Error analysis of operator splitting methods with application to reaction-diffusion problems

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1 Introduction

The dissertation consists of four large parts, which discuss the method of operator splitting applied to abstract nonlinear problems. The investigations are mostly motivated by reaction-diffusion and advection-diffusion problems. The results in the dissertation are as follows.

2 New results

2.1 Convergence of operator splitting for nonlinear problems

The convergence of operator splitting have been investigated in many works of the literature. One approach is based on the concepts of Lie-algebras and Lie-operators, which requires the suboperators to be infinitely many times differentiable. There are studies assuming that the suboperators are only several times differentiable, although in every case the global error is expressed in terms that are rather complicated to evaluate in practice and uses the solution itself.

In [2] I prove that for nonlinear problems with locally Lipschitz-continuous suboperators the sequential and the additive splittings are convergent of first order. These statements are an extension of the existing results for problems with linear or arbitrarily smooth operators, in the case of sequential splitting. In the case of additive splitting only the local order for bounded linear suboperators have been investigated so far.

Definition 1. Let $(X, \|\cdot\|)$ be a Banach space, $D(f), D(g) \subset X$, suppose $D := D(f) \cap D(g)$ is an open connected set. Let $f : D(f) \longrightarrow X$ and $g : D(g) \longrightarrow X$ be continuous operators. Consider the initial value problem

$$u'(t) = (f+g)(u(t)), \quad u(t_0) = u_0, \tag{1}$$

with $u_0 \in D$, $t_0 \in I$, $I \subset \mathbb{R}$ being an open interval.

The study [2] is built on the following two assumptions.

1. Suppose that (1) with $t_0 = 0$ has a unique solution $[0,T] \ni t \mapsto u(t)$, with some $T > 0$.
2. Consider the following neighborhood of the solution

$$\mathbb{S}^r := \bigcup_{t\in[0,T]} \mathbb{S}_r(t), \quad \text{where} \quad \mathbb{S}_r(t) := \{\eta : \|\eta - u(t)\| \leqslant r\}, \quad t \in [0,T],$$

where r is a given positive real number. Suppose that there exists L such that for any $t \in [0,T]$ and any $w, z \in \mathbb{S}_r(t)$ the relations $\|f(w) - f(z)\| \leqslant L\|w - z\|$ and $\|g(w) - g(z)\| \leqslant L\|w - z\|$ hold.

In practice the constant L depends on r, here L is chosen to the fixed value of r. L also depends on f and g, although for the following results no real advantage is taken of this fact, L can be considered as $\max\{L_f, L_g\}$.

Definition 2. Consider $t^\star \in (0,T]$ and the function

$$G_{\text{seq}}(t) := e^{Lt}\left(e^{3Lt}\left(\frac{Lt^2}{2}\frac{e^{2Lt^\star}-1}{e^{2Lt}-1}\|g(u_0)\| + \frac{Lt}{4}\left(t^\star e^{2Lt^\star} - t\frac{e^{2Lt^\star}-1}{e^{2Lt}-1}\right)\|(f+g)(u_0)\|\right) + \right.$$

$$\left. + e^{Lt}\frac{Lt^2}{2}\left(\frac{e^{2Lt^\star}-1}{2}\|(f+g)(u_0)\| + \|g(u_0)\|\right) + t\left(\frac{e^{2Lt^\star}-1}{2}\|(f+g)(u_0)\| + \|f(u_0)\| + \|g(u_0)\|\right)\right)$$

for $t > 0$. As it can be shown to be strictly monotone increasing one can introduce $\tau^\star := G_{\text{seq}}^{-1}(r)$, obviously $G_{\text{seq}}(t) \leqslant r$ for every $t \in [0, \tau^\star]$.

Theorem 1. *The sequential splitting is convergent at every $t^\star \in [0,T]$ of order 1. Furthermore for any $0 < \tau \leqslant \tau^\star$ such that $t^\star/\tau =: n \in \mathbb{N}$ the global error is*

$$E_{seq}(i\tau) \leqslant L\tau\frac{e^{2L\tau}}{4}\left(2\tau\frac{e^{2Li\tau}-1}{e^{2L\tau}-1}\|g(u_0)\| + \left(i\tau e^{2Li\tau} - \tau\frac{e^{2Li\tau}-1}{e^{2L\tau}-1}\right)\|(f+g)(u_0)\|\right),$$

for every $i \in \{0,1,...,n\}$.

Definition 3. Consider $t^\star \in (0,T]$ and the strictly monotone increasing function

$$G_{\text{as}}(t) := t\frac{e^{Lt}}{2}\left(e^{3Lt}Lt\frac{e^{2Lt^\star}-1}{e^{2Lt}-1} + e^{2Lt^\star} + 1\right)(\|f(u_0)\| + \|g(u_0)\|)$$

for $t > 0$. Let us define $\tau^\star := G_{\text{as}}^{-1}(r)$.

