MŰEGYETEM 1782

PhD Thesis

Standard Koszul algebras

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2018

Acknowledgements

I am greatful to all of you who helped me on my way.

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Chapter 1

Introduction

Quasi-hereditary algebras were originally defined by Cline, Parshall and Scott in the context of Lie algebras and algebraic groups during their work on highest weight categories related to Lusztig's conjecture [12]. The concept of quasi-hereditary algebras was formulated in a purely ring theoretical manner as well, and these algebras appear in several situations and applications, both in Lie theory and in the theory of associative rings. For instance, the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ is a categorical sum of blocks, where each block is equivalent to a module category of a quasi-hereditary algebra (see [10]). It is also known that finite dimensional algebras with global dimension at most 2 are all quasi-hereditary (cf. [15]).

In the theory of quasi-hereditary algebras, the (left or right) (co)standard modules play a fundamental role. Namely, an algebra is quasi-hereditary if the regular module is filtered by standard modules, and in addition, all standard modules are Schurian, meaning that their endomorphism rings are division rings. Without the additional assumption on standard modules, we get the concept of standardly stratified algebras.

Inspired by the work of Cline, Parshall and Scott [13], Ágoston, Dlab and Lukács determined the conditions which ensure that a quasi-hereditary algebra has a quasi-hereditary Yoneda extension algebra (cf. [1], [2] and [4]).

In [4], the authors proved that if a quasi-hereditary algebra is standard

Koszul (i.e. every left and right standard module is Koszul), then its extension algebra is quasi-hereditary. Moreover, in [4], the authors also pointed out that the homological duality respects the stratifying structure, in the sense that the natural functor Ext_A^* maps the standard modules of A to the standard modules defined over the extension algebra A^*.

It was also proved in [4] that standard Koszul quasi-hereditary algebras are always Koszul algebras, i.e. the graph of the extension algebra is the same as the graph of the original algebra. It turned out that this implication itself is a useful tool in other situations, too (cf. [11], [16], [30] or [36]). The wide range of applications suggests that it is worth investigating this implication for more general (or different) settings.

Later, the same authors generalized their results to the case of Koszul standardly stratified algebras under the additional assumption that the initial algebra was graded [5]. They showed that the homological dual of such an algebra is standardly stratified, and the homological duality functor maps the standard right A-modules to the corresponding left proper standard modules of A^*, while the left proper standard A-modules are mapped to the right standard A^*-modules. However, since algebras in [5] were assumed to be Koszul algebras, the question if all standard Koszul standardly stratified algebras are Koszul was left open. The neccessity of the graded strucure also remained a question.

In Chapter 2, we investigate standardly stratified algebras along with their extension algebras from a perspective similar to [1], [2] and [4], and extend the above results.

In the second section of the chapter, we prove that every standard Koszul standardly stratified algebra is Koszul. The stratification enables us to do an induction on the number of simple modules, just like in the quasi-hereditary case. But the proof of Ágoston, Dlab and Lukács for quasi-hereditary algebras relied heavily on the finite global dimension of the algebra and the lack of certain extensions resulting from the Schurian property of the standard modules. So here we needed an essentially different approach.

The second part (Sections 2.3 and 2.4) proves that the extension algebra of a standard Koszul standardly stratified algebra is also standardly stratified. For this, we only need to prove that the regular module over the extension algebra is filtered by standard modules (or equivalently, by proper standard modules for the other side). We extended this to a more general question: we found easy-to-check conditions that ensure that a module is mapped by the Ext* functor to a module filtered by standard (or on the other side proper standard) modules. Both the conditions and the methods are different when we look at right or left modules of a standardly stratified algebra.

In Chapter 3, we omit the condition of stratification, and investigate the connection between the Koszul and standard Koszul property for monomial and special biserial algebras. Both classes are defined as quotients of path algebras, and they are frequently used for testing (homological) conjectures. We prove in both cases that standard Koszul algebras are always Koszul, and characterize standard Koszul algebras in these classes in terms of graphs and relations. We close the thesis with a few examples and counterexamples.

The results of the thesis were published in [24], [25], [28] and [29].

1.1 Preliminaries and notation

We will use general results from module theory without presenting proofs here. For the basic results and proofs one may refer to for example [9], [14] or [7]. The fundamentals of homological constructions we will use can be found in [27], [32] or [37]. There are also a few tools from category theory we shall employ. These are covered in [26].

Throughout the work, A always stands for a basic finite dimensional associative algebra over a field K. The Jacobson radical of A will be denoted by $J = \text{rad } A$. All considered modules are unitary and finitely generated. Modules are meant to be right modules, unless otherwise stated. The category of finitely generated left or right A-modules will be denoted by $A\text{-mod}$ and $\text{mod-}A$, respectively.

For the algebra A, we fix a complete ordered set of primitive orthogonal idempotents $\mathbf{e} = (e_1, \ldots, e_n)$. In the canonical decomposition

$$A_A = e_1 A \oplus \ldots \oplus e_n A$$

of the regular module, the ith indecomposable projective module $e_i A$ will be denoted by $P(i)$ and its simple top $P(i)/\text{rad } P(i)$ by $S(i)$. Besides, $\hat{S}$ stands for the semisimple top of A_A, so $\hat{S} = \oplus_{i=1}^{n} S(i)$. The corresponding left modules are denoted by $P^\circ(i), S^\circ(i)$ and $\hat{S}^\circ$, respectively.

If $1 \leq i \leq n$, set $\varepsilon_i = e_i + \ldots + e_n$, and $\varepsilon_{n+1} = 0$. The centralizer algebras $\varepsilon_i A \varepsilon_i$ of A will be denoted by C_i, where the idempotents and their order are naturally inherited from A. The ith *standard* and *proper standard* A-modules are $\Delta(i) = e_i A / e_i A \varepsilon_{i+1} A$ and $\overline{\Delta}(i) = e_i A / e_i (\text{rad } A) \varepsilon_i A$, respectively. That is, the ith standard module is the largest factor module of $P(i)$ which has no composition factor isomorphic to $S(j)$ if $j > i$, while the ith proper standard module is the largest factor module of $P(i)$ whose radical has no composition factor isomorphic to $S(j)$ if $j \geq i$. The left standard and proper standard modules are defined analogously. The ith *costandard* module is $\nabla(i) = D(\Delta^\circ(i))$, and the ith *proper costandard* module is $\overline{\nabla}(i) = D(\overline{\Delta}^\circ(i))$, where D stands for the usual K-duality functor $\text{Hom}_K(-, K)$ of finitely generated modules.

Let $\mathcal{X}$ be a class of modules. We say that a module X is *filtered* by $\mathcal{X}$ if there is a sequence of submodules $X = X^0 \supseteq X^1 \supseteq \ldots$ such that $\bigcap_{i\geq 0} X^i = 0$, and all the factor modules X^i/X^{i+1} are isomorphic to some modules of $\mathcal{X}$. In this case, we write $X \in \mathcal{F}(\mathcal{X})$. Given the ordered set $(e_1, \ldots, e_n)$, we can form the trace filtration of a module X with respect to the projective modules $P(i)$

$$X = X\varepsilon_1 A \supseteq X\varepsilon_2 A \supseteq \ldots \supseteq X\varepsilon_n A \supseteq 0.$$

We will simply refer to this filtration as the *trace filtration* of X. We call an algebra A (with a fixed complete ordered set $\mathbf{e}$ of primitive orthogonal idempotents) *standardly stratified* if the regular module $A_A \in \mathcal{F}(\Delta)$, or equivalently, the left regular module ${}_AA \in \mathcal{F}(\overline{\Delta}^\circ)$, where $\overline{\Delta}^\circ$ consists of the proper standard modules, while Δ consists of the left standard modules. We shall use later the fact that $\mathrm{Ext}^h_A(\Delta(i), S(j)) = 0$ for all $h \geq 0$ and $i \geq j$ when $A_A \in \mathcal{F}(\Delta)$ (cf. [12]), and similarly, $\mathrm{Ext}^h_A(\overline{\Delta}(i), S(j)) = 0$ for all $h \geq 0$ and $i > j$ when $A_A \in \mathcal{F}(\overline{\Delta})$.

A submodule $X \leq Y$ is a *top submodule* ($X \overset{t}{\leq} Y$) whenever $X \cap \operatorname{rad} Y = \operatorname{rad} X$. This is equivalent to the condition that the natural embedding of X into Y induces an embedding of $X/\operatorname{rad} X$ into $Y/\operatorname{rad} Y$ (such embeddings will be called top embeddings), or in other words, the induced map $\mathrm{Hom}_A(Y, \hat{S}) \to \mathrm{Hom}_A(X, \hat{S})$ is surjective. (See [1] for the origin of this concept.) Let

$$P_\bullet(X): \qquad \ldots \to P_h(X) \to \ldots \to P_1(X) \to P_0(X) \to X \to 0$$

be a minimal projective resolution of X with the hth syzygy $\Omega_h = \Omega_h(X)$. Using the concept of top submodules, we introduce the classes $\mathcal{C}^i_A$. The module X belongs to $\mathcal{C}^i_A$ if Ω_h is a top submodule of $\operatorname{rad} P_{h-1}$ for all $h \leq i$. We say that X has a top projective resolution, or X is *Koszul*, if $X \in \mathcal{C}_A := \bigcap_{i=1}^{\infty} \mathcal{C}^i_A$. The algebra A is a *Koszul algebra* if $\hat{S}$ (or equivalently if $\hat{S}^\circ$) has a top projective resolution (cf. [20]). We say that A is standard Koszul if $\Delta(i) \in \mathcal{C}_A$ and $\overline{\Delta}^\circ(i) \in \mathcal{C}_{A^\circ}$ for all i. Examples A.2 and A.3 explain that why we should generalize standard Koszul property this way. Besides, Example A.1 provides an example for a standard Koszul standardly stratified algebra.

The *extension algebra* (or *homological dual*) of A is the positively graded algebra A^* whose underlying vector space is $\oplus_{h\geq 0}(A^*)_h = \oplus_{h\geq 0}\operatorname{Ext}^h_A(\hat{S},\hat{S})$, and the multiplication is given by the Yoneda composition of the extensions (cf. [20]). A *graded (left) A^*-module* $X = \oplus_{h\in\mathbb{Z}}X_h$ is an A^*-module for which $(A^*)_h X_k \subseteq X_{h+k}$, and by an A^*-module homomorphism $f : X \to Y$, we mean a graded A^*-module homomorphism f having any degree $d \in \mathbb{Z}$. In this sense, we say that two graded A^*-modules X and Y are isomorphic if there exists a bijective A^*-homomorphism $f : X \to Y$ (not necessarily degree of 0). The ith graded shift of the graded A^*-module X is denoted by $X[i]$, which is a graded module such that $X[i]_h = X_{h-i}$. For graded modules, we shall also use the notation $X_{\geq i} = \oplus_{h\geq i}X_h$. We shall call the A^*-modules X and Y isomorphic if there is an isomorphism between X and $Y[i]$ in the category of graded A^*-modules with an appropriate integer i.

The functor $\operatorname{Ext}^*_A : \text{mod-}A \to A^*\text{-}\operatorname{grmod}$ is defined as the direct sum of the functors $\operatorname{Ext}^h_A(-,\hat{S})$. Namely, if $X \in \text{mod-}A$, then $\operatorname{Ext}^*_A(X)$ is the graded left module $\oplus_{h\geq 0}\operatorname{Ext}^h_A(X,\hat{S})$. For simplicity, we denote $\operatorname{Ext}^*_A(X)$ by X^*, while for its homogeneous part of degree h we write $(X^*)_h$. We use the notation $\varphi^* = \operatorname{Ext}^*_A(\varphi,\hat{S}) : \operatorname{Ext}^*_A(Y,\hat{S}) \to \operatorname{Ext}^*_A(X,\hat{S})$, where $\varphi : X \to Y$ is a module homomorphism, and we denote by E^h_X the canonical isomorphism between the spaces $\operatorname{Hom}_A(\Omega_h(X),\hat{S})$ and $\operatorname{Ext}^h_A(X,\hat{S})$. Thus we have the commutative diagram

$$\begin{array}{ccc}
\operatorname{Hom}_A(\Omega_h(Y),\hat{S}) & \xrightarrow{(\widetilde{\varphi}_{h-1})^*} & \operatorname{Hom}_A(\Omega_h(X),\hat{S}) \\
\downarrow E^h_Y & & \downarrow E^h_X \\
\operatorname{Ext}^h_A(Y,\hat{S}) & \xrightarrow{\varphi^*} & \operatorname{Ext}^h_A(X,\hat{S})
\end{array}$$

of left $(A^*)_0$-modules, where $\varphi_\bullet : P_\bullet(X) \to P_\bullet(Y)$ is a lifting of φ, while $\widetilde{\varphi}_{h-1}$ is the restriction of φ_{h-1} to the submodule $\Omega_h(X) \subseteq P_{h-1}(X)$.

Note that the module X has a top projective resolution if and only if $(X^*)_h = \operatorname{Ext}^h_A(X,\hat{S}) = (A^*)^h_1 \cdot (X^*)_0$ for all $h \geq 0$. In particular, if A is Koszul, then A^* is *tightly graded*, i.e. $\operatorname{Ext}^h_A(\hat{S},\hat{S}) = (\operatorname{Ext}^1_A(\hat{S},\hat{S}))^h$ for $h \geq 1$ (cf.

[20]).

Let $(e_1, \ldots, e_n)$ be a complete ordered set of primitive orthogonal idempotents of A. The set $\{f_i = \mathrm{id}_{S(i)} \mid 1 \leq i \leq n\}$ defines a complete set of primitive orthogonal idempotents in A^*. We will always consider this set with the opposite order $\mathbf{f} = (f_n, \ldots, f_1)$. In this way, the ith standard A^*-module $\Delta_{A^*}(i)$ is defined as $\Delta_{A^*}(i) = f_iA^*/f_iA^*(f_1 + \ldots + f_{i-1})A^*$, while the ith proper standard module is given by $\overline{\Delta}_{A^*}(i) = f_iA^*/f_i(A^*)_{\geq 1}(f_1 + \ldots + f_i)A^*$. The definitions of left standard and proper standard modules are analogous. The algebra A^* is standardly stratified if $A^*_{A^*}$ is filtered by right standard A^*-modules. In view of Theorem 1 of [5], if A^* is tightly graded, then this is equivalent to the condition that ${}_{A^*}A^*$ is filtered by left proper standard A^*-modules.

Chapter 2

Standardly stratified algebras

This chapter is focusing on the extension algebras of standard Koszul standardly stratified algebras. The main goal is to generalize the results of Ágoston, Dlab and Lukács on quasi-hereditary algebras [4]. As in the quasi-hereditary case, the proofs use an induction on the number of simple modules. In the induction procedure, the centralizer algebras play an important role.

The first section contains some technical results about top embeddings in a somewhat more general setting, in module categories over lean algebras.

2.1 Lean algebras

Among quasi-hereditary algebras, lean algebras are those which satisfy the condition $e_iJ^2e_j = e_iJ\varepsilon_mJe_j$ for every i, j with $m = \min\{i, j\}$, or equivalently, $\varepsilon_iJ^2\varepsilon_i = \varepsilon_iJ\varepsilon_iJ\varepsilon_i$ for all i. It was shown in [1] that this condition is equivalent to saying that $\Delta(i) \in \mathcal{C}^1_A$ and $\Delta(j)^\circ \in \mathcal{C}^1_{A^\circ}$ for every i, j. Closely following the proof, we get the next statement about an analogue of lean algebras in a more general setting. We note that the algebras in the section are not assumed to be standardly stratified.

Lemma 2.1.1. *The algebra A satisfies the condition that $\Delta(i) \in \mathcal{C}^1_A$ and $\overline{\Delta}(j)^\circ \in \mathcal{C}^1_{A^\circ}$ for every i, j if and only if $e_iJ^2e_j = e_iJ\varepsilon_mJe_j$ for every i, j with $m = \min\{i + 1, j\}$.*

Proof. $\Delta(i) \in \mathcal{C}^1_A \iff e_iA\varepsilon_{i+1}A = e_iJ\varepsilon_{i+1}A \overset{t}{\leq} e_iJ \iff e_iJ\varepsilon_{i+1}A \cap e_iJ^2 \subseteq e_iJ\varepsilon_{i+1}J \iff e_iJ\varepsilon_{i+1}Ae_j \cap e_iJ^2e_j \subseteq e_iJ\varepsilon_{i+1}Je_j\ \forall j$. For $j \leq i$, $e_iJ\varepsilon_{i+1}Ae_j = e_iJ\varepsilon_{i+1}Je_j$, so the last inclusion is always true for $j \leq i$. On the other hand, for $j \geq i+1$, we have $e_iJ\varepsilon_{i+1}Ae_j = e_iJe_j \supseteq e_iJ^2e_j$, so

$$\Delta(i) \in \mathcal{C}^1_A \iff e_iJ^2e_j \subseteq e_iJ\varepsilon_{i+1}Je_j\ \forall j.$$

$\overline{\Delta}^{\circ}(j) \in \mathcal{C}^1_{A^\circ} \iff A\varepsilon_jJe_j \overset{t}{\leq} Je_j \iff A\varepsilon_jJe_j \cap J^2e_j \subseteq J\varepsilon_jJe_j \iff e_iA\varepsilon_jJe_j \cap e_iJ^2e_j \subseteq e_iJ\varepsilon_jJe_j\ \forall i$. For $i < j$, $e_iA\varepsilon_jJe_j = e_iJ\varepsilon_jJe_j$, so the last inclusion is always true for $i < j$. On the other hand, for $i \geq j$, we have $e_iA\varepsilon_jJe_j = e_iJe_j \supseteq e_iJ^2e_j$, so

$$\overline{\Delta}^{\circ}(j) \in \mathcal{C}^1_{A^\circ} \iff e_iJ^2e_j \subseteq e_iJ\varepsilon_jJe_j\ \forall i.$$

The combination of the two conditions (and the trivial reverse inclusion) gives the statement of the lemma. □

In particular, standard Koszul algebras satisfy the condition of the previous lemma. As a consequence, we get a useful feature of standard Koszul algebras in terms of the idempotents ε_i.

Corollary 2.1.2. *If $\Delta(i) \in \mathcal{C}^1_A$ and $\overline{\Delta}(j)^\circ \in \mathcal{C}^1_{A^\circ}$ for every i, j (in particular, if A is standard Koszul), then $\varepsilon_iJ^2\varepsilon_i = \varepsilon_iJ\varepsilon_iJ\varepsilon_i$ for every i.*

We call an algebra A with a fixed ordered set $(e_1, \ldots, e_n)$ of primitive idempotents lean if $\varepsilon_iJ^2\varepsilon_i = \varepsilon_iJ\varepsilon_iJ\varepsilon_i$ for every i. The previous corollary showed that all standard Koszul algebras are lean.

The next few lemmas will be useful in finding connection between top embeddings over A and those over its centralizer algebras (cf. [1]).

Lemma 2.1.3. *If $X \leq Y \leq Z$, and $X \overset{t}{\leq} Z$, then*

(1) $X \overset{t}{\leq} Y$;

(2) $Y \overset{t}{\leq} Z \Leftrightarrow Y/X \overset{t}{\leq} Z/X$.

