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PhD Thesis Booklet

Standard Koszul algebras

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Introduction

Quasi-hereditary algebras were originally defined by Cline, Parshall and Scott in the context of Lie algebras and algebraic groups during their work on highest weight categories related to Lusztig's conjecture [12]. The concept of quasi-hereditary algebras was formulated in a purely ring theoretical manner as well, and these algebras appear in several situations and applications, both in Lie theory and in the theory of associative rings. For instance, the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ is a categorical sum of blocks, where each block is equivalent to a module category of a quasi-hereditary algebra (see [10]). It is also known that finite dimensional algebras with global dimension at most 2 are all quasi-hereditary (cf. [15]).

In the theory of quasi-hereditary algebras, the (left or right) (co)standard modules play a fundamental role. Namely, an algebra is quasi-hereditary if the regular module is filtered by standard modules, and in addition, all standard modules are Schurian, meaning that their endomorphism rings are division rings. Without the additional assumption on standard modules, we get the concept of standardly stratified algebras. Inspired by the work of Cline, Parshall and Scott [13], Ágoston, Dlab and Lukács determined the conditions, which ensure that a quasi-hereditary algebra has a quasi-hereditary Yoneda extension algebra (cf. [1], [2] and [4]).

In [4], the authors proved that if a quasi-hereditary algebra is standard Koszul (i.e. every right or left standard module is Koszul), then its extension algebra is quasi-hereditary. Moreover, in [4], the authors also pointed out that the homological duality respects the stratifying structure, in the sense that the natural functor Ext^*_A maps the standard modules of A to the standard modules defined over the extension algebra A^*.

It was also proved in [4] that standard Koszul quasi-hereditary algebras are always Koszul algebras, i.e. the simple modules are Koszul, or for graph algebras, the graph of the extension algebra is the same as the graph of the original algebra. It turned out that this implication itself is a useful tool in other situations, too (cf. [11], [16], [30] or [36]). The wide range of applications suggests that it is worth investigating this implication for more general (or different) settings.

Later, the same authors generalized their results to the case of Koszul standardly stratified algebras under the additional assumption that the initial algebra was graded [5]. They showed that the homological dual of such an algebra is standardly stratified, and the homological duality functor maps the standard right A-modules to the corresponding left proper standard modules of A^*, while the left proper standard A-modules are mapped to the right standard A^*-modules. However, since algebras in [5] were assumed to be Koszul algebras, the question if all standard Koszul standardly stratified algebras are Koszul was left open. The neccessity of the graded strucure also remained a question.

In this booklet, we give a summary of the results presented in the thesis. We follow the outline of the thesis, i.e. in the first section, we recall the notation and basic definitions we use later. Then we discuss the main results in two sections corresponding to Chapter 2 and 3 of the thesis.

In the second section we focus on standardly stratified algebras along with their extension algebras from a perspective similar to [1], [2] and [4], and extend the above results. In Section 3, we omit the condition of stratification, and investigate the connection between the Koszul and standard Koszul property for monomial and special biserial algebras. Both classes are defined as quotients of path algebras, and they are frequently used for testing (homological) conjectures.

The results of the thesis were published in [24], [25] and [28]. In the first article, we proved that the standard Koszul standardly stratified algebras are Koszul. In the second, we were focusing on the extension algebra of a standard Koszul standardly stratified algebra. The main result of the paper is that the extension algebra of a standard Koszul standardly stratified algebra is standardly stratified. The third article contains our results on special biserial algebras.

1 Preliminaries and notation

As in the thesis, A always stands for a basic finite dimensional associative algebra over a field K. The Jacobson radical of A will be denoted by $J = \text{rad } A$. All considered modules are unitary and finitely generated. Modules are meant to be right modules, unless otherwise stated. The category of finitely generated left or right A-modules will be denoted by $A\text{-}\mathrm{mod}$ and $\mathrm{mod}\text{-}A$, respectively.

For the algebra A, we fix a complete ordered set of primitive orthogonal idempotents $\mathbf{e} = (e_1, \ldots, e_n)$. In the canonical decomposition

$$A_A = e_1A \oplus \ldots \oplus e_nA$$

of the regular module, the ith indecomposable projective module e_iA will be denoted by $P(i)$ and its simple top $P(i)/\text{rad } P(i)$ by $S(i)$. Besides, $\hat{S}$ stands for the semisimple top of A_A, so $\hat{S} = \oplus_{i=1}^n S(i)$. The corresponding left modules are denoted by $P^\circ(i), S^\circ(i)$ and $\hat{S}^\circ$, respectively.

If $1 \leq i \leq n$, set $\varepsilon_i = e_i + \ldots + e_n$, and $\varepsilon_{n+1} = 0$. The centralizer algebras $\varepsilon_iA\varepsilon_i$ of A will be denoted by C_i, where the idempotents and their order are naturally inherited from A. The ith *standard* and *proper standard* A-modules are $\Delta(i) = e_iA/e_iA\varepsilon_{i+1}A$ and $\overline{\Delta}(i) = e_iA/e_i(\text{rad } A)\varepsilon_iA$, respectively. That is, the ith standard module is the largest factor module of $P(i)$ which has no composition factor isomorphic to $S(j)$ if $j > i$, while the ith proper standard module is the largest factor module of $P(i)$ whose radical has no composition factor isomorphic to $S(j)$ if $j \geq i$. The left standard and proper standard modules are defined analogously.