Theorem 2. *The additive splitting is convergent at every $t^\star \in [0,T]$ of order 1. Furthermore for any $0 < \tau \leqslant \tau^\star$ such that $t^\star/\tau =: n \in \mathbb{N}$ the global error is*

$$E_{as}(i\tau) \leqslant e^{3L\tau}\frac{L\tau^2}{2}\frac{e^{2Li\tau}-1}{e^{2L\tau}-1}(\|f(u_0)\| + \|g(u_0)\|),$$

for every $i \in \{0,1,...,n\}$.

The global errors are expressed as the function of $L, \|g(u_0)\|, \|f(u_0)\|, \|(f+g)(u_0)\|$ and the step size τ. These results allows a posteriori estimates of the error in practice, thus a suitable step size can be chosen.

2.2 Local order of combined methods

In practice the subproblems are solved numerically, thus it is crucial to investigate the properties of *combined methods* obtained by applying splitting and some numerical method.

Definition 4. Consider the initial value problem

$$u'(t) = Au(t) + R(u(t)), \quad u(0) = u_0, \qquad t \in [0,T], T \in \mathbb{R}^+ \tag{2}$$

where $u_0 \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$ is a bounded linear operator and $R : \mathbb{R}^d \to \mathbb{R}^d$ is an at least twice differentiable nonlinear mapping.

An important class of problems described by (2) can be obtained after spatial discretization of diffusion-reaction and advection-reaction equations. The linear operator A is the spatially discretized analogue of the operator representing diffusion or advection and R is the analogue of chemical reactions, in many practical cases a polynomial.

In [1] I prove that for problem (2) the combined method—obtained by applying classical splittings (sequential, Marchuk–Strang, symmetrically weighted) and with four different explicit numerical methods of order $1,2,3,4$ applied to solve the subproblems—is of order 1 or 2, the minimum of the order of splitting and the order of the applied numerical scheme. This is the extension of the case with linear suboperators.

Definition 5. Consider a division $\{0,\tau,2\tau,...,n\tau=T\}$ of $[0,T]$ and let $u^{(n)}_{\text{comb}}$ be an approximation of the solution u of (2) generated by a combined method. The *local error* of the combined method is $\|u(\tau)-u^{(n)}_{\text{comb}}(\tau)\|$, where u is the solution of (2). The *local order* of the combined method is the largest $q\in\mathbb{N}$ for which there exist positive constants c and $\tau^\star\in(0,T]$ such that for every $\tau\in(0,\tau^\star]$

$$\left\|\frac{u(\tau)-u^{(n)}_{\text{comb}}(\tau)}{\tau}\right\|\leqslant c\tau^q$$

holds.

Theorem 3. *The local order of the method obtained as the combination of sequential splitting and the first order explicit Euler method is* 1.

Theorem 4. *The local order of the method obtained as the combination of symmetrically weighted splitting and the first order explicit Euler method is* 1.

Theorem 5. *The local order of the method obtained as the combination of sequential splitting and the second order improved Euler method is* 1.

The proofs of the above theorems can be adapted to determine the order of combined methods with higher order numerical schemes, although the calculations become very complicated as the order increases. A code was written in *Mathematica* to perform the symbolic calculations in the proofs. I have determined the orders for combinations of the classical splitting methods with four different numerical schemes: the explicit Euler, the improved Euler method which is of second order, the third order Heun and the fourth order Runge-Kutta method. Table 1 contains the orders of combined methods. The order of the splitting and numerical methods are in the parenthesis. The order of the combined

p_{c}	exp. Euler (1)	impr. Euler (2)	Heun (3)	Runge-Kutta (4)
seq (1)	1	1	1	1
M-S (2)	1	2	2	2
sw (2)	1	2	2	2

Table 1: Local orders of combined methods for (2)

method is $p_{\text{c}}=\min\{p_{\text{spl}},p_{\text{num}}\}$, if p_{spl} is the order of splitting and p_{num} denotes the order of the numerical method.

Example 1. Numerical results on the Fisher-equation

$$\begin{cases} \partial_t u(t,x) &= \partial_x^2 u(t,x)+u(t,x)(1-u(t,x)) \qquad x\in[0,4\pi],\ t\geqslant 0,\\ u(0,x) &= 1+0.9\sin x\\ u(t,0) &= 1\\ u(t,4\pi) &= 1 \end{cases} \tag{3}$$

are presented in Table 2.

p_c	exp. Euler (1)	improved Euler (2)	Heun (3)	Runge-Kutta (4)
seq (1)	0.99	0.98	0.98	0.99
M–S (2)	0.99	1.99	1.90	1.98
sw (2)	0.99	1.99	1.98	1.97

Table 2: Local order estimates for the Fisher-equation.

2.3 Local order of iterative splitting

In [4] we prove that in the case when the operator is a sum of an arbitrary number of suboperators the local order of iterative splitting equals to the number of subproblems solved in the process for the *two-level* method. This is the extension of the results for two suboperators. We also define a large class of *multi-level* methods and present a characterization of these methods concerning the local order.