Lemma 2.1.4. *Let ε be an idempotent in A, and $X \leq Y$ be A-modules such that $X = X\varepsilon A$ and $Y = Y\varepsilon A$. Then*

(1) $X \overset{t}{\leq} Y \Leftrightarrow X\varepsilon \overset{t}{\leq} Y\varepsilon$ *in mod-*$\varepsilon A\varepsilon$.

(2) If we also assume that $\varepsilon J^2\varepsilon = \varepsilon J\varepsilon J\varepsilon$, *then* $X \overset{t}{\leq} \operatorname{rad} Y \Leftrightarrow X\varepsilon \overset{t}{\leq} \operatorname{rad} Y\varepsilon$ *in mod-*$\varepsilon A\varepsilon$.

Proof. If $X \cap YJ \subseteq XJ$, then $X\varepsilon \cap Y\varepsilon J\varepsilon = (X \cap YJ)\varepsilon = XJ\varepsilon = \operatorname{rad} X\varepsilon$. Conversely, if $X\varepsilon \cap YJ\varepsilon = X\varepsilon J\varepsilon$, then $(X \cap YJ)\varepsilon = X\varepsilon \cap YJ\varepsilon = X\varepsilon J\varepsilon \subseteq XJ$, while $(X \cap YJ)(1-\varepsilon) \subseteq X\varepsilon A(1-\varepsilon) \subseteq XJ$, so $X \cap YJ \subseteq XJ$.

The second statement is contained in Lemma 1.6 of [4]. □

Lemma 2.1.5. *Let* $\varepsilon \in A$ *be an idempotent element. Suppose that* $X \leq Y$ *are two A-modules such that* $X\varepsilon A = 0$. *Then*

$$(Y/X)\varepsilon A \overset{t}{\leq} Y/X \Leftrightarrow Y\varepsilon A \overset{t}{\leq} Y.$$

Proof. Rewrite the condition $Y\varepsilon A \overset{t}{\leq} Y$ as $Y\varepsilon A \cap YJ \subseteq Y\varepsilon J$ and the condition $(Y/X)\varepsilon A \overset{t}{\leq} Y/X$ as $Y\varepsilon A \cap (YJ+X) \subseteq Y\varepsilon J + X$. Since $Y\varepsilon A(1-\varepsilon) \subseteq Y\varepsilon J$ and $X\varepsilon = 0$, both of the previous inclusions are equivalent to $Y\varepsilon \cap YJ\varepsilon \subseteq Y\varepsilon J\varepsilon$. □

Lemma 2.1.6. *Let* $0 \to X \to Y \to Z \to 0$ *be a short exact sequence with* $X\varepsilon A \overset{t}{\leq} Y$. *Then* $Y\varepsilon A \overset{t}{\leq} Y$ *if and only if* $Z\varepsilon A \overset{t}{\leq} Z$.

Proof. Take the factors of X and Y by $X\varepsilon A$ to get

$$0 \to \overline{X} \to \overline{Y} \to Z \to 0.$$

By Lemma 2.1.3, $Y\varepsilon A \overset{t}{\leq} Y$ if and only if $\overline{Y}\varepsilon A \overset{t}{\leq} \overline{Y}$. Since $\overline{X}\varepsilon A = 0$, the latter is equivalent to $Z\varepsilon A \overset{t}{\leq} Z$ by Lemma 2.1.5. □

We shall need a generalized version of Lemma 2.1.6.

Lemma 2.1.7. *Let* ε *be an idempotent in A. Suppose that the following commutative diagram has exact rows and columns.*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_1 & \longrightarrow & Y_1 & \xrightarrow{\alpha_1} & Z_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \xrightarrow{\alpha} & Z & \longrightarrow & 0
\end{array}$$

If $X_1\varepsilon A \overset{t}{\leq} Y$ *and* $Z_1\varepsilon A \overset{t}{\leq} Z$*, then* $Y_1\varepsilon A \overset{t}{\leq} Y$.

Proof. We may assume that $X_1\varepsilon A = 0$ because otherwise we can substitute the modules X_1, X, Y_1 and Y with their factors by the (top) submodule $X_1\varepsilon A$. In the new diagram, the same embeddings will be top embeddings as in the original by Lemma 2.1.3. Then

$$X_1 \cap Y_1\varepsilon A = X_1(1-\varepsilon) \cap Y_1\varepsilon A \subseteq Y_1\varepsilon A(1-\varepsilon) \subseteq Y_1\varepsilon J.$$

The assumption $Z_1\varepsilon A \cap ZJ \subseteq Z_1\varepsilon J$ implies that

$$(Y_1\varepsilon A \cap YJ)\alpha_1 \subseteq (Y_1\varepsilon A)\alpha_1 \cap (YJ)\alpha = Z_1\varepsilon A \cap ZJ \subseteq Z_1\varepsilon J = (Y_1\varepsilon J)\alpha_1,$$

so $Y_1\varepsilon A \cap YJ \subseteq Y_1\varepsilon J + X_1$, thus

$$Y_1\varepsilon A \cap YJ \subseteq Y_1\varepsilon A \cap (X_1 + Y_1\varepsilon J) = (Y_1\varepsilon A \cap X_1) + Y_1\varepsilon J \subseteq Y_1\varepsilon J,$$

giving that $Y_1\varepsilon A \overset{t}{\leq} Y$. □

Remark 2.1.8. Note that the "reverse" of Lemma 2.1.7 does not hold in general. Let $X \leq Y$ and suppose that X is not a top submodule of Y. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} & & \\
0 & \longrightarrow & X & \longrightarrow & X \oplus Y & \longrightarrow & Y & \longrightarrow & 0
\end{array}$$

with exact rows and columns, where β is the diagonal map and the bottom row splits. Here, β is a top embedding but α is not.

Finally, we would like to recall Lemma 1.7 from [4] about the connection between Koszul A- and $\varepsilon A\varepsilon$-modules.

Lemma 2.1.9. *Suppose that* ε *is an idempotent of* A *such that* $\varepsilon J^2\varepsilon = \varepsilon J\varepsilon J\varepsilon$*, and let* X *be a module with* $\mathrm{Ext}_A^t(X, \mathrm{top}\,((1-\varepsilon)A)) = 0$ *for all* $t \geq 0$*. Then* $X \in \mathcal{C}_A$ *if and only if* $X\varepsilon \in \mathcal{C}_{\varepsilon A\varepsilon}$.

2.2 Standard Koszul standardly stratified algebras

In this section, we turn our attention to standardly stratified algebras, and prove that if such an algebra is standard Koszul then it is also Koszul.

Lemma 2.2.1. *Suppose that A is a standard Koszul standardly stratified algebra. Then its centralizer algebra $C_2 = \varepsilon_2 A \varepsilon_2$ is again standard Koszul and standardly stratified, its standard and left proper standard modules are $\Delta(i)\varepsilon_2$ and $\varepsilon_2 \overline{\Delta}^\circ(i)$ for $i \geq 2$.*

Proof. Observe that $(\varepsilon_2 A \varepsilon_2)e_n(\varepsilon_2 A \varepsilon_2) = \varepsilon_2(Ae_nA)\varepsilon_2$ is a projective C_2-module, since Ae_nA is the direct sum of copies of e_nA, and $\varepsilon_2 e_n A \varepsilon_2 = e_n C_2$. So C_2 is standardly stratified because $\varepsilon_2 A \varepsilon_2 / \varepsilon_2 A e_n A \varepsilon_2 \cong \varepsilon_2(A/Ae_nA)\varepsilon_2$ as algebras. It is also easy to check that the standard modules $\Delta_{C_2}(i)$ and the left proper standard modules $\overline{\Delta}^\circ_{C_2}(i)$ $(i \geq 2)$ over C_2 are isomorphic to the modules $\Delta(i)\varepsilon_2$ and $\varepsilon_2 \overline{\Delta}^\circ(i)$, respectively. The Koszul property of the modules $\Delta(i)\varepsilon_2$ and $\varepsilon_2 \overline{\Delta}^\circ(i)$ follows from Lemma 2.1.9, since $\mathrm{Ext}^t_A(\Delta(i), S(1)) = 0 = \mathrm{Ext}^t_A(\overline{\Delta}^\circ(i), S^\circ(1))$ for any $t \geq 0$ and $i \geq 2$. □

Let S be a semisimple A-module. As in Definition 1.8 of [4], a module X is called *S-Koszul*, if $\mathrm{Ext}^t_A(X, S) \subseteq \mathrm{Ext}^1_A(\hat{S}, S) \cdot \mathrm{Ext}^{t-1}_A(X, \hat{S})$ for all $t \geq 1$, or equivalently, the trace of S in the top of the syzygy $\Omega_t(X)$ is mapped injectively into the top of rad $P_{t-1}(X)$ for every $t \geq 1$. In other words, X is S-Koszul if and only if $\Omega_t(X)\varepsilon_S A \cap P_{t-1}(X)J^2 \subseteq \Omega_t(X)J$ for every $t \geq 1$, where $\varepsilon_S = \sum\{e_i \mid Se_i \neq 0\}$.

Lemma 2.2.2. *A module X is Koszul if and only if X is $S(1)$-Koszul and $\Omega_t(X)\varepsilon_2 A \cap P_{t-1}(X)J^2 \subseteq \Omega_t(X)J$ for all $t \geq 1$, i. e. X is both $S(1)$- and $\oplus_{i \geq 2} S(i)$-Koszul.*

Proof. For $X \leq Y$, the condition $X \cap YJ \subseteq XJ$ holds if and only if $Xe_1A \cap YJ \subseteq XJ$ and $X\varepsilon_2 A \cap YJ \subseteq XJ$. □

Corollary 2.2.3. *If the module X is $S(1)$-Koszul, and $\Omega_t(X)\varepsilon_2 A$ is a top submodule in* rad $P_{t-1}(X)$ *for all $t \geq 0$, then $X \in \mathcal{C}_A$.*

Now let us take the subclass $\mathcal{K}$ of A-modules

$$\mathcal{K} = \left\{ X \mid X \text{ is } S(1)\text{-Koszul}, \ X\varepsilon_2 A \overset{t}{\leq} X, \ X\varepsilon_2 \in \mathcal{C}_{C_2} \right\}.$$

As in the case of quasi-hereditary algebras in [4], we plan to show that all modules in $\mathcal{K}$ are Koszul, and the simple modules belong to $\mathcal{K}$. First we investigate modules without the additional $S(1)$-Koszul property:

$$\mathcal{K}_2 = \left\{ X \mid X\varepsilon_2 A \overset{t}{\leq} X, \ X\varepsilon_2 \in \mathcal{C}_{C_2} \right\}.$$

We fix some notation for the upcoming lemmas. For any A-module X, let $\mathcal{P}(X)$ and $\Omega(X)$ denote the projective cover and the first syzygy of X, respectively, while $\widetilde{X}$ will stand for the submodule $X\varepsilon_2 A$, and $\overline{X}$ for the respective factor module $X/X\varepsilon_2 A$.

Remark 2.2.4. We shall motivate our choice of $\mathcal{K}_2$. We point it out that for a quasi-hereditary algebra A, the classes $\mathcal{K}_2$ and $\mathcal{K}$ are the same if A is standard Koszul. This is not true for standardly stratified algebras as we show in Example A.4.

For the rest of the section, A is always assumed to be a standard Koszul standardly stratified algebra.

Lemma 2.2.5. *If X is an A-module for which $X\varepsilon_2 A = 0$, then $\Omega(X)\varepsilon_2 A$ is a top submodule of* $\operatorname{rad} \mathcal{P}(X)$.

Proof. Since $X\varepsilon_2 A = 0$, the projective cover $\mathcal{P}(X)$ is isomorphic to $\oplus P(1)$, and $\Omega(X)\varepsilon_2 A = (\operatorname{rad} \oplus P(1))\varepsilon_2 A = \oplus P(1)\varepsilon_2 A \overset{t}{\leq} \operatorname{rad} \oplus P(1)$ as $\Delta(1) \in \mathcal{C}_A^1$. □

Lemma 2.2.6. *Let X be an arbitrary A-module. If $X \in \mathcal{K}_2$, then $\Omega(\widetilde{X}) \in \mathcal{K}_2$. Moreover, $\Omega(\widetilde{X})\varepsilon_2 A \overset{t}{\leq} \operatorname{rad} P(\widetilde{X})$.*

Proof. Take the minimal projective resolution $0 \to \Omega(\widetilde{X}) \to \mathcal{P}(\widetilde{X}) \to \widetilde{X} \to 0$ of $\widetilde{X}$, and apply the exact functor $\operatorname{Hom}_A(\varepsilon_2 A, -)$ to get the short exact sequence $0 \to \Omega(\widetilde{X})\varepsilon_2 \to \mathcal{P}(\widetilde{X})\varepsilon_2 \to \widetilde{X}\varepsilon_2 \to 0$ of C_2-modules. Since $\widetilde{X} = \widetilde{X}\varepsilon_2 A$, the projective module $\mathcal{P}(\widetilde{X})\varepsilon_2$ is the projective cover of $\widetilde{X}\varepsilon_2$. But $X \in \mathcal{K}_2$ gives that $\widetilde{X}\varepsilon_2 = X\varepsilon_2$ belongs to $\mathcal{C}_{C_2}$, together with its syzygy $\Omega(\widetilde{X})\varepsilon_2$.

Furthermore, $\Omega(\widetilde{X})\varepsilon_2 \overset{t}{\leq}$ rad $\mathcal{P}(\widetilde{X})\varepsilon_2$, so by Lemma 2.1.4, $\Omega(\widetilde{X})\varepsilon_2 A \overset{t}{\leq}$ rad $\mathcal{P}(\widetilde{X})$, and also $\Omega(\widetilde{X})\varepsilon_2 A \overset{t}{\leq} \Omega(\widetilde{X})$ by Lemma 2.1.3. □

Proposition 2.2.7. *The class $\mathcal{K}_2$ is closed under syzygies, that is if $X \in \mathcal{K}_2$, then $\Omega(X)\varepsilon_2 A \overset{t}{\leq} \Omega(X)$ and $\Omega(X)\varepsilon_2 \in \mathcal{C}_{C_2}$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega(\widetilde{X}) & \longrightarrow & \Omega(X) & \longrightarrow & \Omega(\overline{X}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{P}(\widetilde{X}) & \longrightarrow & \mathcal{P}(X) & \longrightarrow & \mathcal{P}(\overline{X}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{X} & \longrightarrow & X & \longrightarrow & \overline{X} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2.1}$$

with exact rows and columns.

The condition $\widetilde{X} \overset{t}{\leq} X$ implies that top $X \cong$ top $\widetilde{X} \oplus$top $\overline{X}$, so the projective module $\mathcal{P}(X)$ in the middle of the diagram is indeed the projective cover of X.

By Lemma 2.2.6, $\Omega(\widetilde{X})\varepsilon_2 A \overset{t}{\leq}$ rad $\mathcal{P}(\widetilde{X})$ and $\Omega(\widetilde{X})\varepsilon_2 A \overset{t}{\leq} \Omega(\widetilde{X})$. The former implies $\Omega(\widetilde{X})\varepsilon_2 A \overset{t}{\leq}$ rad $\mathcal{P}(X)$ because the middle row splits, so we also have $\Omega(\widetilde{X})\varepsilon_2 A \overset{t}{\leq} \Omega(X)$ by Lemma 2.1.3. We can apply Lemma 2.1.6 to the first row of the diagram to get $\Omega(X)\varepsilon_2 A \overset{t}{\leq} \Omega(X)$, as $\Omega(\overline{X})\varepsilon_2 A \overset{t}{\leq} \Omega(\overline{X})$ holds by Lemma 2.2.5 and Lemma 2.1.3. The first statement of the lemma is now proved.

Let us apply the functor $\mathrm{Hom}_A(\varepsilon_2 A, -)$ to the first row and the third column of diagram (2.1).

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & \Omega(\widetilde{X})\varepsilon_2 & \longrightarrow & \Omega(X)\varepsilon_2 & \longrightarrow & \Omega(\overline{X})\varepsilon_2 & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & \mathcal{P}(\overline{X})\varepsilon_2 & & \\ & & & & & & \downarrow & & \\ & & & & & & \overline{X}\varepsilon_2 & & \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

It is clear that both the row and the column are exact. As $\overline{X}\varepsilon_2 = 0$, the modules $\Omega(\overline{X})\varepsilon_2$ and $\mathcal{P}(\overline{X})\varepsilon_2$ are isomorphic, and the latter can be written in the form $\oplus P(1)\varepsilon_2$. The module $\oplus P(1)\varepsilon_2$ is Koszul because $\Delta(1)$, and also its syzygy, $P(1)\varepsilon_2 A$ are Koszul modules satisfying the conditions of Lemma 2.1.9. So $\Omega(\overline{X})\varepsilon_2$ is a Koszul C_2-module.

Let us observe also that $\Omega(\widetilde{X})\varepsilon_2$ is Koszul by Lemma 2.2.6, so the first and the last terms of the exact sequence

$$0 \to \Omega(\widetilde{X})\varepsilon_2 \to \Omega(X)\varepsilon_2 \to \Omega(\overline{X})\varepsilon_2 \to 0$$

are Koszul. Besides, we have seen that $\Omega(\widetilde{X})\varepsilon_2 A \overset{t}{\leq} \Omega(X)$, thus Lemmas 2.1.3 and 2.1.4 give that the map $\Omega(\widetilde{X})\varepsilon_2 \to \Omega(X)\varepsilon_2$ is a top embedding. So by Lemma 2.4 of [2], the C_2-module $\Omega(X)\varepsilon_2$ is also Koszul. □

Proposition 2.2.8. *All modules in $\mathcal{K}_2$ are $\oplus_{i\geq 2}S(i)$-Koszul.*

Proof. In view of the previous proposition and Corollary 2.2.3, it suffices to prove that $X \in \mathcal{K}_2$ implies $\Omega(X)\varepsilon_2 A \overset{t}{\leq} \operatorname{rad} \mathcal{P}(X)$, and the rest will follow by induction. Let $X \in \mathcal{K}_2$, and take a look at diagram (2.1) again. Since – as we noted – its middle row is split exact, we have the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega(\widetilde{X}) & \longrightarrow & \Omega(X) & \longrightarrow & \Omega(\overline{X}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \text{rad } \mathcal{P}(\widetilde{X}) & \longrightarrow & \text{rad } \mathcal{P}(X) & \longrightarrow & \text{rad } \mathcal{P}(\overline{X}) & \longrightarrow & 0 \end{array}$$

with exact rows and columns, where the vertical arrows are the natural induced homomorphisms.

We saw in the proof of Proposition 2.2.7 that $\Omega(\widetilde{X})\varepsilon_2 A \overset{t}{\leq} \text{rad } \mathcal{P}(X)$, while Lemma 2.2.5 implies $\Omega(\overline{X})\varepsilon_2 A \overset{t}{\leq} \text{rad } \mathcal{P}(\overline{X})$. So $\Omega(X)\varepsilon_2 A \overset{t}{\leq} \text{rad } \mathcal{P}(X)$ by Lemma 2.1.7. □

Corollary 2.2.9. *If $X \in \mathcal{K}$, then X is a Koszul module.*

Theorem 2.2.10. *All standard Koszul standardly stratified algebras are Koszul.*

Proof. We prove the theorem by induction on the number of simple modules. Since C_2 is a standard Koszul standardly stratified algebra by Lemma 2.2.1, C_2 is also Koszul by the induction hypothesis, thus every simple module is in $\mathcal{K}_2$. So by Corollary 2.2.9, we only need to prove that all simple modules are $S(1)$-Koszul.