Let $\mathcal{X}$ be a class of modules. We say that a module X is *filtered* by $\mathcal{X}$ if there is a sequence of submodules $X = X^0 \supseteq X^1 \supseteq \ldots$ such that $\bigcap_{i\geq 0} X^i = 0$, and all the factor modules X^i/X^{i+1} are isomorphic to some modules of $\mathcal{X}$. In this case, we write $X \in \mathcal{F}(\mathcal{X})$. Given the ordered set $\mathbf{e} = (e_1, \ldots, e_n)$, the *trace filtration* of a module X is

$$X = X\varepsilon_1A \supseteq X\varepsilon_2A \supseteq \ldots \supseteq X\varepsilon_nA \supseteq 0.$$

We call an algebra A (with a fixed complete ordered set $\mathbf{e}$ of primitive orthogonal idempotents) *standardly stratified* if the regular module $A_A \in \mathcal{F}(\Delta)$, or equivalently, the left regular module ${}_AA \in \mathcal{F}(\overline{\Delta}^\circ)$, where $\overline{\Delta}^\circ$ consists of the proper standard modules, while Δ consists of the left standard modules. We shall use later the fact that $\mathrm{Ext}_A^h(\Delta(i), S(j)) = 0$ for all $h \geq 0$ and $i \geq j$ when $A_A \in \mathcal{F}(\Delta)$ (cf. [12]), and similarly, $\mathrm{Ext}_A^h(\overline{\Delta}(i), S(j)) = 0$ for all $h \geq 0$ and $i > j$ when $A_A \in \mathcal{F}(\overline{\Delta})$.

A submodule $X \leq Y$ is a *top submodule* ($X \overset{t}{\leq} Y$) whenever $X \cap \mathrm{rad}\ Y = \mathrm{rad}\ X$. This is equivalent to the condition that the natural embedding of X into Y induces an embedding of $X/\mathrm{rad}\ X$ into $Y/\mathrm{rad}\ Y$ (such embeddings will be called top embeddings), or in other words, the induced map $\mathrm{Hom}_A(Y, \hat{S}) \to \mathrm{Hom}_A(X, \hat{S})$ is surjective. (See [1] for the origin of this concept.) Let

$$P_\bullet(X): \qquad \ldots \to P_h(X) \to \ldots \to P_1(X) \to P_0(X) \to X \to 0$$

be a minimal projective resolution of X with the hth syzygy Ω_h. Using the concept of top submodules, we introduce the classes $\mathcal{C}_A^i$. The module X belongs to $\mathcal{C}_A^i$ if Ω_h is a top submodule of rad P_{h-1} for all $h \leq i$. We say that X has a top projective resolution, or X is *Koszul*, if $X \in \mathcal{C}_A := \bigcap_{i=1}^\infty \mathcal{C}_A^i$. The algebra A is a *Koszul algebra* if $\hat{S}$ (or equivalently if $\hat{S}^\circ$) has a top projective resolution (cf. [20]).

The *extension algebra* (or *homological dual*) of A is the positively graded algebra A^* whose underlying vector space is $\oplus_{h\geq 0}(A^*)_h = \oplus_{h\geq 0} \mathrm{Ext}_A^h(\hat{S}, \hat{S})$, and the multiplication is given by the Yoneda composition of the extensions (cf. [20]). A *graded (left) A^*-module* $X = \oplus_{h\in\mathbb{Z}} X_h$ is an A^*-module for which $(A^*)_h X_k \subseteq X_{h+k}$, and by an A^*-module homomorphism $f: X \to Y$, we mean a graded A^*-module homomorphism f having any degree $d \in \mathbb{Z}$. In this sense, we say that two graded A^*-modules X and Y are isomorphic if there exists a bijective A^*-homomorphism $f: X \to Y$ (not necessarily degree of 0). The ith graded shift of the graded A^*-module X is denoted by $X[i]$, which is a graded module such that $X[i]_h = X_{h-i}$. For graded modules, we shall also use the notation $X_{\geq i} = \oplus_{h\geq i} X_h$. We shall call the A^*-modules X and Y isomorphic if there is an isomorphism between X and $Y[i]$ in the category of graded A^*-modules with an appropriate integer i.