Definition 6. Let $k \in \mathbb{N}$ and consider the initial value problem

$$u'(t) = \sum_{j=1}^{k} A_j u(t), \quad u(0) = u_0, \qquad t \in [0,T], T \in \mathbb{R}^+, \tag{4}$$

where $D(A_j) = X$ and A_j is a bounded linear operator for all $j = 1,...,k$.

Consider the division $\{0, \tau, 2\tau, ..., N\tau = T\}$ of $[0,T]$ for some $N \in \mathbb{N}$ and let $\mathcal{N}_0 := \{0,1,...,N-1\}$.

Definition 7. The *two-level iterative splitting* consists of solving the subproblems

$$\begin{cases} v'_{i+j}(t) & = A_j v_{i+l}(t) + (A - A_j) v_{i+j-1}(t), \quad t \in [n\tau, (n+1)\tau], \\ v_{i+j}(n\tau) & = u_{\mathrm{it},2}(n\tau), \end{cases} \qquad j = 1,...,k$$

consecutively for $i = 0,k,2k,...,(m-1)k$ and $n \in \mathcal{N}_0$, with $u_{\mathrm{it},2}(0) := u_0$ and $v_0(t) := u_{\mathrm{it},2}(n\tau)$ for all $t \in [n\tau,(n+1)\tau]$. The approximation for the solution of (4) is $u_{\mathrm{it},2}((n+1)\tau) = v_{mk}((n+1)\tau)$.

Theorem 6. *Consider the initial value problem* (4) *and let* $A_j \in \mathscr{L}(X)$, $j = 1,...,k$ *be bounded linear operators. The local order (Definition 5) of the two-level method is mk.*

Definition 8. The *k-level iterative splitting* can be defined as solving the following subproblems consecutively for $i = 0,k,2k,...,(m-1)k$ and $n \in \mathcal{N}_0$:

$$v'_{i+1}(t) = A_1 v_{i+1}(t) + \sum_{j=2}^{k} A_j v_{i+j-k}(t), \quad v_{i+1}(n\tau) = u_{\mathrm{it,k}}(n\tau), \quad t \in [n\tau,(n+1)\tau], \tag{5a}$$

$$\vdots$$

$$v'_{i+l}(t) = \sum_{j=1}^{l} A_j v_{i+j}(t) + \sum_{j=l+1}^{k} A_j v_{i+j-k}(t), \quad v_{i+l}(n\tau) = u_{\mathrm{it,k}}(n\tau), \quad t \in [n\tau,(n+1)\tau], \tag{5b}$$

$$\vdots$$

$$v'_{i+k}(t) = \sum_{j=1}^{k} A_j v_{i+j}(t), \quad v_{i+k}(n\tau) = u_{\mathrm{it,k}}(n\tau), \quad t \in [n\tau,(n+1)\tau], \tag{5c}$$

with $u_{\mathrm{it,k}}(0) := u_0$ and $v_{2-k}(t) = v_{3-k}(t) = \ldots = v_0(t) = u_{\mathrm{it,k}}(n\tau)$ for all $t \in [n\tau, (n+1)\tau]$. The approximation for the solution of (4) is $u_{\mathrm{it,k}}((n+1)\tau) = v_{mk}((n+1)\tau)$.

Theorem 7. *The local order of the k-level iterative splitting* (5) *is* $\left\lceil \dfrac{mk}{k-1} \right\rceil$, *where* $\lceil\ \rceil$ *denotes the upper integer part.*

We can define a family of iterative splittings—which also includes the ones introduced above—in the following way.

Definition 9. Let us introduce the set $\mathscr{I} := \{1,2,,\ldots,k-1\}$ and the numbers $n_j^l \in \mathscr{I}$, where $j,l = 1,2,\ldots,k$. Consider the following iterative splitting procedure, where $i = 0,k,2k,\ldots,(m-1)k$:

$$v'_{i+l}(t) = A_l v_{i+l}(t) + \sum_{\substack{j=1 \\ j\neq l}}^{k} A_j v_{i+l-n_j^l}(t), \quad v_{i+l}(n\tau) = u_{\mathrm{it,k}}(n\tau), \quad l = 1,\ldots,k \tag{6}$$

with $v_{2-k}(t) = \ldots = v_0(t) = u_{\mathrm{it}}(n\tau)$ and $n \in \mathscr{N}_0$.

This definition of the indices allows us to couple a suboperator with an iteration function of $k-1$ earlier levels in many different ways, thus a large class of iterative splitting methods can be described.