As $S^\circ(1) = \overline{\Delta}^\circ(1)$ is in $\mathcal{C}_{A^\circ}$, for an arbitrary $t \geq 1$,

$$\text{Ext}_A^t(S^\circ(1), \hat{S}^\circ) \subseteq \text{Ext}_A^{t-1}(\hat{S}^\circ, \hat{S}^\circ) \cdot \text{Ext}_A^1(S^\circ(1), \hat{S}^\circ).$$

Applying the K-duality functor, we get that

$$\text{Ext}_A^t(\hat{S}, S(1)) \subseteq \text{Ext}_A^1(\hat{S}, S(1)) \cdot \text{Ext}_A^{t-1}(\hat{S}, \hat{S})$$

for all $t \geq 1$, which finishes the proof. □

Remark 2.2.11. In view of Lemma 2.2.1, we also obtained that a standard Koszul standardly stratified algebra is also *recursively Koszul* in the sense of [4].

2.3 The extension algebra of a standard Koszul standardly stratified algebra

We continue our study with the investigation of the extension algebra of a standard Koszul standardly stratified algebra. Principally, our aim is to show that the extension algebra of a standard Koszul standardly stratified algebra is standardly stratified. The lack of left-right symmetry apparent in this case makes it neccessary to handle left and right modules differently. We explore wide classes of right and left modules including the right standard and left proper standard modules and whose images under the natural functor Ext_A^* are filtered by proper standard and standard modules, respectively.

We will show in Section 2.3.1 that for certain A-modules, the functors $\mathrm{Hom}(\varepsilon_i A, -) : \text{mod-}A \to \text{mod-}\varepsilon_i A \varepsilon_i$ and the trace filtration (corresponding to the projective left A^*-modules) of the Ext_A^*-images of these modules are closely related when A or A° is standard Koszul and standardly stratified. After a short preparatory section (2.3.2), the refinement of this filtration is handled separately for the two cases in subsections 2.3.3 and 2.3.4. In both cases we define sufficiently large classes of modules (which contain simple and standard or proper standard modules, and are closed under top extensions), whose elements are mapped by Ext_A^* to $\overline{\Delta}^\circ$- or Δ-filtered A^*-modules. In particular, ${}_{A^*}A^*$ and $A^*_{A^*}$ prove to be $\overline{\Delta}^\circ$- and Δ-filtered, respectively. For an illustrative example, the reader should refer to Example A.1.

We recall the two fundamental results of the previous section, which comprises the statements of Lemma 2.2.1 and Theorem 2.2.10.

Theorem 2.3.1. *If A is a standard Koszul standardly stratified algebra, then A is Koszul. Furthermore, the centralizer algebras C_i are also standard Koszul and standardly stratified algebras, moreover, $\Delta_{C_i}(j) \cong \Delta_A(j)\varepsilon_i$ and $\overline{\Delta}^\circ_{C_i}(j) \cong \varepsilon_i \overline{\Delta}^\circ_A(j)$ for all $j \geq i$.*

2.3.1 Stratification of modules over A^*

In this section, we examine modules over the extension algebra of a lean algebra, that is an algebra which satisfies the condition $\varepsilon_i J^2 \varepsilon_i = \varepsilon_i J \varepsilon_i J \varepsilon_i$ for all i. In particular, it follows from Corollary 2.1.2 that A is lean if A or A° is standard Koszul. We should also note that the centralizer algebras $\varepsilon_i A \varepsilon_i$ of A are also lean if A is lean.

We recall the definition of subclasses

$$\mathcal{K}_2 = \left\{ X \in \text{mod-}A \mid X\varepsilon_2 A \overset{t}{\leq} X, X\varepsilon_2 \in \mathcal{C}_{C_2} \right\} \text{ and } \mathcal{K} = \mathcal{K}_2 \cap \mathcal{C}_A,$$

from Section 2.2. (We shall use the notation $\mathcal{K}_A$, when we need to specify the algebra.) We also introduce a recursive version $r\mathcal{K} \subset \mathcal{K}$ of $\mathcal{K}$ as

$$r\mathcal{K} = \left\{ X \in \mathcal{K} \mid X\varepsilon_i \in \mathcal{K}_{C_i} \text{ for all } i \right\}.$$

Although $\mathcal{K}_2$ was originally defined for standard Koszul standardly stratified algebras, several useful features are preserved in this more general setting.

For an arbitrary module X, we write $\widetilde{X} = X\varepsilon_2 A$ and $\overline{X} = X/\widetilde{X}$. Let the operator ω : mod-$A \to$ mod-A be defined by $\omega(X) = \Omega(\widetilde{X})$. If $h \geq 1$, then $\omega_h(X)$ stands for $\omega(\omega_{h-1}(X))$, while we denote the submodule $\omega_h(X)\varepsilon_2 A$ by $\widetilde{\omega}_h(X)$, and set $\omega_0(X) = X$.

Lemma 2.3.2. *Suppose that $X = X\varepsilon_2 A \in$ mod-A. Let $P_\bullet(X)$ denote a minimal projective resolution of X, and let $P_\bullet(X\varepsilon_2)$ denote a minimal projective resolution of the C_2-module $X\varepsilon_2$. If $u_\bullet : P_\bullet(X\varepsilon_2) \to P_\bullet(X)\varepsilon_2$ is a lifting of $\text{id}_{X\varepsilon_2}$, then $\widetilde{u}_0 = u_0|_{\Omega(X\varepsilon_2)}\colon \Omega(X\varepsilon_2) \to \Omega(X)\varepsilon_2$ is an isomorphism.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega(X\varepsilon_2) & \longrightarrow & P(X\varepsilon_2) & \longrightarrow & X\varepsilon_2 & \longrightarrow & 0 \\ & & \downarrow \widetilde{u}_0 & & \downarrow u_0 & & \| & & \\ 0 & \longrightarrow & \Omega(X)\varepsilon_2 & \longrightarrow & P(X)\varepsilon_2 & \longrightarrow & X\varepsilon_2 & \longrightarrow & 0 \end{array}$$

with exact rows. As $X = X\varepsilon_2 A$, it follows that $P(X) = P(X)\varepsilon_2 A$, and so $P(X)\varepsilon_2$ is a also projective cover of $X\varepsilon_2$. Thus u_0 and $\widetilde{u}_0$ are isomorphisms. $\square$

Lemma 2.3.3. *Suppose that A is a lean algebra, and $X \leq Y$ are A-modules such that $X\varepsilon_2 \in \mathcal{C}_{C_2}$ and the natural embedding $\varphi : \widetilde{X} \to Y$ is a top embedding. If $\varphi_\bullet : P_\bullet(\widetilde{X}) \to P_\bullet(Y)$ is a lifting of φ, then $\widetilde{\varphi}_0 = \varphi_0|_{\widetilde{\omega}(X)}\colon \widetilde{\omega}(X) \to \Omega(Y)$ is also a top embedding. Consequently, $\widetilde{\omega}(X) \overset{t}{\leq} \omega(X)$.*

Proof. By the horseshoe lemma we have the commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \omega(X) & \longrightarrow & \Omega(Y) & \longrightarrow & \Omega(Z) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P(\widetilde{X}) & \xrightarrow{\varphi_0} & P(Y) & \longrightarrow & P(Z) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{X} & \xrightarrow{\varphi} & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the middle column is also a projective cover because φ is a top embedding. In view of Lemma 2.3.2, $\widetilde{\omega}(X)\varepsilon_2 \cong \Omega(X\varepsilon_2)$, so $X\varepsilon_2 \in \mathcal{C}_{C_2}$ implies that $\widetilde{\omega}(X)\varepsilon_2$ is a top submodule of $P(X\varepsilon_2)(\varepsilon_2 J \varepsilon_2) = P(\widetilde{X})J\varepsilon_2$, thus by Lemma 2.1.4 (2), $\widetilde{\omega}(X) \overset{t}{\leq} P(\widetilde{X})J$. On the other hand, φ_0 is a split monomorphism, so $P(\widetilde{X})J \overset{t}{\leq} P(Y)J$, giving $\varphi_0(\widetilde{\omega}(X)) \overset{t}{\leq} P(Y)J$. Since

$$\varphi_0(\widetilde{\omega}(X)) \subseteq \varphi_0(\omega(X)) \subseteq \Omega(Y) \subseteq P(Y)J,$$

we get $\widetilde{\varphi}_0(\widetilde{\omega}(X)) \overset{t}{\leq} \Omega(Y)$ and $\widetilde{\omega}(X) \overset{t}{\leq} \omega(X)$. □

Corollary 2.3.4. *If A is lean and $X \in \mathcal{K}_2$, then $\omega(X) \in \mathcal{K}_2$.*

Proof. We apply Lemma 2.3.3 with $Y = X$, and Lemma 2.3.2. □

Proposition 2.3.5. *If A is lean, then the classes $\mathcal{K}_2, \mathcal{K}$, and $r\mathcal{K}$ are closed under top extensions. That is, if*

$$0 \to X \overset{t}{\to} Y \to Z \to 0$$

is an exact sequence with top embedding, and both X and Z are in one of these classes, then Y is in the same class.

Proof. Since $\widetilde{X} \overset{t}{\leq} X \overset{t}{\leq} Y$ and $\widetilde{Z} \overset{t}{\leq} Z$, by Lemma 2.1.6, $\widetilde{Y} \overset{t}{\leq} Y$. Besides, $\widetilde{X} \overset{t}{\leq} Y$ also gives that $X\varepsilon_2 \overset{t}{\leq} Y\varepsilon_2$, so $Y\varepsilon_2$ is a top extension of the Koszul modules $X\varepsilon_2$ and $Z\varepsilon_2$, thus $Y\varepsilon_2 \in \mathcal{C}_{C_2}$ by Lemma 2.4 of [2]. Hence we get that the class $\mathcal{K}_2$ is closed under top extensions; and this also implies the same condition for $\mathcal{K} = \mathcal{K}_2 \cap \mathcal{C}_A$. To prove the statement for $r\mathcal{K}$, we can use the previous argument recursively for $X\varepsilon_i$ and $Z\varepsilon_i$. □

Proposition 2.3.6. *Suppose that $\varepsilon_2 J^2 \varepsilon_2 = \varepsilon_2 J \varepsilon_2 J \varepsilon_2$. If $X \in \mathcal{K}_2$, then for every $h \geq 0$ we have an exact sequence*

$$0 \to \widetilde{\omega}_h(X) \xrightarrow{\alpha_h} \Omega_h(X) \xrightarrow{\beta_h} Y_h(X) \to 0 \tag{2.2}$$

with α_h a top embedding.

Proof. Fix an A-module $X \in \mathcal{K}_2$, and consider the embeddings $e^h : \widetilde{\omega}_h(X) \to \omega_h(X)$. For $h \geq 0$ let $e^h_\bullet : P_\bullet(\widetilde{\omega}_h(X)) \to P_\bullet(\omega_h(X))$ denote a lifting of e^h (and also its restriction to $\Omega_{\bullet+1}(\widetilde{\omega}_h(X)) \subseteq P_\bullet(\widetilde{\omega}_h(X))$). Using Lemma 2.3.3 and Corollary 2.3.4, an induction on h shows that α_h as the composition of morphisms

$$\widetilde{\omega}_h(X) \xrightarrow{e^h} \omega_h(X) = \Omega_1(\widetilde{\omega}_{h-1}(X)) \xrightarrow{e_0^{h-1}} \Omega_1(\omega_{h-1}(X)) = \Omega_2(\widetilde{\omega}_{h-2}(X)) \xrightarrow{e_1^{h-2}}$$
$$\dots \xrightarrow{e_{h-2}^1} \Omega_{h-1}(\omega_1(X)) = \Omega_h(\widetilde{\omega}_0(X)) \xrightarrow{e_{h-1}^0} \Omega_h(X), \tag{2.3}$$

is a top embedding. □

Corollary 2.3.7. *Let A be lean and $X \in \mathcal{K}_2$. Using the earlier notation, the degree k part $\mathrm{Ext}^k_A(\alpha_h, \hat{S}) : \mathrm{Ext}^k_A(\Omega_h(X), \hat{S}) \to \mathrm{Ext}^k_A(\widetilde{\omega}_h(X), \hat{S})$ of $\mathrm{Ext}^*_A(\alpha_h)$ can be written as*

$$\mathrm{Ext}^k_A(\alpha_h, \hat{S}) = (\alpha_{h,k-1})^* = \left(E^k_{\widetilde{\omega}_h(X)} \circ (e^h_{k-1})^* \circ \dots \circ (e^0_{h+k-1})^* \circ (E^k_{\Omega_h(X)})^{-1}\right),$$

where $\alpha_{h,\bullet} : P_\bullet(\widetilde{\omega}_h(X)) \to P_\bullet(\Omega_h(X))$ is a lifting of α_h, and $e^h_\bullet$ is the same as in the previous proof.

The functor $\mathrm{Hom}_A(\varepsilon_i A, -)$ maps exact sequences of mod-A to exact sequences of mod-C_i. For $i = 2$, let us denote $\mathrm{Hom}_A(\varepsilon_2 A, -)$ by F. For an

A-module X, we define q_X to be the direct sum of linear maps

$$q_X = \bigoplus_{h \geq 0} (q_X)_h : \mathrm{Ext}^*_A(X) \to \mathrm{Ext}^*_{C_2}(X\varepsilon_2),$$

where $(q_X)_h$ sends every h-fold extension $0 \to \hat{S} \to X_{h-1} \to \ldots \to X_0 \to X \to 0$ to an h-fold extension $0 \to \hat{S}\varepsilon_2 \to X_{h-1}\varepsilon_2 \to \ldots \to X_0\varepsilon_2 \to X\varepsilon_2 \to 0$. The map q_X is well-defined because F preserves the equivalence of extensions. Since the functor F commutes with the Yoneda product of extensions, $q_{\hat{S}}$ provides an algebra homomorphism from A^* to C_2^*. Consequently, q_X can be considered as a left graded A^*-module homomorphism having degree 0.

Lemma 2.3.8. *For $h \geq 1$, the following diagram is commutative:*

$$\begin{array}{ccccc}
\mathrm{Ext}^h_A(X, \hat{S}) & & \xrightarrow{(q_X)_h} & & \mathrm{Ext}^h_{C_2}(X\varepsilon_2, \hat{S}\varepsilon_2) \\
\downarrow{\scriptstyle (E^h_X)^{-1}} & & & & \uparrow{\scriptstyle E^h_{X\varepsilon_2}} \\
\mathrm{Hom}_A(\Omega_h(X), \hat{S}) & \xrightarrow{(q_{\Omega_h(X)})_0} & \mathrm{Hom}_{C_2}(\Omega_h(X)\varepsilon_2, \hat{S}\varepsilon_2) & \xrightarrow{(\widetilde{u}_{h-1})^*} & \mathrm{Hom}_{C_2}(\Omega_h(X\varepsilon_2), \hat{S}\varepsilon_2)
\end{array}$$

where $\widetilde{u}_{h-1} : \Omega_h(X\varepsilon_2) \to \Omega_h(X)\varepsilon_2$ is the restriction of a lifting $u_\bullet : P_\bullet(X\varepsilon_2) \to P_\bullet(X)\varepsilon_2$ of $\mathrm{id}_{X\varepsilon_2}$. That is,

$$(q_X)_h = E^h_{X\varepsilon_2} \circ (\widetilde{u}_{h-1})^* \circ (q_{\Omega_h(X)})_0 \circ (E^h_X)^{-1}.$$

When $h = 0$, the actions of $(q_X)_0$ and F coincide, i.e. $(q_X)_0(\xi) = F(\xi)$ for all $\xi \in \mathrm{Hom}_A(X, \hat{S})$.

Proof. The statement for $h = 0$ is an easy consequence of the construction of q.

For $h \geq 1$, let $\xi \in \mathrm{Ext}^h_A(X, \hat{S})$ and $\xi' = (E^h_X)^{-1}(\xi) \in \mathrm{Hom}_A(\Omega_h(X), \hat{S})$. In the diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \Omega_h(X\varepsilon_2) & \longrightarrow & P_{h-1}(X\varepsilon_2) & \longrightarrow & \cdots & \longrightarrow & X\varepsilon_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \widetilde{u}_{h-1}} & & \downarrow & & & & \downarrow{\scriptstyle \mathrm{id}_{X\varepsilon_2}} & & \\
0 & \longrightarrow & \Omega_h(X)\varepsilon_2 & \longrightarrow & P_h(X)\varepsilon_2 & \longrightarrow & \cdots & \longrightarrow & X\varepsilon_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle F(\xi')} & & \downarrow & & & & \downarrow{\scriptstyle \mathrm{id}_{X\varepsilon_2}} & & \\
0 & \longrightarrow & \hat{S}\varepsilon_2 & \longrightarrow & X_{h-1}\varepsilon_2 & \longrightarrow & \cdots & \longrightarrow & X\varepsilon_2 & \longrightarrow & 0
\end{array}$$

the extensions $(q_X)_h(\xi) = ((q_X)_h \circ E^h_X)(\xi')$ and $(E^h_{X\varepsilon_2} \circ (\widetilde{u}_{h-1})^* \circ F)(\xi')$ are both equivalent to the extension represented by the bottom row. □

Lemma 2.3.9. *The correspondence q_X is natural, that is, if $\varphi : X \to Y$ is an A-module homomorphism, then the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{Ext}^*_A(Y) & \xrightarrow{q_Y} & \mathrm{Ext}^*_{C_2}(Y\varepsilon_2) \\ \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle F(\varphi)^*} \\ \mathrm{Ext}^*_A(X) & \xrightarrow{q_X} & \mathrm{Ext}^*_{C_2}(X\varepsilon_2) \end{array}$$

Proof. Let $u_\bullet : P_\bullet(X\varepsilon_2) \to P_\bullet(X)\varepsilon_2$ denote a lifting of $\mathrm{id}_{X\varepsilon_2}$, and similarly let $v_\bullet : P_\bullet(Y\varepsilon_2) \to P_\bullet(Y)\varepsilon_2$ denote a lifting of $\mathrm{id}_{Y\varepsilon_2}$. In the diagram

$$\begin{array}{ccccc} & & P_\bullet(X\varepsilon_2) & & \\ & \swarrow{\scriptstyle u_\bullet} & & \searrow{\scriptstyle F(\varphi)_\bullet} & \\ P_\bullet(X)\varepsilon_2 & & & & P_\bullet(Y\varepsilon_2) \\ & \searrow{\scriptstyle F(\varphi_\bullet)} & & \swarrow{\scriptstyle v_\bullet} & \\ & & P_\bullet(Y)\varepsilon_2 & & \end{array}$$

the chain maps $F(\varphi_\bullet) \circ u_\bullet$ and $v_\bullet \circ F(\varphi)_\bullet$ are homotopic, since they are both liftings of the map $F(\varphi) \circ \mathrm{id}_{X\varepsilon_2} = \mathrm{id}_{Y\varepsilon_2} \circ F(\varphi)$. Let $\xi \in \mathrm{Ext}^h_A(Y, \hat{S})$ for which $\xi' = (E^h_Y)^{-1}(\xi)$. Then we have

$$\begin{array}{ccccc} E^h_Y(\xi') & \overset{q_Y}{\longmapsto} & (E^h_{Y\varepsilon_2} \circ (\widetilde{v}_{h-1})^*_0 \circ F)(\xi') & \overset{F(\varphi)^*}{\longmapsto} & (E^h_{X\varepsilon_2} \circ (F(\varphi)_{h-1})^*_0 \circ (\widetilde{v}_{h-1})^*_0 \circ F)(\xi') \\ \| & & & & \| \\ E^h_Y(\xi') & \overset{\varphi^*}{\longmapsto} & (E^h_X \circ (\varphi_{h-1})^*)(\xi') & \overset{q_X}{\longmapsto} & (E^h_{X\varepsilon_2} \circ (\widetilde{u}_{h-1})^*_0 \circ F(\varphi_{h-1})^*_0)(F(\xi')). \end{array}$$

□

Remark 2.3.10. We should point out that for any A-module X, the kernel of q_X contains $A^* f_1 X^*$ because any extension $\xi \in \mathrm{Ext}^k_A(X, \hat{S}) \cap A^* f_1 X^*$ can be written as a Yoneda-composite of

$$0 \to \hat{S} \to \ldots \to \oplus S(1) \to 0 \quad \text{and} \quad 0 \to \oplus S(1) \to \ldots \to X \to 0,$$

which has clearly a 0 image with respect to q_X.