The functor Ext^*_A : mod-$A \to A^*$- grmod is defined as the direct sum of the functors $\mathrm{Ext}^h_A(-,\hat{S})$. Namely, if $X \in$ mod-A, then $\mathrm{Ext}^*_A(X)$ is the graded left module $\oplus_{h\geq 0}\mathrm{Ext}^h_A(X,\hat{S})$. For simplicity, we denote $\mathrm{Ext}^*_A(X)$ by X^*, while for its homogeneous part of degree h we write $(X^*)_h$. We use the notation $\varphi^* = \mathrm{Ext}^*_A(\varphi,\hat{S}) : \mathrm{Ext}^*_A(Y,\hat{S}) \to \mathrm{Ext}^*_A(X,\hat{S})$, where $\varphi : X \to Y$ is a module homomorphism, and we denote by E^h_X the canonical isomorphism between the spaces $\mathrm{Hom}_A(\Omega_h(X),\hat{S})$ and $\mathrm{Ext}^h_A(X,\hat{S})$. Thus we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_A(\Omega_h(Y),\hat{S}) & \xrightarrow{(\widetilde{\varphi}_{h-1})^*} & \mathrm{Hom}_A(\Omega_h(X),\hat{S}) \\ \Big\downarrow E^h_Y & & \Big\downarrow E^h_X \\ \mathrm{Ext}^h_A(Y,\hat{S}) & \xrightarrow{\varphi^*} & \mathrm{Ext}^h_A(X,\hat{S}) \end{array}$$

of left $(A^*)_0$-modules, where $\varphi_\bullet : P_\bullet(X) \to P_\bullet(Y)$ is a lifting of φ, while $\widetilde{\varphi}_{h-1}$ is the restriction of φ_{h-1} to the submodule $\Omega_h(X) \subseteq P_{h-1}(X)$.

Note that the module X has a top projective resolution if and only if $(X^*)_h = \mathrm{Ext}^h_A(X,\hat{S}) = (A^*)^h_1 \cdot (X^*)_0$ for all $h \geq 0$. In particular, if A is Koszul, then A^* is *tightly graded*, i.e. $\mathrm{Ext}^h_A(\hat{S},\hat{S}) = (\mathrm{Ext}^1_A(\hat{S},\hat{S}))^h$ for $h \geq 1$ (cf. [20]).

For a semisimple A-module S we say that $X \in$ mod-A is *S-Koszul*, if $\mathrm{Ext}^t_A(S,X) \subseteq \mathrm{Ext}^1_A(\hat{S},S) \cdot \mathrm{Ext}^{t-1}_A(X,\hat{S})$ for all $t \geq 1$. Or equivalently, the trace of S in the top of the syzygy $\Omega_t(X)$ is mapped injectively into the top of rad $P_{t-1}(X)$ for every $t \geq 1$. In other words, X is S-Koszul if and only if $\Omega_t(X)\varepsilon_S A \cap P_{t-1}(X)J^2 \subseteq \Omega_t(X)J$ for every $t \geq 1$, where $\varepsilon_S = \sum\{e_i \mid Se_i \neq 0\}$.

Let $\mathbf{e} = (e_1,\dots,e_n)$ be a complete ordered set of primitive orthogonal idempotents of A. The set $\{f_i = \mathrm{id}_{S(i)} \mid 1 \leq i \leq n\}$ defines a complete set of primitive orthogonal idempotents in A^*. We will always consider this set with the opposite order $\mathbf{f} = (f_n,\dots,f_1)$. In this way, the ith standard A^*-module $\Delta_{A^*}(i)$ is defined as $\Delta_{A^*}(i) = f_iA^*/f_iA^*(f_1+\ldots+f_{i-1})A^*$, while the ith proper standard module is given by $\overline{\Delta}_{A^*}(i) = f_iA^*/f_i(A^*)_{\geq 1}(f_1+\ldots+f_i)A^*$. The definitions of left standard and proper standard modules are analogous. The algebra

A^* is standardly stratified if $A^*_{A^*}$ is filtered by right standard A^*-modules. In view of Theorem 1 of [5], if A^* is tightly graded, then this is equivalent to the condition that ${}_{A^*}A^*$ is filtered by left proper standard A^*-modules.

2 Standardly stratified algebras

The concept of standardly stratified algebras is a natural generalization of quasi-hereditary algebras. We extended the following results on quasi-hereditary algebras to standardly stratified algebras.

Theorem 2.1 (I. Ágoston, V. Dlab, E. Lukács [4] Theroem 1)**.** *Let* $(A, \mathbf{e})$ *be a quasi-hereditary algebra. If both left and right standard modules have top projective resolutions, then* A *is Koszul (i.e. all simple modules have top projective resolutions).*

Theorem 2.2 (I. Ágoston, V. Dlab, E. Lukács [4], Theroem 2)**.** *Let* $(A, \mathbf{e})$ *be a standard Koszul quasi-hereditary algebra. Then the Yoneda extension algebra* $(A^*, \mathbf{f})$ *is also a quasi-hereditary algebra.*

2.1 Standard Koszul standardly stratified algebras

We proved the analogue of Theorem 2.1 in the first part of Chapter 2 of the thesis. The stratification enables us to do an induction on the number of simple modules, just like in the quasi-hereditary case. But the proof of Ágoston, Dlab and Lukács for quasi-hereditary algebras relied heavily on the finite global dimension of the algebra and the lack of certain extensions resulting from the Schurian property of the standard modules. So here we needed an essentially different approach.

