu = T \circ \Phi_{\varepsilon,\mu}$ such that T is a circle map and $\Phi_{\varepsilon,\mu}$ is defined by (3.1). Then the second coordinate of $\mathcal{T}$ describes the evolution of measures we studied in Section 3. By our calculations in Section 3 we can see that the existence of invariant measures, their unicity and stability basically only depends on the regularity (and the expansion rate) of the maps $(\mu, x) \mapsto T_\mu(x)$. So by studying more simple examples of systems where the dynamics changes from step to step subject to some current statistics, we can get a likely picture about the interesting features of coupled map systems as well. In this section present our results of a self-consistent system where the dynamics is *discontinuous* in each step and show that it has multiple absolutely continuous invariant measures, implying that this is likely to hold in case of coupled map systems where the individual dynamics T is discontinuous.

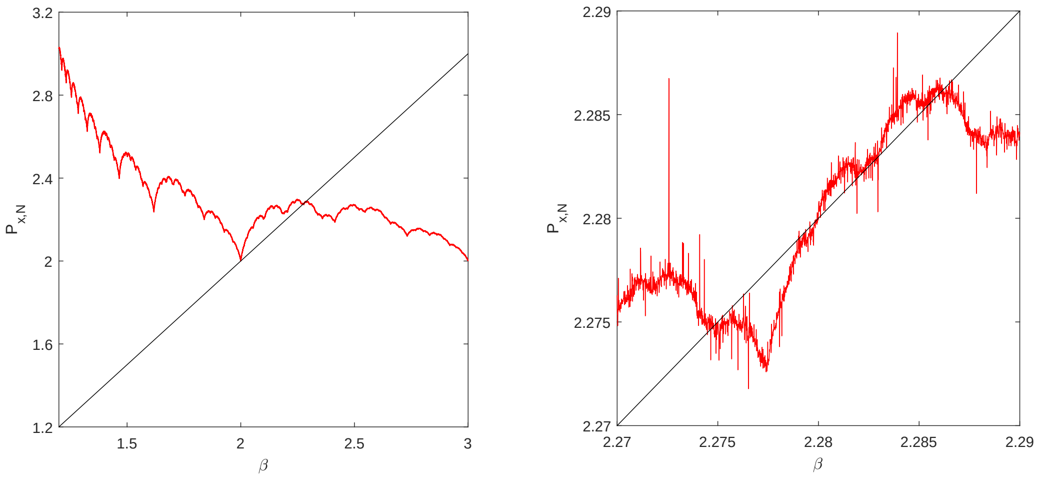
Consider

$$T_\mu(x) = \frac{1}{E_\mu} \cdot x \mod 1. \tag{4.1}$$

We are claim the following:

Proposition 4.2. *The self-consistent system (4.1) has at least four Lebesgue-absolutely continuous invariant measures. One of them is Lebesgue, and the other three are equivalent to Lebesgue.*

The proof builds on two facts: denoting $\beta_\mu = \frac{1}{E_\mu}$, Rényi [55] proved that the map $T_\beta(x) = \beta x \mod 1$, $\beta > 1$ has a unique invariant measure equivalent to the Lebesgue measure.



(a) $\beta \in [1.2, 3]$ with gridsize 10^{-4}, $N = 10^7$.

(b) $\beta \in [2.27, 2.29]$ with gridsize 10^{-5}, $N = 10^9$.

FIGURE 6: Approximation of ψ with ergodic averages. $P_{x,N}(\beta) = \frac{N}{\sum_{n=0}^{N-1} T_\beta^n(x)}$ is plotted for various values of β where x is uniform random on $[0, 1]$.

Consider the map

$$\psi(\beta) = \beta_{\mu_\beta}. \tag{4.2}$$

We can see that $\beta = \beta_{\mu_\beta}$ implies that μ_β is an invariant measure of the self-consistent system. By the result of Rényi, μ_β is equivalent to the Lebesgue measure. This means that the fixed points of ψ are in one-to-one correspondence with absolutely continuous invariant measures of the self-consistent system.

By [40, Proposition 2], the function ψ is continuous (in fact log-Lipschitz). So it is enough to find pairs of values of β for which $\psi(\beta)$ is above and under the line $x = y$. We do this by finding special values of β corresponding to Markovian maps T_β for which the invariant density is easily computable. Since $\psi(2) = \psi(3) = 2$, it is enough to find three appropriate values of β (that is, for which $\psi(\beta_1) > \beta_1$, $\psi(\beta_2) < \beta_2$ and $\psi(\beta_3) > \beta_3$) to complete the proof.

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