Lemma 2.3.11. *Suppose that A is lean, $X \in \mathcal{K}_2$, and $P_\bullet(X)$ is a minimal projective resolution of X. Then there is a lifting*

$$u_\bullet : P_\bullet(X\varepsilon_2) \to P_\bullet(X)\varepsilon_2$$

of $\mathrm{id}_{X\varepsilon_2}$ *such that each* $\widetilde{u}_h : \Omega_{h+1}(X\varepsilon_2) \to \Omega_{h+1}(x)\varepsilon_2$ *is a top embedding, and*

$$\widetilde{u}_h(\Omega_{h+1}(X\varepsilon_2)) = F(\alpha_{h+1})(\widetilde{\omega}_{h+1}(X)\varepsilon_2) \cong \widetilde{\omega}_{h+1}(X)\varepsilon_2. \tag{2.4}$$

Proof. We use induction on h. The case $h = 0$ is proved by Lemma 2.3.2. Suppose that $h > 0$. We define the maps $\eta_h : P_h(X\varepsilon_2) \to P(\widetilde{\omega}_h(X))\varepsilon_2$ recursively as shown in the first two rows of the commutative diagram below.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_{h+1}(X\varepsilon_2) & \longrightarrow & P_h(X\varepsilon_2) & \longrightarrow & \Omega_h(X\varepsilon_2) & \longrightarrow & 0 \\
 & & \big\downarrow \widetilde{\eta}_h & & \big\downarrow \eta_h & & \cong \big\downarrow \widetilde{\eta}_{h-1} & & \\
0 & \longrightarrow & \omega_{h+1}(X)\varepsilon_2 & \to & P(\widetilde{\omega}_h(X))\varepsilon_2 & \longrightarrow & \widetilde{\omega}_h(X)\varepsilon_2 & \longrightarrow & 0 \\
 & & \big\downarrow F(\alpha_{h+1}) & & \big\downarrow & & \big\downarrow F(\alpha_h) & & \\
0 & \longrightarrow & \Omega_{h+1}(X)\varepsilon_2 & \longrightarrow & P_h(X)\varepsilon_2 & \longrightarrow & \Omega_h(X)\varepsilon_2 & \longrightarrow & 0
\end{array}$$

We show by induction that η_h and $\widetilde{\eta}_h$ are isomorphisms for each h. If $\widetilde{\eta}_{h-1}$ is an isomorphism, then η_h is surjective because $P(\widetilde{\omega}_h(X))\varepsilon_2 \to \widetilde{\omega}_h(X)\varepsilon_2$ is a projective cover. As $P(\widetilde{\omega}_h(X))\varepsilon_2$ is projective, η_h splits. But $\ker \eta_h \subseteq \mathrm{rad}\ P_h(X\varepsilon_2)$, so η_h is also injective. Then, by the snake lemma, $\widetilde{\eta}_h$ is an isomorphism, too.

Finally, $\alpha_{h+1} : \widetilde{\omega}_{h+1}(X) \to \Omega_{h+1}(X)$ is a top embedding with $\widetilde{\omega}_{h+1}(X)$ generated by $\varepsilon_2 A$, so $F(\alpha_{h+1})$ and $\widetilde{u}_h := F(\alpha_{h+1}) \circ \widetilde{\eta}_h$ are also top embeddings. □

For the remaining part of this section, let us fix the notation of the previous lemma. That is, for a fixed arbitrary module $X \in \mathcal{K}_2$, let $u_\bullet$ denote a lifting $P_\bullet(X\varepsilon_2) \to P_\bullet(X)\varepsilon_2$ of $\mathrm{id}_{X\varepsilon_2}$ for which $\widetilde{u}_\bullet = F(\alpha_{\bullet+1}) \circ \widetilde{\eta}_\bullet$, and α_h – along with its cokernel β_h – is defined by the exact sequence (2.2).

Proposition 2.3.12. *Let A be lean and $X \in \mathcal{K}_2$. Then $q_X : X^* \to (X\varepsilon_2)^*$ is an epimorphism, whose kernel is $\oplus_{h\geq 0} E^h_X(\mathrm{im}\,(\beta_h)^*_0)$.*

Proof. For an arbitrary index $h \geq 0$,

$$(q_X)_h \circ E_X^h = E_{X\varepsilon_2}^h \circ (\widetilde{\eta}_{h-1})_0^* \circ F(\alpha_h)_0^* \circ (q_{\Omega_h(X)})_0$$

by the definition of $\widetilde{u}_\bullet$ and Lemma 2.3.8. Both $E_{X\varepsilon_2}^h$ and E_X^h are isomorphisms, so we investigate $(\widetilde{\eta}_{h-1})_0^* \circ F(\alpha_h)_0^* \circ (q_{\Omega_h(X)})_0$. By Lemma 2.3.9,

$$(\eta_{h-1})_0^* \circ \left(F(\alpha_h)_0^* \circ (q_{\Omega_h(X)})_0\right) = (\eta_{h-1})_0^* \circ \left((q_{\widetilde{\omega}_h(X)})_0 \circ (\alpha_h)_0^*\right).$$

As $(\eta_{h-1})_0^*$ and $(q_{\widetilde{\omega}_h(X)})_0$ are isomorphisms, $\ker((q_X)_h \circ E_X^h) = \ker(\alpha_h)_0^* = \operatorname{im}(\beta_h)_0^*$. Furthermore, the surjectivity of $(\alpha_h)_0^*$ follows from α_h being a top embedding. Hence $(q_X)_h$ is surjective with kernel $E_X^h(\operatorname{im}(\beta_h)_0^*)$. □

Proposition 2.3.13. *Suppose that A is lean and $X \in \mathcal{K}_2$. If $Y_h(X)$ is $\hat{S}\varepsilon_2 A$-Koszul for all h, then* $\ker q_X = A^* f_1 X^*$.

Proof. In view of Proposition 2.3.12 and Remark 2.3.10, it is enough to show that $\oplus_{h\geq 0} E_X^h(\operatorname{im}(\beta_h)_0^*) \subseteq A^* f_1 X^*$, or equivalently,

$$\left(E_X^h \circ (\beta_h)_0^*\right)\left(\operatorname{Hom}_A(Y_h(X), \hat{S})\right) \subseteq (A^* f_1 X^*)_h$$

for all h. We prove this by induction on h. If $h = 0$, then $Y_0(X) = \overline{X} \in \mathcal{F}(S(1))$, and that implies

$$E_X^0(\operatorname{im}(\beta_0)_0^*) = \operatorname{im}(\beta_0)_0^* = \operatorname{Hom}_A(X, S(1)) \subseteq (A^* f_1 X^*)_0.$$

It is clear that $(E_X^h \circ (\beta_h)_0^*)(\operatorname{Hom}_A(Y_h(X), S(1))) \subseteq A^* f_1 X^*$, so we only have to deal with the image of $\operatorname{Hom}_A(Y_h(X), \hat{S}\varepsilon_2 A)$. Since α_h is a top embedding, we get, using the horseshoe lemma, the short exact sequence of the respective syzygies as the bottom row of the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{\omega}_{h+1}(X) & \xrightarrow{\alpha_{h+1}} & \Omega_{h+1}(X) & \xrightarrow{\beta_{h+1}} & Y_{h+1}(X) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow{\scriptstyle \theta_{h+1}} & & \\ 0 & \longrightarrow & \omega_{h+1}(X) & \longrightarrow & \Omega_{h+1}(X) & \xrightarrow{\widetilde{\beta}_{h,0}} & \Omega(Y_h(X)) & \longrightarrow & 0. \end{array} \tag{2.5}$$

Here the snake lemma yields the exact sequence

$$0 \longrightarrow \bar{\omega}_{h+1}(X) \longrightarrow Y_{h+1}(X) \xrightarrow{\theta_{h+1}} \Omega(Y_h(X)) \longrightarrow 0. \tag{2.6}$$

By (2.5), $(\beta_{h+1})^* \circ (\theta_{h+1})^* = (\widetilde{\beta}_{h,0})^*$. Besides, $\bar{\omega}_{h+1}(X) \in \mathcal{F}(S(1))$ gives the isomorphism $(\theta_{h+1})^* : \mathrm{Hom}_A(\Omega(Y_h(X)), \hat{S}\varepsilon_2 A) \to \mathrm{Hom}_A(Y_{h+1}(X), \hat{S}\varepsilon_2 A)$, so

$$(\beta_{h+1})_0^* \left(\mathrm{Hom}_A(Y_{h+1}(X), \hat{S}\varepsilon_2 A)\right) = (\beta_{h,0})_0^* \left(\mathrm{Hom}_A(\Omega(Y_h(X)), \hat{S}\varepsilon_2 A)\right).$$

Suppose that φ is an element of $\mathrm{Hom}_A(\Omega(Y), \hat{S}\varepsilon_2 A)$. Then from the diagram

$$\begin{array}{cccccccccc}
\xi\colon & 0 & \longrightarrow & \Omega_{h+1}(X) & \longrightarrow & P_h(X) & \longrightarrow & \Omega_h(X) & \longrightarrow & 0 \\
& & & \downarrow{\scriptstyle \beta_{h,0}} & & \downarrow & & \downarrow{\scriptstyle \beta_h} & & \\
& 0 & \longrightarrow & \Omega(Y_{h+1}) & \longrightarrow & P(Y_{h+1}) & \longrightarrow & Y_{h+1} & \longrightarrow & 0 \\
& & & \downarrow{\scriptstyle \varphi} & & & & & & \\
& & & \hat{S}\varepsilon_2 A & & & & & &
\end{array}$$

we get

$$\begin{aligned}
(E_X^{h+1} \circ (\beta_{h,0})_0^*)(\varphi) &\subseteq \varphi * \beta_{h,0} * \xi * \mathrm{Ext}_A^h(X, \Omega_h(X)) \subseteq \\
&\subseteq \varphi * \mathrm{Ext}_A^1(Y_{h+1}, \Omega(Y)) * \beta_h * \mathrm{Ext}_A^h(Y_{h+1}, \Omega_h(X)) \subseteq \\
&\subseteq \mathrm{Ext}_A^1(Y_{h+1}, \hat{S}\varepsilon_2 A) * \beta_h * \mathrm{Ext}_A^h(X, \Omega_h(X)),
\end{aligned}$$

where $*$ stands for the Yoneda product of extensions of arbitrary modules, to emphasize that this product is not necessarily a product in A^*. It was assumed that Y_{h+1} is $\hat{S}\varepsilon_2 A$-Koszul, so the latter is included in

$$\begin{aligned}
(A^*)_1 * \mathrm{Hom}_A(Y_{h+1}, \hat{S}) * \beta_h * \mathrm{Ext}_A^h(X, \Omega_h(X)) &\subseteq \\
\subseteq (A^*)_1 * E_X^h(\mathrm{im}\,(\beta_h)_0^*) &\subseteq (A^* f_1 X^*)_{h+1}.
\end{aligned}$$

□

Remark 2.3.14. To highlight the importance of Propositions 2.3.12 and 2.3.13, we give simple counterexamples in the Appendix (see Examples A.7 and A.8) to show that q_X is not an epimorphism in general, and $\ker q_X$ is not necessarily coincide with $A^* f_1 X^*$.

2.3.2 $\overline{\Delta}$-filtration of modules over an infinite dimensional graded algebra

Suppose that $\Lambda = \oplus_{h\geq 0}\Lambda_h$ is a tightly graded K-algebra, i.e. $\Lambda_h \cdot \Lambda_k = \Lambda_{h+k}$ for all $h, k \geq 0$. Let Λ- grfmod denote the category of left graded Λ-modules $X = \oplus_{h\in\mathbb{Z}}X_h$ such that $\dim_{\mathrm{K}} X_h < \infty$ for every h, and there exists a $t \in \mathbb{Z}$ for which $X_h = 0$ whenever $h < t$. The homomorphisms and isomorphisms in Λ- grfmod will be graded, but not necessarily of degree 0. We assume that $f_1 \in \Lambda_0$ is an idempotent element, and the proper standard module belonging to f_1 is defined as

$$\overline{\Delta}^{\circ}(1) = \Lambda f_1/\Lambda f_1\,(\Lambda_{\geq 1})\,f_1.$$

Clearly, $\mathrm{Ext}^1_\Lambda(\overline{\Delta}^{\circ}(1), S) = 0$ for all simple modules with $f_1 S = 0$. We call a chain of submodules $X = X^0 \supseteq X^1 \supseteq \ldots$ a $\overline{\Delta}^{\circ}(1)$-filtration if $\bigcap_{i=0}^{\infty} X^i = 0$ and $X^i/X^{i+1} \cong \overline{\Delta}^{\circ}(1)$ for each i.

Lemma 2.3.15. *If $X \in \mathcal{F}(\overline{\Delta}^{\circ}(1))$, then X is generated by the projective module Λf_1, i.e. $X = \Lambda f_1 X$.*

Proof. If $X = X^0 \supseteq X^1 \supseteq \ldots$ is a $\overline{\Delta}^{\circ}(1)$-filtration, then $X^i = \Lambda u_i + X^{i+1}$ for some elements $u_i = f_1 u_i$. Then for any h, the finiteness of the dimension of $(X)_{\leq h}$ and the condition $\bigcap X^i = 0$ implies that $(X^i)_h = 0$ for some i, thus

$$X_h = \left(\sum_{j=0}^{i-1} \Lambda u_j\right)_h + (X^i)_h \leq \sum_{j=0}^{\infty} \Lambda u_j \leq \Lambda f_1 X.$$

□

Proposition 2.3.16. *Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence, where $Z \in \mathcal{F}(\overline{\Delta}^{\circ}(1))$, and $Y = \Lambda f_1 Y$. If S is a simple module such that $f_1 S = 0$, then*

$$\mathrm{Ext}^1_\Lambda(Z, S) \cong \mathrm{Hom}_\Lambda(X, S) = 0.$$

As a consequence, X is generated by Λf_1.

Proof. First suppose that $Z \cong \overline{\Delta}^{\circ}(1)$. Then $\mathrm{Ext}^1_\Lambda(Z, S) = 0$, and from the exact sequence

$$\mathrm{Hom}_\Lambda(Y, S) \to \mathrm{Hom}_\Lambda(X, S) \to \mathrm{Ext}^1_\Lambda(Z, S), \tag{2.7}$$

we get $\mathrm{Hom}_\Lambda(X,S)=0$.

Now let $Z=Z^0\supseteq Z^1\supseteq\dots$ be a $\overline{\Delta}^\circ(1)$-filtration and assume that

$$\xi:\ 0\to S\to W\to Z\to 0$$

is a short exact sequence. Let us denote by W^i the preimage of Z^i in W for each i. Then $\bigcap W^i=S$.

If $\Lambda f_1 W\neq W$, then the condition $Z=\Lambda f_1 Z$ (by Lemma 2.3.15) together with the simplicity of S implies that $W=S\oplus\Lambda f_1 W$, so the extension ξ is trivial.

If $\Lambda f_1 W=W$, then we may apply the first step of the proof to the sequences

$$0\to W^{i+1}\to W^i\to W^i/W^{i+1}\to 0$$

to show by induction that $\mathrm{Hom}_\Lambda(W^i,S)=0$ for all i.

On the other hand, the simple module S lies in W_h for some h. But $\bigcap_{i=0}^\infty W^i=S$ yields that $\bigcap_{i=0}^\infty(W^i)_k=0$ for $k\neq h$, and S for $k=h$. So $\dim_{\mathrm{K}} W_k<\infty$ implies that there is an i such that $(W^i)_k=0$ for $k<h$ and S for $k=h$, which contradicts $\mathrm{Hom}_A(W^i,S)=0$. We proved that $\mathrm{Ext}^1_\Lambda(Z,S)=0$, thus (2.7) gives $\mathrm{Hom}_\Lambda(X,S)=0$. □

Proposition 2.3.17. *Let $0\to X\to Y\to Z\to 0$ be a short exact sequence, where $Y\in\mathcal{F}(\overline{\Delta}^\circ(1))$, and X is generated by Λf_1. Then both X and Z are $\overline{\Delta}^\circ(1)$-filtered.*

Proof. Let $Y=Y^0\supseteq Y^1\supseteq\dots$ be a $\overline{\Delta}^\circ(1)$-filtration. To prove that X is $\overline{\Delta}^\circ(1)$-filtered, we can show by induction that the terms in the chain of modules $X=X\cap Y^0\supseteq X\cap Y^1\supseteq\dots$ are generated by Λf_1 and the factors are isomorphic to $\overline{\Delta}^\circ(1)$. Indeed, if $X\cap Y^i$ is generated by Λf_1, then the factor module $(X\cap Y^i)/(X\cap Y^{i+1})\cong(Y^{i+1}+(X\cap Y^i))/Y^{i+1}$, which is also generated by Λf_1, is embeddable into $Y^i/Y^{i+1}\cong\overline{\Delta}^\circ(1)$, so it is either 0, or is isomorphic to $\overline{\Delta}^\circ(1)$. Then Proposition 2.3.16 implies that $X\cap Y^{i+1}$ is generated by Λf_1.

Next we show that the image of the chain $Y=X+Y^0\supseteq X+Y^1\supseteq\dots$ gives a $\overline{\Delta}^\circ(1)$-filtration of Z. The modules $X+Y^i$ are Λf_1-generated, since X and Y^i are Λf_1-generated by Lemma 2.3.15. The factor $(X+Y^i)/(X+Y^{i+1})\cong$

$Y^i/(Y^i \cap (X + Y^{i+1}))$ is a homomorphic image of $Y^i/Y^{i+1} \cong \overline{\Delta}^\circ(1)$, where the kernel is $(Y^i \cap (X + Y^{i+1}))/Y^{i+1} = ((Y^i \cap X) + Y^{i+1})/Y^{i+1}$, and this is, by the first part of the proof, generated by Λf_1. So the kernel can only be 0 or Y^i/Y^{i+1}, consequently the factor is either isomorphic to $\overline{\Delta}^\circ(1)$ or 0.