As a preparation, Section 2.1 contains a few technical lemmas (Lemma 2.1.3–2.1.8 of the thesis) for a wider class of algebras we call lean algebras. An algebra is lean if $\varepsilon_i J^2 \varepsilon_i = \varepsilon_i J \varepsilon_i J \varepsilon_i$ for all i. This concept generalizes the notion of leanness in [1], and it is easy to see that standardly stratified algebras and their opposite algebras are always lean. The lemmas point out strong connections between top embeddings over A and its centralizer algebras $C_i = \varepsilon_i A \varepsilon_i$, and lead us to Lemma 2.3 which plays a fundamental role in the induction procedures.

Lemma 2.3 (E. Lukács, A. Magyar [24], Lemma 2.1)**.** *Suppose that A is a standard Koszul standardly stratified algebra. Then its centralizer algebra $C_2 = \varepsilon_2 A \varepsilon_2$ is again standard Koszul and standardly stratified, its standard and left proper standard modules are $\Delta(i)\varepsilon_2$ and $\varepsilon_2 \overline{\Delta}^\circ(i)$ for $i \geq 2$.*

In [4] the subclass $\mathcal{K}$ of A-modules for a quasi-hereditary algebra A was crucial.

$$\mathcal{K} = \left\{ X \mid X \text{ is } S(1)\text{-Koszul},\ X\varepsilon_2 A \overset{t}{\leq} X,\ X\varepsilon_2 \in \mathcal{C}_{C_2} \right\}.$$

The authors showed that all modules in $\mathcal{K}$ are Koszul. We proved the same for standardly stratified algebras but with a different approach. We investigated first the wider subclass

$$\mathcal{K}_2 = \left\{ X \mid X\varepsilon_2 A \overset{t}{\leq} X,\ X\varepsilon_2 \in \mathcal{C}_{C_2} \right\},$$

showed that it is closed under taking syzygies, and inferred the statement of Proposition 2.4.

Proposition 2.4 (E. Lukács, A. Magyar [24], Proposition 2.7)**.** *All modules in $\mathcal{K}_2$ are $\oplus_{i\geq 2} S(i)$-Koszul.*

As $\overline{\Delta}^\circ(1) = S^\circ(1)$ is a Koszul module, all simple modules over a standardly stratified algebra must be $S(1)$-Koszul. On the other hand, all simple A-modules are Koszul modules over the centralizer algebra C_2 by induction. Hence we get that all simple A-modules belong to $\mathcal{K}$, which along with Proposition 2.4 gives Theorem 2.5.

Theorem 2.5 (E. Lukács, A. Magyar [24], Theorem 2.9)**.** *Every standard Koszul standardly stratified algebra is Koszul.*

2.2 The extension algebra of a standard Koszul standardly stratified algebra

The second part of Chapter 2 (Sections 2.3 and 2.4) of the thesis proves that the extension algebra of a standard Koszul standardly stratified algebra is also standardly stratified. For this, one needs to prove that the regular module over the extension algebra is filtered by standard modules (or equivalently, by proper standard modules for the other side). We extended this to a more general question: we found easy-to-check conditions that ensure that a module is mapped by the Ext* functor to a module filtered by standard (or on the other side proper standard) modules. Both the conditions and the methods are different when we look at right or left modules of a standardly stratified algebra. The main idea is to perform an induction, using centralizer algebras and build a correspondence between the trace filtration of $\mathrm{Ext}^*_A(X)$ and the Ext^*-images $\mathrm{Ext}^*_{C_i}(X)$ (considered as A^*-modules) for certain A-modules.

As the initial step, we investigated modules over the extension algebra of a lean algebra. In particular, we defined the classes $\mathcal{K}$ and $\mathcal{K}_2$ of modules for lean algebras and showed that several useful properties are preserved in this more general setting. We also introduced a "recursive" version $r\mathcal{K}$ of $\mathcal{K}$ as

$$r\mathcal{K} = \left\{ X \in \mathcal{K} \,|\, X\varepsilon_i \in \mathcal{K}_{C_i} \text{ for all } i \right\}.$$

We showed that these classes are closed under top extensions. We were investigating the action of the operator ω_h on modules belonging to these subclasses. For an A-module X, let $\widetilde{X} = X\varepsilon_2 A$ and $\omega(X) = \Omega(\widetilde{X})$, while $\omega_h(X) = \omega(\omega_{h-1}(X))$ recursively. In particular, it turns out that for modules in $\mathcal{K}_2$ over a lean algebra, $\widetilde{\omega}_h(X) = \omega_h(X)\varepsilon_2 A \overset{t}{\leq} \omega_h(X)$ for all h, and $\mathcal{K}_2$ is closed under ω. Proposition 2.6 shows the benefits of these features.

Proposition 2.6 (E. Lukács, A. Magyar [25], Proposition 3.5)**.** *Suppose that* $\varepsilon_2 J^2 \varepsilon_2 = \varepsilon_2 J \varepsilon_2 J \varepsilon_2$. *If* $X \in \mathcal{K}_2$, *then for every* $h \geq 0$ *we have an exact sequence*

$$0 \to \widetilde{\omega}_h(X) \xrightarrow{\alpha_h} \Omega_h(X) \xrightarrow{\beta_h} Y_h(X) \to 0 \tag{1}$$

with α_h a top embedding.