Theorem 8. *The local order of the method* (6) *is p, where* $\left\lceil \dfrac{mk}{k-1} \right\rceil \leqslant p \leqslant mk$ *holds.*

Example 2. Consider the system of ordinary differential equations with constant coefficients:

$$u'(t) = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} u(t), \qquad u(0) = u_0 \tag{7}$$

with $u(t) = (u_1(t), u_2(t)), t \in \mathbb{R}^+$. Let us decompose the coefficient matrix into the sum of three matrices $A = A_1 + A_2 + A_3$ the following way,

$$A = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

With this choice the exponential of the suboperators can be calculated symbolically,

$$e^{tA_1} = \begin{pmatrix} e^t & 0 \\ 0 & e^{3t} \end{pmatrix}, \quad e^{tA_2} = \begin{pmatrix} 1 & -2t \\ 0 & 1 \end{pmatrix}, \quad e^{tA_3} = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}.$$

Equation (7) was solved with the two-level and the k-level iterative splittings symbolically with *Mathematica*. The leading term of the errors are presented in Table 3 with the number of iterations m and the order of the method p. Based on Definition 5 for small t we have $\|u(t) - u_{\mathrm{it}}(t)\| \approx ct^{p+1}$ where the order is p. The results are in accordance with the theoretical results: $p = mk$ for the two-level and $p = \left\lceil \dfrac{mk}{k-1} \right\rceil$ for the k-level method.

Example 3. Consider the three-dimensional diffusion equation for $u : \mathbb{R}_0^+ \times \Omega \to \mathbb{R}$, where $\Omega = [-\pi,\pi] \times [-\pi,\pi] \times [-\pi,\pi]$ which is a cube in $\mathbb{R}^3$.

$$\begin{aligned} \partial_t u(t,\mathbf{x}) &= \Delta u(t,\mathbf{x}) \\ u(0,\mathbf{x}) &= \sin x \sin y \sin z \\ u(t,\mathbf{x}_b) &= 0, \qquad \forall \mathbf{x}_b \in \partial\Omega, \end{aligned} \tag{8}$$

Table 3: The leading term of the error for the two-level and the k-level methods, $k = 3$.

m	$u(t) - u_{\mathrm{it},2}(t)$	$p = 3m$	$u(t) - u_{\mathrm{it,k}}(t)$	$p = \left\lceil \frac{3m}{2} \right\rceil$
1	$\begin{pmatrix} \frac{1}{2} & -\frac{5}{4} \\ -\frac{5}{8} & \frac{3}{2} \end{pmatrix} t^4 u_0$	3	$\begin{pmatrix} \frac{2}{3} & -\frac{5}{3} \\ -\frac{1}{2} & 1 \end{pmatrix} t^3 u_0$	2
2	$\begin{pmatrix} \frac{9}{280} & -\frac{11}{140} \\ -\frac{11}{280} & \frac{27}{280} \end{pmatrix} t^7 u_0$	6	$\begin{pmatrix} \frac{1}{12} & -\frac{1}{6} \\ 0 & 0 \end{pmatrix} t^4 u_0$	3
3	$\begin{pmatrix} \frac{1}{1680} & -\frac{7}{480} \\ -\frac{7}{9600} & \frac{1}{560} \end{pmatrix} t^{10} u_0$	9	$\begin{pmatrix} \frac{1}{60} & -\frac{7}{180} \\ -\frac{1}{120} & \frac{1}{60} \end{pmatrix} t^6 u_0$	5
4	$O(t^{13})$	12	$\begin{pmatrix} \frac{1}{1260} & -\frac{1}{630} \\ 0 & 0 \end{pmatrix} t^7 u_0$	6
5	$O(t^{16})$	15	$\begin{pmatrix} \frac{1}{11340} & -\frac{1}{5040} \\ -\frac{1}{30240} & -\frac{1}{15120} \end{pmatrix} t^9 u_0$	8

Table 4: The estimates of the local order of combined methods with the two-level and the 3-level method applied for (8).

two-level method			3-level method	
p_{num}	$m=1$	$m=2$	$m=1$	$m=2$
1	0.921	0.921	0.921	0.921
2	1.954	1.954	1.954	1.954
3	2.973	2.973	2.922	2.94
4	3.981	3.96	3.895	3.933

where $\mathbf{x} = (x,y,z) \in \mathbb{R}^3$ and $\Delta u = \partial_x^2 u + \partial_y^2 u + \partial_z^2 u$. The iterative splitting procedures are based on the decomposition

$$A_1 u := \Delta_x u, \qquad A_2 u := \Delta_y u, \qquad A_3 u := \Delta_z u.$$

Local order estimates can be found in Table 4. The subproblems were solved numerical with four different methods of order $p_{\text{num}} = 1,2,3,4$. The last row – except the one in the second column – and the third term in the third column (it should be ≈ 2) indicate higher order of accuracy than it can be expected based on the theoretical results. They should be $3,4,2,3$. The explanation for this is that the subequations were solved in one step, in other words the time step of the numerical schemes and the iterative splittings were the same. One can expect to get back the numbers predicted by the theory if the numerical step size is taken much smaller than the splitting time step. Such interference can be observed when a combination of different methods is applied to solve a problem.

2.4 Waveform relaxation method for semi-linear problems

In [3] give an explicit error estimate for the waveform relaxation method applied directly on the abstract, continuous semilinear problem. This error estimate is better then the ones in the literature, that also contain the unknown solution itself, thus do not provide a quantitative estimation of the error. I extend the results for the case, when the iteration is applied on time windows, which has not been quantitatively studied before. Furthermore I present the error estimate for the combined method, that is, when the iterative subproblems are solved numerically.