It remains to be shown that $\bigcap(X + Y^i) = X$. Let x be an element of the intersection, which is in Y_h. Since the homogeneous parts of the graded module Y are finite dimensional, there is an i such that $Y^i \subseteq (Y)_{>h}$, hence $x \in X + Y^i$ implies that $x \in X$. □

Lemma 2.3.18. *A module $X \in \Lambda\text{-}\operatorname{grfmod}$ is $\overline{\Delta}^\circ(1)$-filtered if and only if the factors of the sequence*

$$X = \Lambda f_1(X)_{\geq t} \supseteq \Lambda f_1(X)_{\geq t+1} \supseteq \ldots \supseteq \Lambda f_1(X)_{\geq h} \supseteq \ldots \tag{2.8}$$

have finite $\overline{\Delta}^\circ(1)$-filtrations, or equivalently,

$$\Lambda f_1(X)_{\geq h}/\Lambda f_1(X)_{\geq h+1} \cong \oplus\overline{\Delta}^\circ(1) \text{ for every } h.$$

Proof. If the factors have finite $\overline{\Delta}^\circ(1)$-filtrations, then the chain of modules in (2.8) can be refined to a $\overline{\Delta}^\circ(1)$-filtration of X.

On the other hand, if X is $\overline{\Delta}^\circ(1)$-filtered, then every factor of the sequence (2.8) is $\overline{\Delta}^\circ(1)$-filtered as well by Proposition 2.3.17, while $\dim_{\mathrm{K}} f_1X_h = \dim_{\mathrm{K}} f_1\big(\Lambda f_1(X)_{\geq h}/\Lambda f_1(X)_{\geq h+1}\big) < \infty$ shows that the factors, in fact, have finite $\overline{\Delta}^\circ(1)$-filtrations.

For the second equivalence, let $0 \to \Omega \to P \to Z \to 0$ be the projective cover of a factor Z of the sequence (2.8). Then $Z = \Lambda f_1 Z$ gives $P = \oplus\Lambda f_1$, where $\Omega \subseteq (P)_{\geq 1}$ is generated by Λf_1 according to Proposition 2.3.16. So $\Omega \subseteq \Lambda f_1(P)_{\geq 1}$, while $\Lambda f_1(Z)_{\geq 1} = 0$ yields $\Lambda f_1(P)_{\geq 1} \subseteq \Omega$, thus $Z \cong P/\Lambda f_1(P)_{\geq 1} \cong \oplus\overline{\Delta}^\circ(1)$. □

Proposition 2.3.19. *If $0 \to X \to Y \to Z \to 0$ is a short exact sequence with X and Z both $\overline{\Delta}^\circ(1)$-filtered, then Y is also $\overline{\Delta}^\circ(1)$-filtered.*

Proof. We need to show that the factors of the chain of modules

$$Y = \Lambda f_1(Y)_{\geq t} \supseteq \Lambda f_1(Y)_{\geq t+1} \supseteq \ldots \supseteq \Lambda f_1(Y)_{\geq h} \supseteq \ldots.$$

have finite $\overline{\Delta}^\circ(1)$-filtrations.

For every index $h \geq 0$, we can form the short exact sequence

$$0 \to (X)_{\geq h} \cap \Lambda f_1(Y)_{\geq h} \to \Lambda f_1(Y)_{\geq h} \to \Lambda f_1(Z)_{\geq h} \to 0. \tag{2.9}$$

Since $\Lambda f_1(Z)_{\geq h}$ belongs to $\mathcal{F}(\overline{\Delta}^\circ(1))$ and $\Lambda f_1(Y)_{\geq h}$ is generated by Λf_1, Proposition 2.3.16 gives $(X)_{\geq h} \cap \Lambda f_1(Y)_{\geq h} = \Lambda f_1((X)_{\geq h} \cap \Lambda f_1(Y)_{\geq h}) = \Lambda f_1(X)_{\geq h}$. Therefore, we can rewrite (2.9) as

$$0 \to \Lambda f_1(X)_{\geq h} \to \Lambda f_1(Y)_{\geq h} \to \Lambda f_1(Z)_{\geq h} \to 0,$$

so we get the short exact sequences

$$0 \to \Lambda f_1(X)_{\geq h}/\Lambda f_1(X)_{\geq h+1} \to \Lambda f_1(Y)_{\geq h}/\Lambda f_1(Y)_{\geq h+1} \to \\ \Lambda f_1(Z)_{\geq h}/\Lambda f_1(Z)_{\geq h+1} \to 0,$$

where the first and third modules have finite $\overline{\Delta}^\circ(1)$-filtrations, providing finite $\overline{\Delta}^\circ(1)$-filtrations for the middle terms. By Lemma 2.3.18, this proves that Y is $\overline{\Delta}^\circ(1)$-filtered. □

2.3.3 Δ-filtered algebras

In this section, we shall prove that the Ext^*_A-images of the modules of $r\mathcal{K}$ are filtered by left proper standard modules of A^*, when A is a standard Koszul standardly stratified algebra.

For an easier reference, let us quote two lemmas from [2], which will be used repeatedly in the sequel.

Lemma 2.3.20. *Let $0 \to X \to Y \to Z \to 0$ be exact with the map $X \to Y$ a top embedding. If $X \in \mathcal{C}$, then the induced sequence of graded left A^*-modules $0 \to \mathrm{Ext}^*_A(Z) \to \mathrm{Ext}^*_A(Y) \to \mathrm{Ext}^*_A(X) \to 0$ is also exact with morphisms of degree 0.*

Lemma 2.3.21. *Let $0 \to X \to Y \to Z \to 0$ be exact with $X \subseteq \mathrm{rad}\, Y$. If $Y \in \mathcal{C}$, then the induced sequence of graded left A^*-modules $0 \to \mathrm{Ext}^*_A(X)[1] \to \mathrm{Ext}^*_A(Z) \to \mathrm{Ext}^*_A(Y) \to 0$ is also exact with morphisms of degree 0.*

Proposition 2.3.22. *If A is standard Koszul standardly stratified and $X \in \mathcal{K}_2$, then $X^*/A^*f_1X^* \cong (X\varepsilon_2)^*$.*

Proof. In view of Propositions 2.3.12 and 2.3.13, we only need to show that the modules $Y_h(X)$ defined in Proposition 2.3.6 by the short exact sequences

$$0 \to \widetilde{\omega}_h(X) \xrightarrow{\alpha_h} \Omega_h(X) \xrightarrow{\beta_h} Y_h(X) \to 0 \tag{2.10}$$

are in $\mathcal{K}_2$, since by Proposition 2.2.8 this will imply that $Y_h(X)$ is $\hat{S}\varepsilon_2A$-Koszul.

Since $X \in \mathcal{K}_2$, its hth syzygy $\Omega_h(X)$ also lies in $\mathcal{K}_2$ by Proposition 2.2.7. In particular, $\Omega_h(X)\varepsilon_2A$ is a top submodule of $\Omega_h(X)$. Hence, we can apply Lemma 2.1.6 to the sequence (2.10) to get $Y_h(X)\varepsilon_2A \overset{t}{\leq} Y_h(X)$. Note that $Y_0(X) = \overline{X} \in \mathcal{F}(S(1)) \subseteq \mathcal{K}_2$, so it suffices to prove that $Y_h(X) \in \mathcal{K}_2$ implies $Y_{h+1}(X)\varepsilon_2 \in \mathcal{C}_{C_2}$.

In the short exact sequence (2.6) of Proposition 2.3.13:

$$0 \to \bar{\omega}_{h+1}(X) \to Y_{h+1}(X) \to \Omega(Y_h(X)) \to 0,$$

$\bar{\omega}_{h+1}(X) \in \mathcal{F}(S(1))$, so $Y_{h+1}(X)\varepsilon_2 \cong \Omega(Y_h(X))\varepsilon_2$. By the inductive hypothesis $Y_h(X) \in \mathcal{K}_2$, thus $\Omega(Y_h(X)) \in \mathcal{K}_2$ by Proposition 2.2.7, consequently $Y_{h+1}(X)\varepsilon_2 \cong \Omega(Y_h(X))\varepsilon_2 \in \mathcal{C}_{C_2}$. □

Applying Proposition 2.3.22 recursively, we immediately get the trace filtration of X^* for modules X of $r\mathcal{K}$.

Theorem 2.3.23. *If A is standard Koszul standardly stratified and $X \in r\mathcal{K}$, then $X^*/A^*(f_1 + \ldots + f_{i-1})X^* \cong (X\varepsilon_i)^*$ for all $i \geq 1$.*

Lemma 2.3.24. *If A is a standard Koszul standardly stratified algebra, then $A^*/A^*f_1A^* \cong C_2^*$ as algebras.*

Proof. By Theorem 2.3.1, the module $\hat{S}$ belongs to $\mathcal{K}_2$, so we can apply Proposition 2.3.22 to this module to get the isomorphism ${}_{A^*}A^*/A^*f_1{}_{A^*}A^* \cong {}_{A^*}C_2^*$ of (left) A^*-modules, which implies the required isomorphism of algebras. □

Lemma 2.3.25. *If A is standard Koszul standardly stratified and $X \in \mathcal{F}(S(1))$, then $X^* = A^*f_1X^*$.*

Proof. Clearly, $X \in \mathcal{K}_2$, so $X^*/A^*f_1X^* \cong (X\varepsilon_2)^* = 0$ by Proposition 2.3.22. $\square$

Theorem 2.3.26. *If A is standard Koszul standardly stratified, then right standard A-modules are mapped to left proper standard A^*-modules, and left proper standard A-modules are mapped to right standard A^*-modules by the functor Ext_A^*, that is, $\mathrm{Ext}_A^*(\Delta(i)) \cong \overline{\Delta}^\circ_{A^*}(i)$ and $\mathrm{Ext}_A^*(\overline{\Delta}^\circ(i)) \cong \Delta_{A^*}(i)$.*

Proof. We provide here the proof only for right standard modules. The statement about the left proper standard modules can be proved similarly. Applying Theorem 2.3.1, we use induction on the number of simple modules.

For a local algebra, the module $\Delta(1)$ is projective, and $\mathrm{Ext}_A^*(\Delta(1)) = S^\circ_{A^*}(1) = \overline{\Delta}^\circ_{A^*}(1)$. So we may assume that A is not local and the statement holds for C_2. We recall that $\mathrm{Ext}_A^h(\Delta(i), S(j)) = 0$ for all $h \geq 0$, and $i \geq j$. Besides, it is easy to see that $\Delta(i) \in \mathcal{K}$.

Suppose that $i \geq 2$. Then $\mathrm{Ext}_{C_2}^*(\Delta(i)\varepsilon_2) \cong \mathrm{Ext}_{C_2}^*(\Delta_{C_2}(i))$, and they are isomorphic to $\overline{\Delta}^\circ_{C_2^*}(i)$ by the inductive hypothesis. On the other hand, $A^*f_1\mathrm{Ext}_A^*(\Delta(i)) = 0$ because $\mathrm{Ext}_A^h(\Delta(i), S(1)) = 0$ for all $h \geq 0$, so we get $\overline{\Delta}^\circ_{C_2^*}(i) \cong \overline{\Delta}^\circ_{A^*}(i)$ as A^*-modules, since $C_2^* \cong A^*/A^*f_1A^*$ by Lemma 2.3.24. Finally, Proposition 2.3.22 yields $\mathrm{Ext}_A^*(\Delta(i)) \cong \overline{\Delta}^\circ_{A^*}(i)$.

It is left to be shown that $\mathrm{Ext}_A^*(\Delta(1)) \cong \overline{\Delta}^\circ_{A^*}(1)$. Since $\Delta(1) \in \mathcal{K}$, the module $\mathrm{Ext}_A^*(\Delta(1))$ is a graded module generated in degree 0. It is also clear that it has a one-dimensional degree 0 part, and since $\mathrm{Ext}_A^h(\Delta(1), S(1)) = 0$ if $h \geq 1$, we see that $\mathrm{Ext}_A^*(\Delta(1))$ is a homomorphic image of $\overline{\Delta}^\circ_{A^*}(1)$. Consider the Ext_A^*-image of the short exact sequence $0 \to \mathrm{rad}\ \Delta(1) \to \Delta(1) \to S(1) \to 0$, which is the exact sequence

$$0 \to \mathrm{Ext}_A^*(\mathrm{rad}\ \Delta(1))[1] \to \mathrm{Ext}_A^*(S(1)) \to \mathrm{Ext}_A^*(\Delta(1)) \to 0$$

in A^*-grmod by Lemma 2.3.21. This sequence shows that there is an epimorphism $P^\circ_{A^*}(1) \to \mathrm{Ext}_A^*(\Delta(1))$, whose kernel is isomorphic to $\mathrm{Ext}_A^*(\mathrm{rad}\ \Delta(1))$. By Lemma 2.3.25, $\mathrm{Ext}_A^*(\mathrm{rad}\ \Delta(1)) = A^*f_1\,\mathrm{Ext}_A^*(\mathrm{rad}\ \Delta(1))$ because $\mathrm{rad}\ \Delta(1)$ is in $\mathcal{F}(S(1))$. Thus $\mathrm{Ext}_A^*(\Delta(1)) \cong \overline{\Delta}^\circ_{A^*}(1)$. $\square$

Next, we want to show that $r\mathcal{K}$ is mapped into $\mathcal{F}(\overline{\Delta}^{\circ}_{A^*})$. In particular, this will imply that A^* is a standardly stratified algebra with respect to the opposite order of idempotents. In the proof, we use induction on the number of simple modules, so for the induction step, we need to show that for $X \in \mathcal{K}$, the trace of the first projective A^*-module in X^* is filtered by $\overline{\Delta}^{\circ}_{A^*}(1)$.

Lemma 2.3.27. *If A is standard Koszul standardly stratified and $X \in \mathcal{F}(S(1))$, then $A^* f_1 X^*$ is filtered by $\overline{\Delta}^{\circ}_{A^*}(1)$.*

Proof. First, we observe that if $X \in \mathcal{F}(S(1))$, then X has a Δ-cover. That is, there exists an epimorphism $\Delta(X) \to X$ such that its kernel is contained in $\operatorname{rad} \Delta(X)$, and $\Delta(X)$ is isomorphic to a direct sum of copies of $\Delta(1)$. Indeed, if we take the projective cover $P(X) \to X$, then it factors through $P(X) \to P(X)/P(X)\varepsilon_2 A \cong \oplus \Delta(1)$.

Let us apply the functor Ext^*_A to the short exact sequence

$$0 \to X' \to \Delta(X) \to X \to 0.$$

This yields the exact sequence

$$0 \to (X')^* [1] \to X^* \to (\Delta(X))^* \to 0$$

by Lemma 2.3.21. Since X' also belongs to $\mathcal{F}(S(1))$, we can continue the procedure to get

$$X^* \supseteq (X')^* \supseteq (X'')^* \supseteq \ldots \supseteq \left(X^{(i)}\right)^* \supseteq \ldots,$$

where $\left(X^{(i)}\right)^*$ is identified with its image in $(X^*)_{\geq i}$. Thus the intersection of the chain is 0, and the factors are isomorphic to $\operatorname{Ext}^*_A(\Delta(X^{(i)})) \cong \oplus \overline{\Delta}^{\circ}_{A^*}(1)$. $\square$

Proposition 2.3.28. *Suppose that A is standard Koszul standardly stratified and $X \in \mathcal{K}_2$. Then the short exact sequence $0 \to \widetilde{X} \to X \to \overline{X} \to 0$ yields an exact sequence in $A^*\text{-}\operatorname{grfmod}$*

$$0 \to N[1] \to A^* f_1 \overline{X}^* \to A^* f_1 \operatorname{Ext}^*_A X^* \to A^* f_1 \widetilde{X}^* \to N \to 0 \qquad (2.11)$$

with morphisms of degree 0 and $N = A^* f_1 N$.

Proof. We apply $\mathrm{Hom}_A(-, \hat{S})$ to $0 \to \widetilde{X} \to X \to \overline{X} \to 0$, and get the long exact sequence

$$\dots \xrightarrow{\delta_h} \mathrm{Ext}^h_A(\overline{X}, \hat{S}) \to \mathrm{Ext}^h_A(X, \hat{S}) \to \mathrm{Ext}^h_A(\widetilde{X}, \hat{S}) \xrightarrow{\delta_{h+1}} \mathrm{Ext}^{h+1}_A(\overline{X}, \hat{S}) \to \dots.$$

The sequence $\overline{X}^* \to X^* \to \widetilde{X}^*$ is exact, and we may add to it the respective kernel and cokernel to get

$$0 \to N[1] \to \overline{X}^* \to X^* \to \widetilde{X}^* \to N \to 0,$$

where N is the graded left A^*-module whose degree h part is

$$N_h = \mathrm{coker}\left(\mathrm{Ext}^h_A(X, \hat{S}) \to \mathrm{Ext}^h_A(\widetilde{X}, \hat{S})\right) = \\ = \ker\left(\mathrm{Ext}^{h+1}_A(\overline{X}, \hat{S}) \to \mathrm{Ext}^{h+1}_A(X, \hat{S})\right).$$

We still need to show that $A^*f_1N = N$. Since both X and $\widetilde{X}$ are in $\mathcal{K}_2$, we can apply Proposition 2.3.13 to get $X^*/A^*f_1X^* \cong (X\varepsilon_2)^* \cong (\widetilde{X}\varepsilon_2)^* \cong \widetilde{X}^*/A^*f_1\widetilde{X}^*$. Hence we have the following commutative exact diagram:

$$\begin{array}{ccccccc}
 & & A^*f_1X^* & \longrightarrow & A^*f_1\widetilde{X}^* & \longrightarrow & N & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & X^* & \longrightarrow & \widetilde{X}^* & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle q_X} & & \downarrow{\scriptstyle q_{\widetilde{X}}} & & \downarrow & & \\
0 & \longrightarrow & (X\varepsilon_2)^* & \longrightarrow & (\widetilde{X}\varepsilon_2)^* & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The snake lemma gives us that $A^*f_1\widetilde{X}^* \to N$ is an epimorphism, and so $N = A^*f_1N$. Finally, we can extend the upper row to get

$$0 \to N[1] \to \overline{X}^* \to A^*f_1X^* \to A^*f_1\widetilde{X}^* \to N \to 0,$$

where $\overline{X}^* \in \mathcal{F}(\overline{\Delta}^\circ(1))$ by Lemma 2.3.27, so Lemma 2.3.15 gives $\overline{X}^* = A^*f_1\overline{X}^*$. □

Theorem 2.3.29. *If A is standard Koszul standardly stratified and $X \in \mathcal{K}_2$, then $A^*f_1X^* \in \mathcal{F}(\overline{\Delta}^\circ_{A^*}(1))$.*

Proof. Consider the following chain of submodules:

$$A^*f_1X^* \supseteq A^*f_1(X^*)_{\geq 1} \supseteq \ldots \supseteq A^*f_1(X^*)_{\geq h} \supseteq \ldots.$$

We claim that the factor modules

$$A^*f_1(X^*)_{\geq h}/A^*f_1(X^*)_{\geq h+1} \cong A^*f_1\Omega_h(X)^*/A^*f_1(\Omega_h(X)^*)_{\geq 1}$$

are isomorphic to finite direct powers of $\overline{\Delta}^{\circ}_{A^*}(1)$. As Proposition 2.2.7 implies that $\Omega_h(X) \in \mathcal{K}_2$ for all $h \geq 0$, it suffices to deal with the case $h = 0$. For this, we show the isomorphism

$$A^*f_1X^*/A^*f_1(X^*)_{\geq 1} \cong A^*f_1\overline{X}^*/A^*f_1(\overline{X}^*)_{\geq 1}. \tag{2.12}$$

Consider the sequence (2.11) for the module X. Then $(N[1])_0 = 0$, and by Proposition 2.3.28, $N[1] = A^*f_1N[1]$, so we have $N[1] \subseteq A^*f_1(\overline{X}^*)_{\geq 1} \cong A^*f_1\Omega(\overline{X})^*$. The space $(A^*f_1\widetilde{X}^*)_0 = \mathrm{Hom}_A(\widetilde{X}, S(1))$ is zero, thus the map $A^*f_1\overline{X}^* \to A^*f_1X^*$ induces an isomorphism

$$A^*f_1\overline{X}^*/(A^*f_1\overline{X}^*)_{\geq 1} \cong A^*f_1X^*/(A^*f_1X^*)_{\geq 1},$$

and these modules are isomorphic to a direct power $(S^{\circ}_{A^*}(1))^t$. Thus the projective cover $(P^{\circ}_{A^*})^t \to A^*f_1X^*/A^*f_1(X^*)_{\geq 1}$ can be factored through $(\overline{\Delta}^{\circ}_{A^*}(1))^t$, which is isomorphic to $A^*f_1\overline{X}^*/A^*f_1(\overline{X}^*)_{\geq 1}$ by Lemmas 2.3.27 and 2.3.18. So

$$A^*f_1\overline{X}^*/A^*f_1(\overline{X}^*)_{\geq 1} \longrightarrow A^*f_1X^*/A^*f_1(X^*)_{\geq 1} \tag{2.13}$$

is a graded epimorphism of degree 0.