This means that for modules in $\mathcal{K}_2$, one can obtain information about $\mathrm{Ext}_h(X, \hat{S})$ by looking at the top of $\widetilde{\omega}_h(X)$ and $Y_h(X)$.

In the next step, we constructed the natural homomorphism q formally defined as

$$q_X = \bigoplus_{h\geq 0}(q_X)_h : \mathrm{Ext}^*_A(X) \to \mathrm{Ext}^*_{C_2}(X\varepsilon_2),$$

which sends every h-fold extension $0 \to \hat{S} \to X_{h-1} \to \ldots \to X_0 \to X \to 0$ to an h-fold extension $0 \to \hat{S}\varepsilon_2 \to X_{h-1}\varepsilon_2 \to \ldots \to X_0\varepsilon_2 \to X\varepsilon_2 \to 0$. Notably, $q_{\hat{S}} : A^* \to C_2^*$ is an algebra homomorphism, and consequently q_X is can be considered as a (left) graded A^*-module homomorphism. It is easy to see, that $\ker q_X \supseteq A^* f_1 X^*$ is always true. We studied q_X for a module $X \in \mathcal{K}_2$ over a lean algebra. The next proposition summarizes our key observations.

Proposition 2.7 (E. Lukács, A. Magyar [25] Proposition 3.11 and 3.12)**.** *Let A be lean and $X \in \mathcal{K}_2$. Then $q_X : X^* \to (X\varepsilon_2)^*$ is an epimorphism. If $Y_h(X)$ is $\hat{S}\varepsilon_2 A$-Koszul for all h, then $\ker q_X = A^* f_1 X^*$.*

To apply the above general results, we divided our discussion into two parts, i.e. we separately investigated algebras that are standard Koszul and standardly stratified, and algebras whose opposite algebra is standard Koszul and standardly stratified. In both cases, we checked the condition given by Proposition 2.7, and use induction to get the stratification of ${}_{A^*}A^*$ and $A^*_{A^*}$, respectively. More precisely, we examined the recursively defined classes $r\mathcal{K}$ and

$$r\mathcal{K}^+ = \Big\{X \in \mathcal{K}^+ \,|\, X\varepsilon_i \in \mathcal{K}^+_{C_i} \text{ for all } i\Big\},$$

where

$$\mathcal{K}^+ = \Big\{X \in \mathcal{K} \mid \widetilde{\omega}_h(X) \in \mathcal{C}_A, \text{ and } \bar{\omega}_h(X) \cong \oplus S(1) \text{ for all } h \geq 0\Big\},$$

respectively. With the additional assumption of the stratification and standard Koszul property, these smaller classes are also closed under ω and top extensions. Finally, we deduced the following results.

Theorem 2.8 (E. Lukács, A. Magyar [25], Theorem 5.4 and 5.10)**.** *If A is standard Koszul standardly stratified and $X \in r\mathcal{K}$, then $X^*/A^*(f_1+\ldots+f_{i-1})X^* \cong (X\varepsilon_i)^*$ for all $i \geq 1$. Moreover, $A^*f_1X^* \in \mathcal{F}(\overline{\Delta}^{\circ}_{A^*}(1))$.*

Since simple and standard modules over standard Koszul standardly stratified algebras belong to $r\mathcal{K}$, the above theorem implies one of our main results.

Theorem 2.9 (E. Lukács, A. Magyar [25], Theorem 5.7 and 5.12)**.** *If A is standard Koszul standardly stratified, then its homological dual A^* is a standardly stratified algebra. Moreover, its right standard A-modules are mapped to left proper standard A^*-modules, and left proper standard A-modules are mapped to right standard A^*-modules by the functor Ext^*_A, that is, $\mathrm{Ext}^*_A(\Delta(i)) \cong \overline{\Delta}^{\circ}_{A^*}(i)$ and $\mathrm{Ext}^*_A(\overline{\Delta}^{\circ}(i)) \cong \Delta_{A^*}(i)$.*

Although using different methods, we were able to show that analogue statements hold for the opposite algebra of standard Koszul standardly stratified algebras. The main results are the following.

Theorem 2.10 (E. Lukács, A. Magyar [25], Theorem 6.4 and 6.11)**.** *If A° is standard Koszul standardly stratified and $X \in r\mathcal{K}^+$, then*

(i) $X^*/A^*(f_1 + \ldots + f_{i-1})X^* \cong (X\varepsilon_i)^*$ *for all* $i \geq 1$;

(ii) all simple A^*-modules are in $r\mathcal{K}^+$;

(iii) consequently, $(A^*)^{\circ}$ is standardly stratified.

It is worth mentioning that beyond proving that the extension algebra A^* of a standard Koszul standardly stratified algebra A is always standardly stratified, we explored a relevant class of submodules both in mod-A and A-mod whose images with respect to the functor Ext^*_A are stratified with standard or proper standard A^*-modules.

3 Monomial and special biserial algebras

We investigated the standard Koszul property for some combinatorially defined classes of algebras as well. Both monomial and special biserial algebras are given by their graphs and relations. (For a thorough introduction to the construction and properties of graph algebras, one may refer to [9].)