Definition 10. Let $(X, \|\cdot\|)$ be a Banach space, $D(A), D(F) \subset X$, suppose that $\Omega := D(A) \cap D(F)$ is an open connected set. Consider the initial value problem

$$u'(t) = Au(t) + F(u(t)), \quad u(0) = u_0, \qquad t \in [0,T],\ T \in \mathbb{R}^+, \tag{9}$$

where $u_0 \in \Omega$. Let $A : D(A) \longrightarrow X$ be linear and $F : D(F) \longrightarrow X$ a nonlinear operator.

The study [3] is based on the following two **assumptions**. Suppose that

1. The operator A generates a strongly continuous semigroup $(S(t))_{t \geqslant 0}$, with $\|S(t)x\| \leqslant Me^{\omega t}\|x\|$, for every $x \in X$ and $t \in [0,T]$, where M and ω are nonnegative constants.
2. There is a closed ball $B_\delta(u_0)$ with $\delta \in \mathbb{R}^+$ and there is a constant L such that $\|F(v) - F(w)\| \leqslant L\|v - w\|$ for every $v, w \in B_\delta(u_0)$.

Definition 11. Let the waveform relaxation method for (9) be defined as

$$v_i'(t) = Av_i(t) + F(v_{i-1}(t)), \quad v_i(0) = u_0, \quad t \in [0,T], \tag{10}$$

where $i \in I := \{1,2,...,m\}$, with some $m \in \mathbb{N}$ (the number of iterations) and the starting iteration function $v_0(t) = u_0$ for every $t \in [0,T]$.

Definition 12. Suppose that the solution of problem (9) is approximated by the sequence of functions obtained as the solutions of (10). Then the iteration error is

$$e_i(t) := \|u(t) - v_i(t)\|.$$

For notational convenience let us define

$$\rho(t) := \alpha_1 M \frac{e^{(\omega+ML)t} - 1}{\omega + ML}, \quad \text{with} \quad \alpha_1 := \|(A+F)(u_0)\|,$$

furthermore let $t_\delta := \rho^{-1}(\delta)$.

Theorem 9. *Consider the assumptions, then for the unique solution u of* (9)

$$\|u(t) - u_0\| \leqslant \rho(t)$$

for any $t \in [0, t_\delta]$.

Theorem 9 provides a strong base to reformulate the iteration error.

Theorem 10. *Under the assumptions of Theorem 9 and* $\omega > 0$*, with the same notations for every* $t \in [0,T]$*, with* $T \leqslant t_\delta$ *the iteration error is*

$$e_i(t) \leqslant \frac{(MLt)^i}{i!}\rho(t).$$

Theorem 11. *Under the assumptions and notations of Theorem 9 and* $M = 1, \omega = 0$ *for every* $t \in [0,T]$

$$e_i(t) \leqslant \left(e^{Lt} - \sum_{k=0}^{i} \frac{(Lt)^k}{k!} \right) \frac{\alpha_1}{L}. \tag{11}$$

The fact that the iteration error decreases at a higher rate for small time intervals inspires the study of the question whether better approximations can be obtained by dividing the time interval into small subintervals and apply the waveform relaxation method repeatedly on the subintervals. This procedure is called *windowing*.

Definition 13. Let

1. $N \in \mathbb{N}^+$ be fixed, $\mathcal{N} := \{1,2,...,N\}$,
2. $v_m^{(n)}$ be the mth iteration function in the nth time window, $v_m^{(n)}(0) = v_m^{(n-1)}(\tau)$, $n \in \mathcal{N}$ and
3. u_n be the solution of $u_n' = (A+F)(u_n)$, for which $u_n(0) = v_m^{(n-1)}(\tau)$, $n \in \mathcal{N}$,

where $v_m^{(0)} := u_0$.

Theorem 12. *With the assumptions on page 7 consider a partition of* $[0,T]$ *into N subintervals of length* τ : $[(n-1)\tau, n\tau]$, $n \in \mathcal{N}$. *Then the method consisting of applying WR on every subinterval is convergent, that is, it approximates the solution of* (9) *with an arbitrarily small error as the number of iterations tends to infinity. Furthermore*

$$\|u(n\tau) - v_m^{(n)}(\tau)\| \leqslant \frac{(ML\tau)^m}{m!} M^2 \frac{e^{(\omega+ML)n\tau} - 1}{\omega + ML} \alpha_N \tag{12}$$

for any $n \in \mathcal{N}$, *with a suitable* α_N *independent of the number of iterations.*