Since $\overline{X} \in \mathcal{F}(S(1)) \subset \mathcal{K}_2$, its syzygy $\Omega(\overline{X}) \in \mathcal{K}_2$ according to Proposition 2.2.7, so $\Omega(\overline{X})^*/A^*f_1\Omega(\overline{X})^* \cong \big(\Omega(\overline{X})\varepsilon_2\big)^*$ by Proposition 2.3.22.

For the sequence $0 \to \widetilde{X} \to X \to \overline{X} \to 0$ (with $\widetilde{X} \overset{t}{\leq} X$), the horseshoe lemma gives the exact sequence $0 \to \omega(X) \to \Omega(X) \to \Omega(\overline{X}) \to 0$ of the syzygies. Apply $\mathrm{Hom}(\varepsilon_2 A, -)$ to get $0 \to \omega(X)\varepsilon_2 \to \Omega(X)\varepsilon_2 \to \Omega(\overline{X})\varepsilon_2 \to 0$, where $\omega(X)\varepsilon_2 = \widetilde{\omega}(X)\varepsilon_2 \cong \Omega(X\varepsilon_2) \in \mathcal{C}_{C_2}$. Since $\widetilde{\omega}(X) \to \Omega(X)$ is a top embedding by Lemma 2.3.3, $\widetilde{\omega}(X)\varepsilon_2$ is a top submodule of $\Omega(X)\varepsilon_2$ according to Lemma 2.1.4. By Lemma 2.3.20, the last sequence is mapped by $\mathrm{Ext}^*_{C_2}$ to the exact sequence

$$0 \to \left(\Omega(\overline{X})\varepsilon_2\right)^* \to (\Omega(X)\varepsilon_2)^* \to (\Omega(X\varepsilon_2))^* \to 0.$$

Thus, we found an injective graded morphism of degree 0 from

$$\left(\Omega(\overline{X})\varepsilon_2\right)^* \cong \Omega(\overline{X})^*/A^* f_1 \Omega(\overline{X})^* \cong \left(A^* f_1 \overline{X}^*/A^* f_1(\overline{X}^*)_{\geq 1}\right)_{\geq 1}$$

to

$$(\Omega(X)\varepsilon_2)^* \cong \Omega(X)^*/A^* f_1 \Omega(X)^* \cong \left(A^* f_1 X^*/A^* f_1(X^*)_{\geq 1}\right)_{\geq 1}.$$

But the epimorphism in (2.13) induces an epimorphism from the former to the latter, so taking into account that all levels of the modules have finite dimension, these factor modules must be isomorphic as stated in (2.12). Then Lemmas 2.3.27 and 2.3.18 finish the proof. □

Theorem 2.3.30. *If A is a standard Koszul standardly stratified algebra and $X \in r\mathcal{K}$, then $X^* \in \mathcal{F}(\overline{\Delta}^{\circ}_{A^*})$. In particular, if X is a top extension of simple and standard modules, then X^* is $\overline{\Delta}^{\circ}_{A^*}$-filtered.*

Proof. The first statement follows by induction, using Theorem 2.3.23 and Proposition 2.3.29, while the second is a consequence of Proposition 2.3.5 because simple and standard modules obviously belong to $r\mathcal{K}$. □

Theorem 2.3.31. *If A is a standard Koszul standardly stratified algebra, then its homological dual A^* is a standardly stratified algebra.*

Proof. Semisimple A-modules belong to $r\mathcal{K}$, thus ${}_{A^*}A^* = \hat{S}^* \in \mathcal{F}(\overline{\Delta}^{\circ}_{A^*})$. □

2.3.4 $\overline{\Delta}$-filtered algebras

In this section, we focus on the left module category of a standard Koszul standardly stratified algebra. To keep our notation simple, we investigate the right modules over an algebra A, whose opposite algebra A° is a standard Koszul standardly stratified algebra, so $A_A \in \mathcal{F}(\overline{\Delta})$.

We would like to prove theorems analogous to those of the previous section. However, to handle the asymmetry of the left and the right module category of A, we have to consider a narrower subclass $\mathcal{K}^+ \subseteq \mathcal{K}$ of modules. It is defined with additional restrictions as

$$\mathcal{K}^+ = \Big\{ X \in \mathcal{K} \mid \widetilde{\omega}_h(X) \in \mathcal{C}_A, \;\text{ and }\; \overline{\omega}_h(X) \cong \oplus S(1) \;\text{ for all }\; h \geq 0 \Big\}.$$

We also introduce the recursive version of $\mathcal{K}^+$ as

$$r\mathcal{K}^+ = \Big\{ X \in \mathcal{K}^+ \,|\, X\varepsilon_i \in \mathcal{K}^+_{C_i} \;\text{ for all } i \Big\}.$$

We shall prove that the functor Ext^*_A maps the subclass $r\mathcal{K}^+$ into $\mathcal{F}(\Delta^\circ_{A^*})$. Furthermore, we show that $r\mathcal{K}^+$ is closed under top extensions, and also that simple and proper standard modules belong to this class.

Lemma 2.3.32. *If A° is standard Koszul standardly stratified and $X \in \mathcal{K}^+$, then $\omega(X)$ and $\widetilde{\omega}(X)$ also belong to $\mathcal{K}^+$.*

Proof. According to Corollary 2.3.4, both modules $\omega(X)$ and $\widetilde{\omega}(X)$ are in $\mathcal{K}_2$. By definition, $\widetilde{\omega}(X)$ is also Koszul, and it is a top submodule of $\omega(X)$. So we have the exact sequence

$$0 \to \widetilde{\omega}(X) \to \omega(X) \to \overline{\omega}(X) \to 0,$$

with a top embedding, where $\widetilde{\omega}(X)$ and $\overline{\omega}(X) \cong \oplus S(1)$ are Koszul, so their top extension $\omega(X)$ is also Koszul by Lemma 2.4 of [2]. The remaining conditions hold by the recursive definition of ω_h. □

Proposition 2.3.33. *If A° is standard Koszul standardly stratified, the classes $\mathcal{K}^+$ and $r\mathcal{K}^+$ are closed under top extensions.*

Proof. Suppose that $X, Z \in \mathcal{K}^+$, and we have the short exact sequence

$$0 \to X \xrightarrow{t} Y \to Z \to 0$$

with a top embedding. First we show that in this case, $\widetilde{Y}$ is a top extension of $\widetilde{Z}$ by $\widetilde{X}$. As $\widetilde{X} \overset{t}{\leq} Y$, the sequence $0 \to X/\widetilde{X} \to Y/\widetilde{X} \to Z \to 0$ is a top extension (cf. Lemma 2.1.3). The first term is a direct sum of copies of $S(1)$, so the sequence splits, and we get $Y/\widetilde{X} \cong \overline{X} \oplus Z$. This yields $\overline{Y} \cong \overline{X} \oplus \overline{Z} \cong \oplus S(1)$, and it also implies $\widetilde{Y}/\widetilde{X} \cong \widetilde{Z}$. That is, the sequence

$$0 \to \widetilde{X} \to \widetilde{Y} \to \widetilde{Z} \to 0 \tag{2.14}$$

is exact, where $\widetilde{X} \overset{t}{\leq} \widetilde{Y}$, so $\widetilde{Y} \in \mathcal{C}_A$ according to Lemma 2.4 of [2]. The application of the horseshoe lemma to the sequence (2.14) gives the short exact sequence $0 \to \omega(X) \to \omega(Y) \to \omega(Z) \to 0$ of the syzygies. By the Koszul property of $\widetilde{X}$, it is a top extension. Using Lemma 2.3.32, we can show by induction that $\widetilde{\omega}_h(Y)$ and $\omega_h(Y)$ satisfy the prescribed conditions of $\mathcal{K}^+$ for every h. Finally, (2.14) gives a top extension $0 \to X\varepsilon_2 \to Y\varepsilon_2 \to Z\varepsilon_2 \to 0$ by Lemma 2.1.4, so a recursive argument shows that $Y \in r\mathcal{K}^+$. □

Proposition 2.3.34. *If A° is standard Koszul standardly stratified and $X \in \mathcal{K}^+$, then $X^*/A^*f_1X^* \cong (X\varepsilon_2)^*$.*

Proof. In view of Propositions 2.3.12 and 2.3.13, it is enough to show that the modules $Y_h(X)$ defined in Proposition 2.3.6 by the short exact sequences

$$0 \to \widetilde{\omega}_h(X) \xrightarrow{\alpha_h} \Omega_h(X) \xrightarrow{\beta_h} Y_h(X) \to 0 \tag{2.15}$$

are Koszul for all h. We prove this by induction on h. The module $Y_0(X) = \overline{\omega}_0(X) = \overline{X}$ is semisimple, hence Koszul. Now we assume that $Y_h(X) \in \mathcal{C}_A$. By assumption, $X \in \mathcal{K}^+$, so $\widetilde{\omega}_h(X)$ is Koszul for all h. If we apply Lemma 2.3.20 to the sequence (2.15), we get that $\Omega_h(X)^* \to \widetilde{\omega}_h(X)^*$ is an epimorphism, in particular, $\mathrm{Hom}_A(\Omega_{h+1}(X), \hat{S}) \to \mathrm{Hom}_A(\omega_{h+1}(X), \hat{S})$ is surjective. It means that in the induced sequence of the syzygies

$$0 \to \omega_{h+1}(X) \to \Omega_{h+1}(X) \to \Omega(Y_h(X)) \to 0$$

we also get a top embedding. If we factor out the submodule $\widetilde{\omega}_{h+1}(X)$ (which is a top submodule both in the first and the middle terms), then by Lemma 2.1.3, we get that the sequence

$$0 \to \overline{\omega}_{h+1}(X) \to Y_{h+1}(X) \to \Omega(Y_h(X)) \to 0$$

also has a top embedding. The first term is semisimple, hence Koszul, and $\Omega(Y_h(X)) \in \mathcal{C}_A$ follows from the inductive hypothesis. By Lemma 2.4 of [2], their top extension $Y_{h+1}(X)$ is also in $\mathcal{C}_A$. $\square$

Applying the proposition recursively, we immediately get the trace filtration of X^* for modules X of $r\mathcal{K}^+$.

Theorem 2.3.35. *If A° is standard Koszul standardly stratified and $X \in r\mathcal{K}^+$, then $X^*/A^*(f_1 + \ldots + f_{i-1})X^* \cong (X\varepsilon_i)^*$ for all $i \geq 1$.*

Lemma 2.3.36. *Suppose that A° is standard Koszul standardly stratified, and $X, Y \in$ mod-A with $Y \in \mathcal{F}(\nabla)$, i.e. Y is filtered by costandard modules. Then the map $\operatorname{Ext}_A^h(X,Y) \to \operatorname{Ext}_A^h(\widetilde{X},Y)$ induced by the natural embedding $\widetilde{X} \to X$ is an isomorphism for $h \geq 1$.*

Proof. We take the short exact sequence $0 \to \widetilde{X} \to X \to \overline{X} \to 0$, and apply the functor $\operatorname{Hom}_A(-,Y)$. In the long exact sequence

$$\ldots \to \operatorname{Ext}_A^h(\overline{X},Y) \to \operatorname{Ext}_A^h(X,Y) \to \operatorname{Ext}_A^h(\widetilde{X},Y) \to \operatorname{Ext}_A^{h+1}(\overline{X},Y) \to \ldots,$$

$\operatorname{Ext}_A^h(\overline{X},Y) = 0$ for $h \geq 0$ because $\operatorname{Ext}_A^h(S(1),\nabla(1)) = \operatorname{Ext}_A^h(\overline{\Delta}(1),\nabla(1)) = 0$ if A° is standardly stratified (cf. Theorem 3.1 of [3]). $\square$

Lemma 2.3.37. *Let $h \geq n$, where n is the number of simple A-modules. If A° is standard Koszul standardly stratified, and the A-module X belongs to $\mathcal{K}_2$, then $\operatorname{Hom}_A(\omega_h(X), S(1)) = 0$. Consequently, $A^* f_1 \omega_n(X)^* = 0$.*

Proof. As $\mathcal{K}_2$ is closed under ω, we only have to deal with the case when $h = n$. Let $0 \to \omega_n(X) \to P(\widetilde{\omega}_{n-1}(X)) \to \widetilde{\omega}_{n-1}(X) \to 0$ be the first step of a projective resolution of $\widetilde{\omega}_{n-1}(X)$. Then $\operatorname{Hom}_A(P(\widetilde{\omega}_{n-1}(X)), \nabla(1)) = 0$, and so $\operatorname{Ext}_A^1(\widetilde{\omega}_{n-1}(X), \nabla(1)) \cong \operatorname{Hom}_A(\omega_n(X), \nabla(1))$. This and Lemma 2.3.36 yield

$$\begin{aligned}\operatorname{Hom}_A(\omega_n(X), \nabla(1)) \cong \operatorname{Ext}_A^1(\widetilde{\omega}_{n-1}(X), \nabla(1)) \cong \operatorname{Ext}_A^1(\omega_{n-1}(X), \nabla(1)) \cong \ldots \\ \ldots \cong \operatorname{Ext}_A^{n-1}(\omega(X), \nabla(1)) \cong \operatorname{Ext}_A^n(\widetilde{X}, \nabla(1)).\end{aligned}$$

Since A° is standardly stratified, the injective dimension of $\nabla(1)$ is less than n (cf. Lemma 3.2 of [6]), giving $\operatorname{Hom}_A(\omega_n(X), \nabla(1)) \cong \operatorname{Ext}_A^n(\widetilde{X}, \nabla(1)) = 0$. Thus $\operatorname{Hom}_A(\omega_n(X), S(1)) = 0$.

We obtained that $\omega_h(X) = \widetilde{\omega}_h(X)$ for all $h \geq n$, hence $\mathrm{Ext}_A^t(\omega_n(X), S(1)) \cong \mathrm{Hom}_A(\Omega_t(\omega_n(X)), S(1)) = \mathrm{Hom}_A(\omega_{n+t}(X), S(1)) = 0$ for $t \geq 0$, proving the second statement. □

Theorem 2.3.38. *If A° is standard Koszul standardly stratified and $X \in r\mathcal{K}^+$, then $X^* \in \mathcal{F}(\Delta^\circ_{A^*})$.*

Proof. In view of Theorem 2.3.35, we only have to show that $A^* f_1 X^*$ is projective, when $X \in \mathcal{K}^+$. Applying the functor Ext_A^* to the short exact sequence $0 \to \widetilde{X} \to X \to \overline{X} \to 0$ gives the exact sequence

$$0 \to \overline{X}^* \to X^* \to \widetilde{X}^* \to 0.$$

Since $\overline{X} = \oplus S(1)$, we have the exact sequence

$$0 \to A^* f_1 \overline{X}^* \to A^* f_1 X^* \to A^* f_1 \widetilde{X}^* \to 0,$$

where $A^* f_1 \overline{X}^*$ is projective. Furthermore, $\mathrm{Hom}_A(\widetilde{X}, S(1)) = 0$, so $A^* f_1 \widetilde{X}^* \cong A^* f_1 \Omega(\widetilde{X})^* = A^* f_1 \omega(X)^*$. We get that $A^* f_1 X^*$ is projective if $A^* f_1 \omega(X)^*$ is projective. We have seen in Lemma 2.3.32 that $\mathcal{K}^+$ is closed under ω, while $A^* f_1 \omega_n(X)^*$ is zero by Proposition 2.3.37. By induction, $A^* f_1 \omega_h(X)^*$ is also projective for all $0 \leq h \leq n$. □

In the remaining part of this section, we want to show that $\overline{\Delta}(i) \in r\mathcal{K}^+$ and $S(i) \in r\mathcal{K}^+$ for all $i \geq 1$.

Theorem 2.3.39. *If A° is standard Koszul standardly stratified, then all proper standard modules are in $r\mathcal{K}^+$.*

Proof. The centralizer algebras of A° are standard Koszul standardly stratified algebras, and $\overline{\Delta}(i)\varepsilon_2 \cong \overline{\Delta}_{C_2}(i)$ for all i (see Theorem 2.3.1). This means that it is enough to see that $\overline{\Delta}(i) \in \mathcal{K}^+$ for all indices i.

If $i = 1$, then $\overline{\Delta}(1) = S(1) \in \mathcal{C}_A$, and $\omega_h(S(1)) = 0$ for $h \geq 1$. If $i \geq 2$, then $\mathrm{Ext}_A^h(\overline{\Delta}(i), S(1)) = 0$ for $h \geq 0$, so $\widetilde{\omega}_h(\overline{\Delta}(i)) = \omega_h(\overline{\Delta}(i)) = \Omega_h(\overline{\Delta}(i))$, which is Koszul by assumption, and we also have $\overline{\omega}_h(\overline{\Delta}(i)) = 0$. □

Now, we focus on simple modules. Since $\overline{\Delta}(1) \cong S(1)$, it suffices to deal with simple modules S which are not isomorphic to $S(1)$. All simple A-modules belong to $\mathcal{K}_2$, so by Corollary 2.3.4, $\omega_h(S) \in \mathcal{K}_2$ for all h.