In the thesis, we showed that for monomial and self-injective (or symmetric) special biserial algebras, the Koszul and standard Koszul properties can be described by combinatorial conditions. Using these, we could prove, even without the assumptions of standard stratification, that in these classes, standard Koszul algebras are always Koszul. Furthermore, we gave a characterization of standard Koszul algebras in these classes in terms of graphs and relations.

3.1 Monomial algebras

Let Γ be a finite oriented graph possibly with loops and multiple edges. The path algebra $K\Gamma$ is a vector space whose basis consists of all oriented paths in Γ, and the product of two paths is given by concatenation if it is possible, and zero otherwise. We say that $I \triangleleft K\Gamma$ is an *admissible ideal*, if it consists of linear combinations of paths with length at least 2, and there exists an integer m such that all paths with length at least m are in I. In this case, we call the factor algebra $A = K\Gamma/I$ a finite dimensional *graph algebra*. A graph algebra $A = K\Gamma/I$ is *monomial* if I is generated by paths.

Theorem 5.2 of [2] implies that a standard Koszul monomial algebra is Koszul when all standard modules are Schurian (i.e. $\mathrm{End}(\Delta(i))$ and $\mathrm{End}(\Delta^\circ(i))$ are division rings for all i). The proof we presented is only slightly more general but it is a good introduction to the more complicated case of special biserial algebras.

In the proof, we heavily relied on earlier results about monomial algebras. For instance, the projective resolutions of simple modules are described in terms of vertices of Γ and non-zero minimal paths of I (cf. [19]). On the other hand, we also have a useful characterization of Koszul algebras.

Theorem 3.1 (E. Green, D. Happel, D. Zacharia [19] and A. Magyar [29]). *Let $A = K\Gamma/I$ be a monomial algebra. Then the following are equivalent*

(i) A is Koszul;

(ii) Every simple A-module S belongs to $\mathcal{C}_A^2$;

(iii) A is quadratic.

Theorem 3.1 gives us alternatives to check whether a monomial algebra is Koszul. Especially, the conditions *(ii)* and *(iii)* are proved to be useful. We introduced the concept of *valleys*, i.e. paths $u_j \rightsquigarrow u_k \rightsquigarrow u_m$ which contain vertices u_j, u_k and u_m such that $u_k \leq u_j$, and $u_k < u_m$. Proposition 3.2 shows the connection between valleys and the standard Koszul property.

Proposition 3.2 (A. Magyar [29]). *Let $A = K\Gamma/I$ be a standard Koszul monomial algebra. Then the non-zero paths of A do not contain valleys.*

Using the proposition, we can conclude that if A is a standard Koszul monomial algebra, then there are no maximal non-zero paths in I with length ≥ 3. This proves the following theorem.

Theorem 3.3 (A. Magyar [29]). *Every standard Koszul monomial algebra is a Koszul algebra.*

Additionally, we gave a characterization of standard Koszul algebras using the notion of valleys.

Theorem 3.4 (A. Magyar [29]). *The monomial algebra $A = K\Gamma/I$ is standard Koszul if and only if A is Koszul and no non-zero path of A contains a valley.*

3.2 Special biserial algebras

The concept of biserial algebras was introduced by Tachikawa in [35], while the significant subclass of special biserial (SB) algebras was first studied in the work of Skowroński and Waschbüsch [33]. Self-injective, in particular, symmetric special biserial (SSB) algebras appear in the theory of modular representations of finite groups (see [23] and [31]), and play an important role in complex representations of the Lorentz group [22]. These algebras are often used to testing various conjectures (cf [17], [18] or [34]).

An algebra $A \cong K\Gamma/I$ is said to be *special biserial*, or SB for short, if for each vertex v of Γ, there are at most two arrows starting, and at most two arrows ending at v, furthermore, for each arrow α there exists at most one arrow β and at most one arrow γ such that $\beta\alpha, \alpha\gamma \notin I$. An algebra A is *self-injective* if A_A is an injective A-module, while A is a *Frobenius algebra* if $A_A \cong \mathrm{Hom}_K({}_AA, K)$ as right modules. Frobenius algebras are always self-injective, on the other hand, every self-injective basic algebra is Frobenius [14]. If A is a Frobenius algebra, then there exists a linear function $\varphi : A \to K$ such that $\ker\varphi$ does not contain any nontrivial right or left ideal of A. We call a Frobenius algebra *symmetric* if the above *Frobenius function* is symmetric, i.e. $\varphi(ab) = \varphi(ba)$ for all $a, b \in A$. By an *SSB algebra*, we mean an SB algebra with such a fixed *symmetric form* φ.

First, we observed that if A is standard Koszul, then apart from a few exceptions, the indecomposable projective A-modules cannot be uniserial (cf. Lemmas 2.1–2.3 [28]). Even more, we precisely determined those cases when A has a uniserial projective module.