Relation (12) allows us to make a comparison between the iteration error obtained with windowing and without it. Since we have only estimations for the upper error bounds our results on this matter are more of a heuristic nature, nevertheless the numerical tests confirm our conclusions. To have a better approximation with windowing one needs

$$\|u(T) - v_m^{(N)}(T)\| \leqslant \|u(T) - v_m(T)\|.$$

This heuristically gives

$$\frac{(ML\tau)^m}{m!} M^2 \frac{e^{(\omega+ML)T} - 1}{\omega + ML} \alpha_N \lesssim \frac{(MLN\tau)^m}{m!} M \frac{e^{(\omega+ML)T} - 1}{\omega + ML} \alpha_1,$$

$$M \frac{\alpha_N}{\alpha_1} \lesssim N^m,$$

which means that exceeding a specific number of iterations windowing will result in smaller error. Furthermore supposing that the error estimations are close to the actual values of the errors

$$\frac{\|u(T) - v_m(T)\|}{\|u(T) - v_m^{(N)}(T)\|} \approx N^m \frac{\alpha_1}{\alpha_N} \frac{1}{M}$$

yields

$$\ln \|u(T) - v_m(T)\| - \ln \|u(T) - v_m^{(N)}(T)\| \approx m \ln N + \ln \left(\frac{\alpha_1}{\alpha_N} \frac{1}{M} \right). \tag{13}$$

In practice the subproblems of (10) are solved numerically, therefore it is crucial to show the convergence of the *combined method*: the combination of the waveform relaxation method and a numerical method. Let us fix T, such that $0 < T < t_\delta$ and consider the mapping $\phi : C^1([0,T], B_\delta(u_0)) \to C^1([0,T], B_\delta(u_0))$ for which

$$v_i(t) = \phi(v_{i-1})(t).$$

Then the iteration function in (10) can be written as $v_i(t) = \phi^i(v_0)(t)$. Let $\hat{\phi}(v)$ be a numerical approximation of $\phi(v)$ generated by a convergent numerical method. Suppose that there is an appropriate interpolation operator P and that the parameters of the applied numerical method can be chosen in such a way that the mapping $\tilde{\phi} := P \circ \hat{\phi}$ is $\tilde{\phi}(v)(t) \in B_\delta(u_0)$ for every $t \in [0,T]$. With these definitions the result of the combined method can be described with the *numerical iteration function* $\tilde{v}_i := \tilde{\phi}^i(v_0)$. The numerical error arising in the i-th iteration step of the combined method is $\|\phi(\tilde{\phi}^{i-1}(v_0)) - \tilde{\phi}(\tilde{\phi}^{i-1}(v_0))\|_\infty$. Besides the choice of the numerical method the numerical error depends on the choice of the discretization parameters. Thus formally $\|\phi(\tilde{\phi}^{i-1}(v_0)) - \tilde{\phi}(\tilde{\phi}^{i-1}(v_0))\|_\infty \leqslant c(p)$, where p is the vector of the discretization parameters and the proper choice of p ensures the numerical error to be arbitrarily small.

Definition 14. Consider the above notations. Then the cumulative numerical error of the combined method is

$$CNE_i(t) := \|v_i(t) - \tilde{v}_i(t)\| = \|\phi^i(v_0)(t) - \tilde{\phi}^i(v_0)(t)\|.$$

Theorem 13. *Suppose that for every $i \in I$ there is a number $c_i(p)$ such that $\|\phi(\tilde{\phi}^{i-1}(v_0)) - \tilde{\phi}(\tilde{\phi}^{i-1}(v_0))\|_\infty \leqslant c_i(p)$, then with $c(p) := \max\{c_i(p), i \in I\}$*

$$CNE_i(t) \leqslant \sum_{j=0}^{i-1} \left(\frac{ML}{\omega}\right)^j e^{\omega t}(-1)^j \left(e^{-\omega t} - \sum_{k=0}^{j-1} \frac{(-\omega t)^k}{k!}\right) c(p)$$

for every $i \in I$, where $\sum_{k=0}^{-1}(\cdot)$ is defined to be zero.

Proposition 1. *Suppose that for every $i \in I$ there is a number $c_i(p)$ such that $\|\phi(\tilde{\phi}^{i-1}(v_0)) - \tilde{\phi}(\tilde{\phi}^{i-1}(v_0))\|_\infty \leqslant c_i(p)$, then with $c(p) := \max\{c_i(p), i \in I\}$*

$$CNE_i(t) \leqslant \frac{MLe^{(\omega+ML)t} + \omega}{\omega + ML} c(p)$$

for every $i \in I$ and $t \in [0,T]$.

Theorem 14. *Consider the assumptions on page 7 and suppose that every iteration function is approximated by a convergent numerical method. Then the combination of WR and the numerical method provides a convergent method to approximate the solution u of* (9)*. Furthermore with the above notations*

$$\|u(t) - \tilde{v}_i(t)\| \leqslant \frac{(MLt)^i}{i!}\rho(t) + \frac{MLe^{(\omega+ML)t} + \omega}{\omega + ML} c(p)$$

for every $i \in I$ and $t \in [0,T]$.