We consider the canonical embeddings $e^h : \widetilde{\omega}_h(S) \to \omega_h(S)$ and $i : S(1) \to \nabla(1)$. These morphisms give rise for every h to a commutative diagram:

$$
\begin{array}{ccccccc}
(\omega_{h+1}(S), S(1))^0 & \xrightarrow{\cong} & (\widetilde{\omega}_h(S), S(1))^1 & \overset{e'}{\twoheadleftarrow} & (\omega_h(S), S(1))^1 & \xrightarrow{\cong} & \\
\cong \downarrow i' & & \downarrow \widetilde{i} & & \downarrow i' & & \cdots \\
(\omega_{h+1}(S), \nabla(1))^0 & \xrightarrow{\cong} & (\widetilde{\omega}_h(S), \nabla(1))^1 & \underset{\cong}{\overset{\widetilde{e}}{\twoheadleftarrow}} & (\omega_h(S), \nabla(1))^1 & \xrightarrow{\cong} &
\end{array}
$$

$$
\begin{array}{ccccc}
& \xleftarrow{e'} & (\omega_1(S), S(1))^h & \xrightarrow{\cong} & (S, S(1))^{h+1} \\
\cdots & & \downarrow i' & & \downarrow \widetilde{i} \\
& \underset{\cong}{\xleftarrow{\widetilde{e}}} & (\omega_1(S), \nabla(1))^h & \xrightarrow{\cong} & (S, \nabla(1))^{h+1}
\end{array}
\tag{2.16}
$$

where $(X, Y)^k$ stands for $\mathrm{Ext}_A^k(X, Y)$ if $k > 0$, while $(X, Y)^0$ denotes the space $\mathrm{Hom}_A(X, Y)$. For simplicity, we also omit the indices of the maps in the diagram. Proposition 2.3.40 shows that in diagram (2.16), the marked morphisms are indeed epimorphism and isomorphisms, respectively.

Proposition 2.3.40. *If A° is standard Koszul standardly stratified, then the induced maps of the diagram (2.16) have the following properties:*

1. $\widetilde{e} : \mathrm{Ext}_A^k(\omega_j(S), \nabla(1)) \to \mathrm{Ext}_A^k(\widetilde{\omega}_j(S), \nabla(1))$ *is an isomorphism for all* $k \geq 1$ *and* $j \geq 0$.

2. *The maps* $\mathrm{Ext}_A^k(\omega_{j+1}(S), X) \to \mathrm{Ext}_A^{k+1}(\widetilde{\omega}_j(S), X)$ *are isomorphisms for all* $k, j \geq 0$ *if* $X \in \mathcal{F}(S(1))$, *in particular, when* $X = S(1)$ *or* $\nabla(1)$. *Consequently, the map* $\widetilde{i} : \mathrm{Ext}_A^1(\widetilde{\omega}_h(S), S(1)) \to \mathrm{Ext}_A^1(\widetilde{\omega}_h(S), \nabla(1))$ *is injective for all* $h \geq 0$.

3. $\widetilde{i} : \mathrm{Ext}_A^k(\widetilde{\omega}_j(S), S(1)) \to \mathrm{Ext}_A^k(\widetilde{\omega}_j(S), \nabla(1))$ *and* $i' : \mathrm{Ext}_A^k(\omega_j(S), S(1)) \to \mathrm{Ext}_A^k(\omega_j(S), \nabla(1))$ *are epimorphisms for all* $j \geq 0$ *and* $k \geq 0$.

4. $e' : \mathrm{Ext}^1_A(\omega_h(S), S(1)) \to \mathrm{Ext}^1_A(\widetilde{\omega}_h(S), S(1))$ *is surjective for all* $h \geq 0$.

Proof. 1. The first statement follows immediately from Lemma 2.3.36.

2. Apply $\mathrm{Hom}_A(-, X)$ to $0 \to \omega_{j+1}(S) \to P(\widetilde{\omega}_j(S)) \to \widetilde{\omega}_j(S) \to 0$, which is the first step of the minimal projective resolution of $\widetilde{\omega}_j(S)$, to get

$$\ldots \to \mathrm{Ext}^k_A(P(\widetilde{\omega}_j(S)), X) \to \mathrm{Ext}^k_A(\omega_{j+1}(S), X) \to \\ \to \mathrm{Ext}^{k+1}_A(\widetilde{\omega}_j(S), X) \to \mathrm{Ext}^{k+1}_A(P(\widetilde{\omega}_j(S)), X) \to \ldots .$$

Here $\mathrm{Ext}^k_A(P(\widetilde{\omega}_j(S), X) = 0$ if $k \geq 1$ because $P(\widetilde{\omega}_j(S))$ is projective, and $\mathrm{Hom}_A(P(\widetilde{\omega}_j(S)), X) = 0$ since $P(\widetilde{\omega}_j(S)) = P(\widetilde{\omega}_j(S))\varepsilon_2 A$. These give the required isomorphisms, while the left exactness of $\mathrm{Hom}_A(\omega_{j+1}(S), -)$ implies the second part.

3. First, we note that, as $\widetilde{e}$ is an isomorphism, the surjectivity of i' implies the surjectivity of $\widetilde{i}$ for every pair (k, j). Thus, we may prove the surjectivity of the two maps simultaneously. We use induction on j.

The algebra A° is standard Koszul, so the left module $\Delta^\circ(1)$ lies in $\mathcal{C}_{A^\circ}$. In view of Proposition 2.7 of [2] (or rather its "K-dual version"), $\Delta^\circ(1) \in \mathcal{C}_{A^\circ}$ implies that the natural maps $\mathrm{Ext}^k_A(S, S(1)) \to \mathrm{Ext}^k_A(S, \nabla(1))$ are epimorphisms for all k. This provides the base case $(k, 0)$ of the induction.

Suppose that the statement is proved for the pair $(k+1, j-1)$. The inductive hypothesis gives the surjectivity of $\widetilde{i}$, and hence the surjectivity of i' in the diagram below.

$$\begin{array}{ccc} \mathrm{Ext}^k_A(\omega_j(S), S(1)) & \xrightarrow{\cong} & \mathrm{Ext}^{k+1}_A(\widetilde{\omega}_{j-1}(S), S(1)) \\ \Big\downarrow i' & & \Big\downarrow \widetilde{i} \\ \mathrm{Ext}^k_A(\omega_j(S), \nabla(1)) & \xrightarrow{\cong} & \mathrm{Ext}^{k+1}_A(\widetilde{\omega}_{j-1}(S), \nabla(1)) \end{array}$$

4. The fourth statement is a consequence of the first three. □

Proposition 2.3.41. *Let* A° *be standard Koszul standardly stratified, and* S *a simple* A*-module not isomorphic to* $S(1)$. *The homomorphism* $\alpha_{k-1,0}$:

$\omega_{k+h}(S) \to \Omega_k(\omega_h(S))$, *induced by* α_{k-1} *of formula (2.15) applied to* $X = \omega_h(S)$ *is a top embedding for all* k.

Proof. Let $k \geq 1$ be arbitrary. The map $\alpha_{k-1} : \widetilde{\omega}_{k+h-1}(S) \to \Omega_{k-1}(\omega_h(S))$ is a top embedding by Proposition 2.3.6, and this implies that $\Omega(\widetilde{\omega}_{k+h-1}(S)) = \omega_{k+h}(S)$ is mapped into $\Omega_k(\omega_h(S))$ injectively.

To see that $\alpha_{k-1,0}$ is a top embedding, we will show that the induced map $\alpha^*_{k-1,0} : \mathrm{Hom}_A(\Omega_k(\omega_h(S)), \hat{S}) \to \mathrm{Hom}_A(\omega_{k+h}(S), \hat{S})$ is surjective. By Proposition 2.3.6, the restriction of $\alpha_{k-1,0}$ to $\widetilde{\omega}_{k+h}(S) \subseteq \omega_{k+h}(S)$ is a top embedding, or what is equivalent, $\mathrm{Hom}_A(\Omega_k(\omega_h(S)), \hat{S}\varepsilon_2 A) \stackrel{\alpha^*_{k-1,0}}{\longrightarrow} \mathrm{Hom}_A(\omega_{k+h}(S), \hat{S}\varepsilon_2 A)$ is an epimorphism. Thus, we only need to show that $\mathrm{Hom}_A(\Omega_k(\omega_h(S)), S(1)) \stackrel{\alpha^*_{k-1,0}}{\longrightarrow} \mathrm{Hom}_A(\omega_{k+h}(S), S(1))$ is an epimorphism. Consider the following commutative diagram.

$$\begin{array}{ccccccccc}
\stackrel{\cong}{\longleftarrow} & (\widetilde{\omega}_j(S), S(1))^\ell & \stackrel{e'}{\longleftarrow} & (\omega_j(S), S(1))^\ell & \stackrel{\cong}{\longleftarrow} & (\widetilde{\omega}_{j-1}(S), S(1))^{\ell+1} & \stackrel{e'}{\longleftarrow} & \\
\cdots & \uparrow E^\ell_{\widetilde{\omega}_j(S)} & & \uparrow E^\ell_{\omega_j(S)} & & \uparrow E^{\ell+1}_{\widetilde{\omega}_{j-1}(S)} & & \cdots \\
= & (\Omega_\ell(\widetilde{\omega}_j(S)), S(1))^0 & \stackrel{(e^j_{\ell-1})^*}{\longleftarrow} & (\Omega_\ell(\omega_j(S)), S(1))^0 & = & (\Omega_{\ell+1}(\widetilde{\omega}_{j-1}(S)), S(1))^0 & \leftarrow &
\end{array}$$

By Corollary 2.3.7, $\mathrm{Hom}_A(\Omega_k(\omega_h(S)), S(1)) \stackrel{\alpha^*_{k-1,0}}{\longrightarrow} \mathrm{Hom}_A(\omega_{k+h}(S), S(1))$ is surjective, if the bottom row of the diagram is surjective. This is equivalent to the surjectivity of the top row, which comes from the top row of diagram (2.16) by reversing the isomorphisms. Hence it can be factored as

$$\begin{aligned}
&\mathrm{Ext}^k_A(\omega_h(S), S(1)) \stackrel{i'}{\to} \mathrm{Ext}^k_A(\omega_h(S), \nabla(1)) \stackrel{\widetilde{e}}{\to} \mathrm{Ext}^k_A(\widetilde{\omega}_h(S), \nabla(1)) \stackrel{\cong}{\to} \\
&\qquad \stackrel{\cong}{\to} \mathrm{Ext}^{k-1}_A(\omega_{h+1}(S), \nabla(1)) \stackrel{\widetilde{e}}{\to} \dots \stackrel{\widetilde{e}}{\to} \mathrm{Ext}^1_A(\widetilde{\omega}_{k+h-1}(S), \nabla(1)) \stackrel{\widetilde{i}^{-1}}{\to} \\
&\qquad\qquad \stackrel{\widetilde{i}^{-1}}{\to} \mathrm{Ext}^1_A(\widetilde{\omega}_{k+h-1}(S), S(1)),
\end{aligned}$$

where i' is an epimorphism, while the other maps are isomorphisms, so the composition is surjective. $\square$

Theorem 2.3.42. *If* A° *is standard Koszul standardly stratified, then the simple* A*-modules are in* $r\mathcal{K}^+$.

Proof. In view of Theorem 1.1, it suffices to show that simple A-modules belong to $\mathcal{K}^+$. We also know that $S(1) \in r\mathcal{K}^+$ by Theorem 2.3.39. So we only have to prove the statement for a simple module S, which is not isomorphic to $S(1)$.

We show first that $\widetilde{\omega}_h(S) \in \mathcal{C}_A^1$ for all h. Applying Proposition 2.3.41 to $\alpha_{h,0} : \Omega(\widetilde{\omega}_h(S)) = \omega_{h+1}(S) \to \Omega_{h+1}(S)$, and using $S \in \mathcal{C}_A$, we get $\alpha_{h,0}(\Omega(\widetilde{\omega}_h(S))) \overset{t}{\leq} \Omega_{h+1}(S) \overset{t}{\leq} \text{rad } P_h(S)$. As $\alpha_{h,0}(\Omega(\widetilde{\omega}_h(S))) \subseteq \alpha_{h,0}(\text{rad } P(\widetilde{\omega}_h(S))) \subseteq \text{rad } P_h(S)$, it follows that $\Omega(\widetilde{\omega}_h(S))$ is a top submodule of rad $P(\widetilde{\omega}_h(S))$.

To prove that $\bar{\omega}_h(S) = \omega_h(S)/\widetilde{\omega}_h(S)$ is semisimple, in fact, isomorphic to $\oplus S(1)$, we only need that $\text{Hom}_A(\omega_h(S), S(1)) \to \text{Hom}_A(\omega_h(S), \nabla(1))$ is surjective, and this was proved in the third part of Proposition 2.3.40.

Finally, we show that $\omega_h(S) \in \mathcal{C}_A$ by backwards induction. For $h \geq n$, Lemma 2.3.37 gives that $\Omega(\widetilde{\omega}_h(S)) = \omega_{h+1}(S) = \widetilde{\omega}_{h+1}(S)$, so every syzygy of $\omega_h(S)$ is in $\mathcal{C}_A^1$. Thus $\widetilde{\omega}_h(S) = \omega_h(S) \in \mathcal{C}_A$ if $h \geq n$. On the other hand, if $\omega_h(S) \in \mathcal{C}_A$, then in the exact sequence $0 \to \widetilde{\omega}_h(S) \to \omega_h(S) \to \bar{\omega}_h(S) \to 0$ (with top embedding) both the first and the third terms are Koszul. Hence by Lemma 2.4 of [2], $\omega_h \in \mathcal{C}_A$. Together with the first part of the proof, this gives $\widetilde{\omega}_{h-1} \in \mathcal{C}_A$. □

We point out that Theorem 2.3.35 and 2.3.42 imply that ${}_{A^*}A^*$ is filtered by standard modules. Actually, this gives an alternative proof for Theorem 2.3.31. Besides, we also note that simple modules on the Δ-filtered side are not necessarily belong to $\mathcal{K}^+$. A counterexample is given in Example A.5. This fact highlights that the study of two sides requires different approaches.

Finally, the combination of the results of Proposition 2.3.33 and Theorems 2.3.38, 2.3.39 and 2.3.42 provides the following theorem.

Theorem 2.3.43. *If A° is a standard Koszul standardly stratified algebra, and X is a top extension of proper standard and simple modules, then X^* is filtered by standard A^*-modules.*

Chapter 3

Monomial and self-injective special biserial algebras

In this chapter, we investigate the standard Koszul property for some combinatorially defined classes of algebras. Both monomial and special biserial algebras are given by their graphs and relations. (For a thorough introduction to the construction and properties of graph algebras, one may refer to [9].) The Koszul and standard Koszul properties will be described by combinatorial conditions. Using these, we shall be able to prove, even without the assumptions of standard stratification, that in these classes, standard Koszul algebras are always Koszul.

3.1 Monomial algebras

Let Γ be a finite oriented graph possibly with loops and multiple edges. The algebra $K\Gamma$ is a vector space whose basis consists of all oriented paths in Γ, and the product of two paths is given by concatenation if it is possible, and zero otherwise. We say that $I \triangleleft K\Gamma$ is an *admissible ideal*, if it consists of linear combinations of paths with length at least 2, and there exists an integer m such that all paths with length at least m are in I. In this case, we call the factor algebra $A = K\Gamma/I$ a *graph algebra*. A graph algebra $A = K\Gamma/I$ is *monomial* if

I is generated by paths.

Theorem 5.2 of [2] implies that a standard Koszul monomial algebra is Koszul when all standard modules are Schurian (i.e. $\mathrm{End}(\Delta(i))$ and $\mathrm{End}(\Delta^{\circ}(i))$ are division rings for all i). The proof we present here is only slightly more general but it is a good introduction to the more complicated case of special biserial algebras.

We shall see that all modules appearing in the projective resolutions of simple, standard or proper standard modules are direct sums of submodules of A_A generated by paths. We simply refer to these submodules of A_A as *path-generated modules.* In particular, $\mathrm{rad}^i A_A$, $\Delta(i)$, $\overline{\Delta}(i)$ are clearly path-generated.

Lemma 3.1.1. *Every path-generated module U over the monomial algebra A can be decomposed as*

$$U_K = \oplus u_i K \quad \textit{as a vector space and} \quad U_A = \oplus v_j A \quad \textit{as a module,}$$

where u_i are all the nonzero paths in U, and v_j are all the right minimal paths in U, i.e. those which have no proper initial segment contained in U.

Proof. The first decomposition follows from the fact that I is generated by paths, so the nonzero paths of A form a basis in A_K. Then for the second it is enough to observe that $v_j A$ and $v_k A$ have no common nonzero path for $j \neq k$, since neither of the paths v_j and v_k can be an initial segment of the other. □

Lemma 3.1.2. *The syzygy of a path-generated module over the monomial algebra A is a direct sum of path-generated modules.*

Proof. By Lemma 3.1.1 it is enough to prove the statement for cyclic path-generated modules. Let $u : i \rightsquigarrow j$ be a nonzero path in A. Then

$$\varphi : e_i A \longrightarrow u e_j A$$
$$x \longmapsto ux$$

is a projective cover. If $x = \sum \lambda_i x_i \in \ker \varphi$ is a linear combination of different nonzero paths x_i of A with all $\lambda_i \neq 0$, then the terms of $\sum \lambda_i u x_i$ are all different or 0, so by Lemma 3.1.1, $ux_i = 0$, that is, $x_i \in \ker \varphi$ for all i. □

Theorem 3.1.3. *A monomial algebra A is Koszul if and only if every simple module is in $\mathcal{C}_A^2$.*

Proof. One implication is trivial, so let us assume that all simple modules are in $\mathcal{C}_A^2$, or what is equivalent, $\operatorname{rad} A_A \in \mathcal{C}_A^1$.

By Lemma 3.1.1, we only need to show that if a cyclic path-generated module uA is in $\mathcal{C}_A^1$, then its syzygy is also a path-generated module in $\mathcal{C}_A^1$. By the condition, the syzygy $\Omega(uA)$ is a top submodule of $\operatorname{rad} A_A$, and by Lemma 3.1.2, it is also path-generated. Then the right-minimal paths of $\Omega(uA)$ must be also right-minimal in $\operatorname{rad} A$, otherwise they would be in $\Omega(uA) \cap \operatorname{rad}^2 A = \operatorname{rad}(uA)$. This implies by Lemma 3.1.1 that $\Omega(uA)$ is a direct summand of $\operatorname{rad} A$, so it is also in $\mathcal{C}_A^1$. □

As a consequnce of this theorem, we can formulate another useful characterization of Koszul algebras. Theorem 3.1.5 reveals a connection between structural and homological properties of monomial Koszul algebras.

Definition 3.1.4. Let $A = K\Gamma/I$ be a graph algebra. We say that A is quadratic if I is generated by homogeneous polinomials of degree 2.

Theorem 3.1.5. *Let $A = K\Gamma/I$ be a monomial algebra. Then the following are equivalent*

(i) A is Koszul;

(ii) Every simple A-module S belongs to $\mathcal{C}_A^2$;

(iii) A is quadratic.

Proof. Theorem 3.1.3 provides the equivalence of (i) and (ii), while the equivalence of (i) and (iii) is proven in [21]. □

We will utilize the threeway equivalence of Theorem 3.1.5 to arrive at the main result of this section presented in Theorem 3.1.8. We start by introducing the concept of *valleys*. Valleys are certain paths in a graph algebra with respect to a fixed order of its vertices (idempotents).