Proposition 3.5 (A. Magyar [28], Lemma 2.6)**.** *If A is a non-simple connected standard Koszul self-injective SB algebra having a uniserial projective module, then every indecomposable projective A-module is two-dimensional.*

Corollary 3.6 (A. Magyar [28], Proposition 2.7)**.** *If A is a standard Koszul self-injective SB algebra that has a uniserial projective module, then A is Koszul.*

Corollary 3.6 lets us to restrict our attention to algebras without uniserial projective modules. We followed ideas similar to what Antipov and Generalov used in [8] to give a complete description of the projective resolutions of simple modules for this case. This led us to a combinatorial (or rather a graphical) construction (See Fig. 3.1 and Proposition 3.2.8 of the thesis). The construction gives us equivalent conditions for a standard Koszul self-injective SB algebra to be Koszul. This is similar to what we inferred in Theorem 3.1 for monomial algebras.

Proposition 3.7 (A. Magyar [28], Corollary 2.10)**.** *Let A be a self-injective SB algebra, then A is Koszul if and only if all of its simple modules are in $\mathcal{C}_A^2$.*

As the next step, we investigated non-zero paths in the projective A-modules e_iA. We defined valleys in a similar way as we did for monomial algebras, and showed that no non-zero paths with length at least 3 can contain a valley, if A is standard Koszul. This implies that standard Koszul self-injective SB algebras are close to being "quadratic" in the following sense.

Proposition 3.8 (A. Magyar [28])**.** *Let A be a non-simple connected standard Koszul self-injective SB algebra. For all i, at least one of the maximal nonzero paths in e_iA has length* 2*.*

As a consequence of Propositions 3.5 and 3.8, and Corollary 3.6 we got Theorem 3.9.

Theorem 3.9. *If A is a self-injective standard Koszul SB algebra, then A is Koszul.*

Finally, we gave a characterization of standard Koszul SSB algebras in terms of quivers and relations.

Theorem 3.10 (A. Magyar [28], Theorem 2.23)**.** *$A = K\Gamma/I$ is a connected standard Koszul SSB algebra if and only if either*

(a) A is isomorphic to one of the K-algebras: K, $K[x]/(x^2)$, $K[x,y]/(x^2,y^2)$, $K\langle x,y\rangle/(xy,yx,x^2-\lambda y^m)$, or

(b) Γ has the shape shown below and I is generated by the relations of

$$I = \big(\alpha_i\alpha_{i+1},\ \beta_{i+1}\beta_i,\ \gamma\alpha_1,\ \alpha_{n-1}\delta,\ \beta_1\gamma,\ \delta\beta_{n-1},\ \alpha_{i+1}\beta_{i+1} - \beta_i\alpha_i,$$
$$\gamma^k - \alpha_1\beta_1,\ \lambda\delta^m - \beta_{n-1}\alpha_{n-1} \mid i = 1, \ldots, n-2\big), \lambda \in K \setminus \{0\},\ k, m \geq 2. \quad (2)$$

Related publications of the author

[24] Lukács, E., Magyar, A., Standard Koszul standardly stratified algebras, *Communications in Algebra*, **45(3)**:1270–1277, 2017.
Zbl 06561311, MR 3573378

[25] Lukács, E., Magyar, A., Stratified modules over an extension algebra, *Czechoslovak Mathematical Journal*, **68(2)**:523–551, 2018.

[28] Magyar, A., Standard Koszul self-injective special biserial algebras, *Journal of Algebra and Its Applications*, **15(03)** 1650044 pp. 15, 2016.
Zbl 06561311, MR 3454706

References

[1] Ágoston, I., Dlab, V., Lukács, E., Lean quasi-hereditary algebras, *in: Representations of Algebras, Sixth International Conference, Ottawa, 1992, Can. Math. Soc. Conf. Proc. Ser.*, **14**:1–14, 1993. Zbl 0820.16006, MR 1265273

[2] Ágoston, I., Dlab, V., Lukács, E., Homological duality and quasi-heredity, *Canadian Journal of Mathematics*, **48**:897–917, 1996. Zbl 0868.16009, MR 1414.062

[3] Ágoston, I., Dlab, V., Lukács, E., Stratified algebras, *Mathematical Reports of the Academy of Science, Canada*, **20**:22–28, 1998. Zbl 0914.16010, MR 1619048

[4] Ágoston, I., Dlab, V., Lukács, E., Quasi-hereditary extension algebras, *Algebras and Representation Theory*, **6**:97–117, 2003. Zbl 1503.16007, MR 1960515

[5] Ágoston, I., Dlab, V., Lukács, E., Standardly stratified extension algebras, *Communications in Algebra*, **33**:1357–1368, 2005. Zbl 1081.16019, MR 2149063

[6] Ágoston, I., Happel, D., Lukács, E., Unger, L., Finitistic dimension of standardly stratified algebras, *Communications in Algebra*, **28(6)**:2745–2752, 2000. Zbl 0964.16005, MR 1757427

[7] Anderson, F., Fuller, K., Rings and Categories of Modules, *Graduate Text in Mathematics* **13**, 1974.

[8] Antipov, M. A., Generalov, A. I.., The Yoneda algebras of symmetric special biserial algebras are finitely generated, *St. Petersburg Mathematical Journal*, **17**:377–392, 2006. Zbl 0964.16005, MR 1757427

[9] Assem, I., Simson, D., Skowronski, A., Elements of the Representation Theory of Associative Algebras I., *London Mathematical Society* **65**, 2006.