Theorem 15. *Suppose that $\omega = 0, M = 1$ and there is a constant $c(p)$ independent of t such that for every $i \in I$ $\|\phi(\tilde{\phi}^{i-1}(v_0)) - \tilde{\phi}(\tilde{\phi}^{i-1}(v_0))\|_\infty \leqslant c(p)$ holds. Then*

$$CNE_i(t) = \|\phi^i(v_0)(t) - \tilde{\phi}^i(v_0)(t)\| \leqslant \sum_{k=0}^{i-1} \frac{(Lt)^k}{k!} c(p)$$

for every $i \in I$ and $t \in [0,T]$.

Remark 1. The limit of the cumulative numerical error as $i \to \infty$ is $e^{Lt}c(p)$, thus the result of Proposition 1 remains valid with $\omega = 0$ and $M = 1$.

Theorem 16. *Consider the assumptions on page 7 with $\omega = 0, M = 1$ and suppose that every iteration function is approximated by a convergent numerical method. Then the combination of the waveform relaxation method and the numerical method provides a convergent method that approximates the solution u of* (9)*. Furthermore*

$$\|u(t) - \tilde{v}_i(t)\| \leqslant \left(e^{Lt} - \sum_{k=0}^{i} \frac{(Lt)^k}{k!}\right) \frac{\alpha_1}{L} + \sum_{k=0}^{i-1} \frac{(Lt)^k}{k!} c(p) \tag{14}$$

for every $i \in I$ and $t \in [0,T]$.

Example 4. The initial value problem

$$\begin{cases} \partial_t u(t,x,y) &= -\partial_x u(t,x,y) - \partial_y u(t,x,y) + u^2(t,x,y) \qquad (x,y) \in \mathbb{R}^2,\ t \geqslant 0 \\ u(0,x,y) &= \frac{1}{\pi} e^{-x^2-y^2}, \end{cases} \tag{15}$$

describes second–order autocatalysis with advection. Its solution is

$$u_a(t,x,y) = \frac{1}{\pi e^{(x-t)^2+(y-t)^2} - t}.$$

Problem (15) is of type (9) with $Au = -\partial_x u - \partial_y u$ and $F(u) = u^2$. The solution blows up at time $t = \pi$, in $(x,y) = (\pi,\pi)$. The solution was approximated for $t \in [0,1]$, in a sufficiently small neighborhood of the solution $L = 1$ holds and $X = C_b^2(\mathbb{R}^2,[0,1])$ can be condsidered.

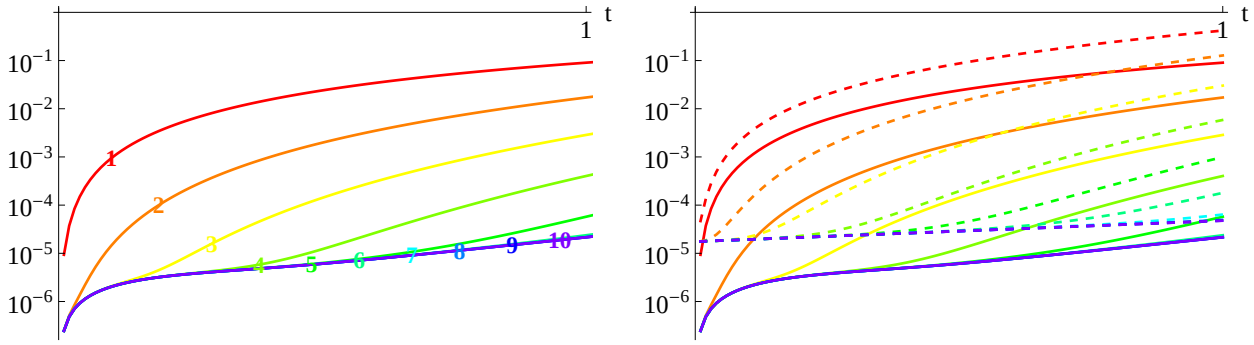

Figure 1: The error of approximations—$\|u(t) - \tilde{v}_m(t)\|$—generated by waveform relaxation method combined with a convergent numerical method is plotted on logarithmic scale versus time for $m = 1,...,10$ iterations on the left. The theoretical estimation (dashed) is added on the right.

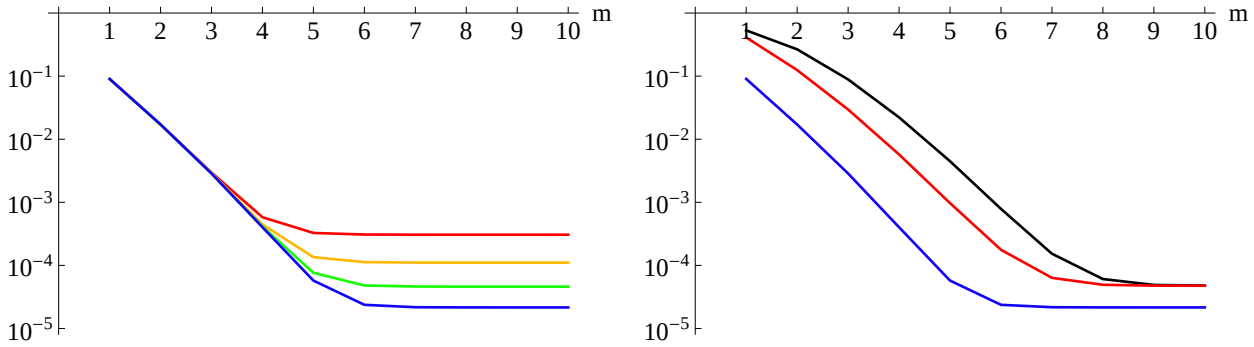

Figure 2: Left: the error of approximate solutions of (15) generated by waveform relaxation method combined with fourth order Runge-Kutta methods are plotted on logarithmic scale versus the number of iterations. The discretization parameter Δx was chosen as $1/5$–red, $1/6$–orange, $1/7$–green, $1/8$–blue. Right: the result with $1/8$ (blue), the theoretical estimation (red) according to (14) with $t = T$ and the classical estimate in the literature (black) are plotted.

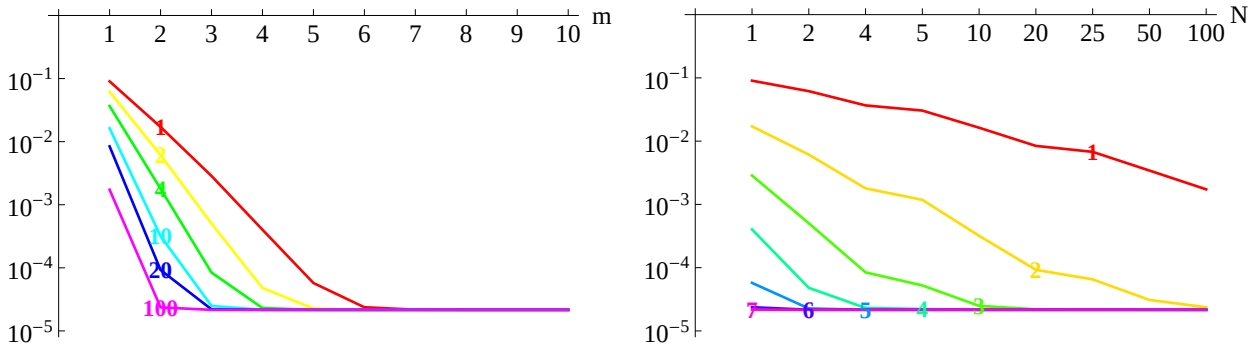

Figure 3: The error $\|u-\tilde{v}_m^{(N)}\|_\infty$ is plotted on logarithmic scale versus the number of iterations $m=1,\ldots,10$; and $N\in\mathscr{W}:=\{1,2,4,10,20,100\}$ on the left; on the right the same expression is plotted versus the number of windows that were taken as the divisors of 100 and $m=1,\ldots,7$.

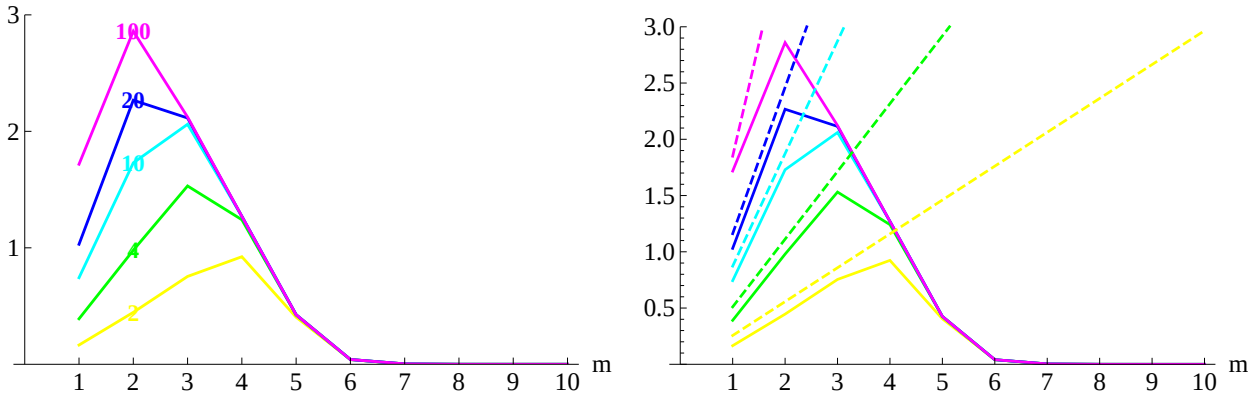

Figure 4: The expression $\ln\|u-\tilde{v}_m^{(1)}\|_\infty-\ln\|u-\tilde{v}_m^{(N)}\|_\infty$ in (13) is plotted versus the number of iterations on the left; on the right the theoretical estimation $m\ln N+\ln(\alpha_1/\alpha_N)$ is added (dashed lines) as given in (13), $N\in\mathscr{W}$.

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