[10] Bernstein, J., Gelfand, I.M., Gelfand, S.I., A category of $\mathfrak{g}$–modules, *Funct. Anal. Appl.*, **10**:87–92, 1976. Zbl 0353.18013, MR 0407097.

[11] J. Brundan, C. Stroppel. Highest weight categories arising from Khovanov's diagram algebra II: Koszulity. *Transformation Groups*, **15**:1–45, 2010. Zbl 1243.17004, MR 2881300

[12] Cline, E., Parshall, B.J., Scott, L.L., *Stratifying Endomorphism Algebras*, Memoirs of the AMS **591**, 1996. Zbl 0888.16006, MR 1350891

[13] Cline, E., Parshall, B.J., Scott, L.L., The homological dual of highest weight categories, *Proc. of LMS*, **s3-68**: 294–316, 1994. Zbl 0819.20045, MR 1253506.

[14] Curtis, C. W., Reiner, I., *Representation Theory of Finite Groups and Associative Algebras*, John Wiley and Sons, Inc., 1962.

[15] Dlab, V., Ringel, C.M., Quasi-hereditary algebras, *Illinois Journal of Mathematics*, **33**:280–291, 1989. Zbl 0666.16014, MR 0987824

[16] Ehring, M., Stroppel, C., Algebras, coideal subalgebras and categorified skew Hodge duality, (*preprint*) `http://arxiv.org/abs/1310.1972`

[17] Erdmann, K. Blocks of Tame Representation Type and Related Algebras, *Lecture Notes in Mathematics*, **1428**, 1990.

[18] Erdmann, K., Holm, T., Iyama, O., Schröer, J., Radical embeddings and representation dimension, *Advances in Mathematics* **185**: 159–177, 2004. Zbl 1062.16006, MR 2058783.

[19] Green, E. L., Happel, D. and Zacharia, D., Projective resolutions over Artin algebras with zero relations, *Illinois Journal of Mathematics*, **29**:180–190, 1985. Zbl 0551.16008, MR 0769766

[20] Green, E., Martínez-Villa, R., Koszul and Yoneda algebras, *Representation Theory of Algebras*, **18**:247–297, 1996. Zbl 0860.16009, MR 1388055

[21] Green, E. L., Zacharia, D., The cohomology ring of a monomial algebra, *Manuscripta Mathematica*, **85**:11–23, 1994. Zbl 0820.16004, MR 1299044

[22] Gelfand, I. M., Ponomarev, V. A., Indecomposable representations of the Lorentz group, *Russian Mathematical Surveys*, **23**:1–58, 1985. Zbl 0236.22012, MR 0229751

[23] Janusz, P., Indecomposable modules for finite groups, *Annals of Mathematicss*, **89**:209–241, 1969. Zbl 0197.02302, MR 0248242.

[24] Lukács, E., Magyar, A., Standard Koszul standardly stratified algebras, *Communications in Algebra*, **45(3)**:1270–1277, 2017. Zbl 06561311, MR 3573378

[25] Lukács, E., Magyar, A., Stratified modules over an extension algebra, *Czechoslovak Mathematical Journal*, 2017.

[26] MacLane, S., Categories for the working mathematician, *Graduate text in mathematics*, **5**, 1971.

[27] MacLane, S., Homology, *Classics in mathematics*, 1963.

[28] Magyar, A., Standard Koszul self-injective special biserial algebras, *Journal of Algebra and Its Applications*, **15(03)** 1650044 pp. 15, 2016. Zbl 06561311, MR 3454706

[29] Magyar, A., Extension algebras and the Koszul property, *Master's thesis (in Hungarian)*, 2013.

[30] Mazorchuk, V., Ovsienko, S. (with Stroppel, C.),

A pairing in homology & the category of linear complexes of tilting modules for a quasi-hereditary algebra. *J. Math. Kyoto Univ.*, **45**:711–741, 2005. Zbl 1147.16010, MR 2226627

[31] Ringel, C. M., The indecomposable representations of dihedral 2-groups, *Mathematische Annalen*, **37**:191–205, 1975. Zbl 0299.20005, MR 0364426

[32] Rotman, J., An introduction to homological algebra, *Universitext*, 2009.

[33] Skowronski, A., Waschbusch, J., Representation-finite biserial algebras, *J. Reine Angew. Math*, **345**:172–181, 1983. Zbl 0511.16021, MR 0717892

[34] J. Stovícek. Telescope conjecture, idempotent ideals, and the transfinite radical, Trans. Amer. Math. Soc., **362**:1475–1489, 2010. Zbl 1242.16007, MR 256373.

[35] Tachikawa, H., On algebras of which every indecomposable representation has an irreducible one as the top or the bottom Loewy constituent, *Mathematische Zeitschrift*, **75**:215–227, 1961. Zbl 0104.03202, MR 0124356.

[36] Webster, B., Canonical bases and higher representation theory. *Composito Mathematica*, **151**:121–166, 2015. Zbl 06417584, MR 3305310

[37] Weibel, C. An introduction to homological algebra, *Cambridge studies in advanced mathematics* **38**, 1994.