Definition 3.1.6. Let $A = K\Gamma/I$ a graph algebra, and u an element of A represented by the path $u = \alpha_1 \dots \alpha_k$ where α_i is an arrow from u_i to u_{i+1}. Thus the path u can be visualized as

$$\underset{u_1}{\bullet} \xrightarrow{\alpha_1} \underset{u_2}{\bullet} \xrightarrow{\alpha_2} \quad \cdots \quad \xrightarrow{\alpha_{k-1}} \underset{u_k}{\bullet} \xrightarrow{\alpha_k} \underset{u_{k+1}}{\bullet} \tag{3.1}$$

We say that u contains a valley, if there are indices j, k and m satisfying

$$j < k < m \qquad \text{such that} \qquad u_k \leq u_j, \text{ and } u_k < u_m.$$

It is clear that if u contains a valley, then it contains a short valley, i.e. a valley of length 2: let the middle vertex u_{k_0} be one of the smallest vertices in u such that k_o is the largest of the possible indices.

Proposition 3.1.7. *Let $A = K\Gamma/I$ be a standard Koszul monomial algebra. Then the non-zero paths of A do not contain valleys.*

Proof. Let us suppose indirectly that $u: \ u_1 \xrightarrow{\beta} u_2 \xrightarrow{\gamma} u_3$ is a short valley such that $u \neq 0$ in A.

Case 1: $u_1 < u_3$. We show that $\Delta(u_1)$ is not a Koszul module, in particular, $\Delta(u_1)$ fails to be in $\mathcal{C}^1_A$. Consider the projective cover of $\Delta(u_1)$:

$$0 \to \Omega_1(\Delta(u_1)) \to P(u_1) \to \Delta(u_1) \to 0.$$

Note that by Lemma 3.1.2, $\Omega_1(\Delta(u_1))$ is a path-generated module, and $\beta\gamma$ is right minimal in $\Omega_1(\Delta(u_1))$, so $\beta\gamma A \overset{\oplus}{\leq} \Omega_1(\Delta(u_1))$ by Lemma 3.1.1. If $\Omega_1(\Delta(u_1)) \overset{t}{\leq} \text{rad } P(u_1)$, then $\beta\gamma A$ would be a top submodule in rad $P(u_1)$, which is clearly a contradiction, as $0 \neq \beta\gamma A \leq \text{rad}^2\ P(u_1)$.

Case 2: $u_1 \geq u_3$. In this case we can show that $\overline{\Delta}^\circ(u_3) \notin \mathcal{C}^1_{A^\circ}$. For the projective cover of $\overline{\Delta}^\circ(u_3)$:

$$0 \to \Omega_1(\overline{\Delta}^\circ(u_3)) \to P^\circ(u_3) \to \overline{\Delta}^\circ(u_3) \to 0,$$

we have $A\beta\gamma \overset{\oplus}{\leq} \Omega_1(\overline{\Delta}^\circ(u_3))$, contradicting $\Omega_1(\overline{\Delta}^\circ(u_3)) \overset{t}{\leq} \text{rad } P^\circ(u_3)$. $\square$

Theorem 3.1.8. *Every standard Koszul monomial algebra is a Koszul algebra.*

Proof. To confirm the statement of the theorem we show that if $A = K\Gamma/I$ is a standard Koszul monomial algebra, then A is quadratic. The rest will follow from Theorem 3.1.5. Suppose that there is a minimal path $u \in I$ which is of length $k \geq 3$:

$$u: \quad \underset{u_1}{\bullet} \xrightarrow{\alpha_1} \underset{u_2}{\bullet} \xrightarrow{\alpha_2} \quad \cdots \quad \xrightarrow{\alpha_{k-1}} \underset{u_k}{\bullet} \xrightarrow{\alpha_k} \underset{u_{k+1}}{\bullet}$$

Case 1: $u_k < u_{k+1}$. We may assume that

$$u_1 < u_2 < \ldots < u_k < u_{k+1}$$

otherwise one of the subpaths $u_1 \to \ldots \to u_k$ and $u_2 \to \ldots \to u_{k+1}$ would be a non-zero path containing a valley. To show that $\Delta(u_1)$ is not a Koszul module, we take the minimal projective resolution of $\Delta(u_1)$

$$\ldots \to P_1(\Delta(u_1)) \to P_0(\Delta(u_1)) \to \Delta(u_1) \to 0.$$

Since $u_1 < u_2$, we get $\alpha_1 A \leq \Omega_1(\Delta(u_1))$. Moreover, $\alpha_1 A \overset{\oplus}{\leq} \Omega_1(\Delta(u_1))$ because α_1 is a right minimal path in the path-generated module $\Omega_1(\Delta(u))$.

So it is enough to show that $\alpha_1 A \notin \mathcal{C}_A$. By the choice of u, it follows that $\alpha_2 \ldots \alpha_k A \overset{\oplus}{\leq} \Omega_1(\alpha_1 A)$ because $\alpha_2 \ldots \alpha_k$ is a right minimal non-zero path in $\Omega_1(\alpha_1 A)$. On the other hand, $k \geq 3$, so $\alpha_2 \ldots \alpha_k A \leq \operatorname{rad}^2 P(u_2)$, hence $\alpha_2 \ldots \alpha_k A \overset{t}{\not\leq} \operatorname{rad}^2 P(u_2)$, which contradicts $\Omega_1(\alpha_2 A) \overset{t}{\leq} \operatorname{rad} P(u_2)$.

Case 2: $u_k \geq u_{k+1}$. We follow an idea similar to Case 1, and show that in this case $\overline{\Delta}^\circ(u_{k+1})$ cannot be Koszul. We look at the minimal projective resolution of $\overline{\Delta}^\circ(u_{k+1})$:

$$\ldots \to P_1(\overline{\Delta}^\circ(u_{k+1})) \to P_0(\overline{\Delta}^\circ(u_{k+1})) \to \overline{\Delta}^\circ(u_{k+1}) \to 0.$$

We see that $A\alpha_k \overset{\oplus}{\leq} \Omega_1(\overline{\Delta}^\circ(u_{k+1}))$, and also that

$$\begin{aligned} \varphi : Ae_{u_k} &\longrightarrow A\alpha_k \\ x &\longmapsto x\alpha_k \end{aligned}$$

is a projective cover of $A\alpha_k$. Since the path $\alpha_1 \dots \alpha_{k-1}$ is left-minimal, we get $A\alpha_1 \dots \alpha_{k-1} \overset{\oplus}{\leq} \ker \varphi$. But $k \geq 3$ gives

$$A\alpha_1 \dots \alpha_{k-1} \leq \operatorname{rad}^2 \, P^\circ(u_k),$$

which contradicts $\ker \varphi \overset{t}{\leq} \operatorname{rad} \, P^\circ(u_k)$. □

Finally, we can formulate a characterization of monomial standard Koszul algebras in terms of their underlying graphs and admissible ideals.

Theorem 3.1.9. *Let $A = K\Gamma/I$ be a quadratic monomial algebra. If no non-zero path of A contains a valley, then A is standard Koszul.*

Proof. We need to confirm that for every index j both $\Delta(j)$ and $\overline{\Delta}^\circ(j)$ are Koszul modules. Let us begin with the standard right modules. Let j be arbitrary, and take the projective cover of $\Delta(j)$:

$$0 \to \Omega_1(\Delta(j)) \to P(j) \to \Delta(j) \to 0.$$

It suffices to show that $\Omega_1(\Delta(j))$ is Koszul. By Lemmas 3.1.1 and 3.1.2,

$$\Omega_1(\Delta(j)) = \bigoplus_{v \in V} vA,$$

where V consists of paths from j to j' such that $j < j'$ and j' is the only vertex in with this property along the path. Clearly, if all direct summands vA are Koszul, then $\Omega_1(\Delta(j))$ is also Koszul.

If $v \in V$ is an arrow, then vA is a direct component of rad $P(j)$ which is a Koszul modul since A is Koszul Theorem 3.1.5. If v is not an arrow, then there exists an vertex j'' inside the path v such that $j'' \leq j$. In this case, $j'' < j'$ also holds, and this would mean that v is a valley, contradicting the conditions.

We turn our attention to the left proper standard modules. We use the fact that A° is also monomial and quadratic, hence Koszul. We consider the projective cover

$$0 \to \Omega_1(\overline{\Delta}^\circ(j)) \to P^\circ(j) \to \overline{\Delta}^\circ(j) \to 0.$$

Here $\Omega_1(\overline{\Delta}^\circ(j))$ can be decomposed as

$$\Omega_1(\overline{\Delta}^\circ(j)) = \bigoplus_{v \in V^\circ} Av,$$

where V° consists of paths v which end at j, start at j' such that if j'' is an inner vertex along the path v, then $j'' < j$ and $j'' < j'$. If $v \in V^\circ$ is an arrow, then $Av \overset{\oplus}{\leq} \operatorname{rad} P^\circ(j)$. If v is not an arrow then v is a short valley. $\square$

3.2 Special biserial algebras

The concept of biserial algebras was introduced by Tachikawa in [35], while the significant subclass of special biserial (SB) algebras was first studied in the work of Skowroński and Waschbüsch [33]. Self-injective, in particular, symmetric special biserial (SSB) algebras appear in the theory of modular representations of finite groups (see [23] and [31]), and play an important role in complex representations of the Lorentz group [22]. These algebras are often used to testing various conjectures (cf [17], [18] or [34]).

In the process, we give a full description of the (minimal) projective resolutions of simple modules over self-injective SB algebras that have no uniserial projective modules (Section 3.2.1). Here we use a path-building technique following the work of Antipov and Generalov [8] on SSB algebras, which is similar to the one that was used for monomial algebras (cf. [19]).

Finally, we conclude with the description of standard Koszul SSB algebras in terms of graphs and relations in Section 3.2.2.

3.2.1 Self-injective special biserial algebras

Let Γ be a graph and I an admissible ideal of $K\Gamma$. We write the product of two arrows $\alpha : i \to j$ and $\beta : j \to k$ as $\alpha\beta : i \to j \to k$. An algebra $A \cong K\Gamma/I$ is said to be *special biserial*, or SB for short, if for each vertex v of Γ, there are at most two arrows starting, and at most two arrows ending at v, furthermore, for each arrow α there exists at most one arrow β and at most one arrow γ such that $\beta\alpha, \alpha\gamma \notin I$.

An algebra A is *self-injective* if A_A is an injective A-module. A is a *Frobenius algebra* if $A_A \cong \mathrm{Hom}_K({}_AA, K)$ as right modules. Frobenius algebras are always self-injective, on the other hand, every self-injective basic algebra is Frobenius [14]. If A is a Frobenius algebra, then there exists a linear function $\varphi : A \to K$ such that $\ker\varphi$ does not contain any nontrivial right or left ideal of A. We call a Frobenius algebra *symmetric* if the above *Frobenius function* is symmetric, i.e. $\varphi(ab) = \varphi(ba)$ for all $a, b \in A$. By an *SSB algebra*, we mean an SB algebra with such a fixed *symmetric form* φ.

Similarly to the ideas of Antipov and Generalov in [8] about SSB algebras, we introduce a function δ, which operates on the (scalar multiples of) paths of a self-injective SB algebra A. Since A is self-injective, $\operatorname{soc} P(i)$ and $\operatorname{soc} P^\circ(i)$ are simple modules for all i. So for every path $u : i \rightsquigarrow j$, there exist paths v, w such that uv, wu are maximal nonzero paths, which generate $\operatorname{soc} P(i)$ and $\operatorname{soc} P^\circ(j)$, respectively. The function φ cannot vanish on the socle of any indecomposable projective summand, so there is a unique scalar multiple $\delta(u)$ of v such that $\varphi(u\delta(u)) = 1$; and this definition extends naturally to nonzero scalar multiples of paths. Note that in the symmetric case $v = w$, and uv is always a cycle [8].

Let $A = K\Gamma/I$ denote a self-injective SB algebra. There are two types of indecomposable projective modules over A. The module e_iA can be either *uniserial* (i. e. its submodules form a chain), or by the definition of biserial algebras (cf. [33]) and self-injectivity, its radical can be written as a sum $U + V$ of two uniserial submodules U and V such that $U \cap V = \operatorname{soc} e_iA$ is a simple module. Moreover, we see that each of the modules U and V is generated by an arrow, that is, $U = \alpha A$ and $V = \beta A$, where α and β are the two distinct arrows starting at the vertex i in the graph Γ.

We are going to prove that, apart from a few trivial exceptions, the indecomposable projective modules of a Koszul or standard Koszul self-injective SB algebra cannot be uniserial, and we construct the (minimal) projective resolutions of simple modules in these cases.

In the lemmas and propositions of the Section, we always assume that $A = K\Gamma/I$ is a self-injective connected SB algebra.

Lemma 3.2.1. *Let αA be the submodule of e_iA generated by the arrow α starting at the vertex i. Then αA is a top submodule of* rad e_iA. *Moreover, if $\alpha A \leq X \overset{t}{\leq} \operatorname{rad} e_iA$, then X is either αA or* rad e_iA.

Proof. If e_iA is uniserial, then the statement clearly holds. Suppose that e_iA is not uniserial, and let β be the other arrow starting at i. Then

$$\operatorname{rad}^2 e_iA \cap \alpha A = \operatorname{rad}(\alpha A + \beta A) \cap \alpha A = \operatorname{rad} \alpha A + \underbrace{\operatorname{rad} \beta A \cap \alpha A}_{\operatorname{soc} e_iA} = \operatorname{rad} \alpha A.$$

So $\alpha A \overset{t}{\leq} \operatorname{rad} e_iA$. Let $X \geq \alpha A$ be a top submodule of rad e_iA. Using Lemma 1.1 of [1], we get

$$\widetilde{X} := X/\alpha A \overset{t}{\leq} \operatorname{rad} e_iA/\alpha A,$$

but since $\operatorname{rad} e_iA = \alpha A + \beta A$ and $\alpha A \cap \beta A = \operatorname{soc} e_iA$,

$$\widetilde{X} \overset{t}{\leq} \operatorname{rad} e_iA/\alpha A \cong \beta A/\operatorname{soc} \beta A.$$

The module $\beta A/\operatorname{soc} \beta A$ is uniserial, so if $\widetilde{X}$ is a top submodule of $\operatorname{rad} e_iA/\alpha A$, then $\widetilde{X} = 0$ or $\widetilde{X} = \operatorname{rad} e_iA/\alpha A$, that is, $X = \alpha A$ or $X = \operatorname{rad} e_iA$. □

Lemma 3.2.2. *Suppose that e_iA is an at least three-dimensional uniserial module for some i. If its simple top $S(i)$ is in $\mathcal{C}_A^m$, then $\Omega_k(i)$ is also uniserial for all $1 \leq k \leq m$, moreover, for all such k, the syzygy $\Omega_k(i)$ is generated by an arrow, and* $\dim_{\mathrm{K}} \Omega_k(i) \geq 2$.

Proof. We prove the lemma by induction on k. The first step is trivial. For the induction step, we need the following. Take a submodule $U = \alpha A$ generated by the arrow $\alpha : j \to \ell$ in the indecomposable projective module e_jA such that $U \in \mathcal{C}_A^1$ and $\dim_{\mathrm{K}} U \geq 2$. Let $\varphi : \mathcal{P}(U) \to U$ be its projective cover. U is uniserial by the SB property but $\mathcal{P}(U) = e_\ell A$ is not, since otherwise $\ker \varphi \leq \operatorname{rad}^2 \mathcal{P}(U)$, contradicting $U \in \mathcal{C}_A^1$. By the SB property, $\alpha\beta = 0$ for one arrow β starting at ℓ. Thus $\beta A \leq \ker \varphi \overset{t}{\leq} \operatorname{rad} e_\ell A$, so Lemma 3.2.1 gives that $\ker \varphi$ is either βA or $\operatorname{rad} e_\ell A$. But the latter is impossible, since $\dim_{\mathrm{K}} U \geq 2$. Finally, βA is at least two-dimensional, since $\mathcal{P}(U)$ is not uniserial. □

Lemma 3.2.3. *Suppose that e_iA is an at least three-dimensional uniserial module for some i. Then $S(i) \notin \mathcal{C}_A$.*

Proof. First, observe that if $X, Y \in$ mod-A and

$$0 \to Y \to P_m \to \ldots \to P_0 \to X \to 0$$

is a non-split exact sequence having indecomposable projective intermediate terms, then we have $\Omega_{m+1}(X) \cong Y$ and by the self-injectivity $\Omega^{-(m+1)}(Y) \cong X$, where $\Omega^{-k}(Y)$ is the kth cosyzygy of the module Y.

Now, assume that $S(i) \in \mathcal{C}_A$, consequently, $\Omega_1(i) = \text{rad } P(i) \in \mathcal{C}_A$. Lemma 3.2.2 yields that the submodule $\Omega_k(i)$ is generated by an arrow in $P_{k-1}(i)$ for each k, moreover, $\dim_{\mathrm{K}} \Omega_k(i) \geq 2$. This also implies that every projective term in the projective resolution of $S(i)$ is indecomposable. But there exist only finitely many modules of the form αA, so there exists a smallest index k and a smallest integer h for which $\Omega_k(i) \cong \Omega_{k+h}(i)$. By our previous observation,

$$S(i) \cong \Omega^{-k}(\Omega_k(i)) \cong \Omega^{-k}(\Omega_{k+h}(i)) \cong \Omega_h(i),$$

which contradicts $\dim_{\mathrm{K}} \Omega_h(i) \geq 2$. □

We have seen in Lemma 3.2.3 that e_iA cannot be uniserial if $\dim_{\mathrm{K}} e_iA \geq 3$ and A is Koszul. However, there are cases when e_iA is uniserial. If A is local, then A_A may be uniserial: K and $K[x]/(x^2)$ are both Koszul and standard Koszul algebras. If A is non-local and there exists at least one index i for which e_iA is uniserial, then the structure of the algebra is very special as we will see in the next proposition.

Proposition 3.2.4. *Suppose that A is non-local and Koszul. Assume that there exists an index i for which the module e_iA is uniserial. Then all the indecomposable projective modules are uniserial. Furthermore, $A \cong K\Gamma/I$, where Γ is a directed cycle on n vertices, and I is generated by all the paths of length 2.*

Proof. First, observe that if e_iA is uniserial for some i, then $\dim_{\mathrm{K}} e_iA = 2$ by Lemma 3.2.3 and the connectedness of A. Suppose that there exists a non-uniserial indecomposable projective module. Let $\mathcal{U} = \{e_iA \mid e_iA \text{ is uniserial}\}$

and $\mathcal{N} = \{e_jA \mid e_jA$ is not uniserial$\}$. The elements of $\mathcal{U}$ are two-dimensional projective-injective modules, so the image of every non-trivial morphism between elements of $\mathcal{U}$ and $\mathcal{N}$ (in either direction) is simple. Let us consider the bijection $f : \mathcal{U} \cup \mathcal{N} \to \mathcal{U} \cup \mathcal{N}$ which maps each projective module to the injective envelope of its top. Since A is connected, $f(\mathcal{U}) \neq \mathcal{U}$ and $f(\mathcal{N}) \neq \mathcal{N}$. So there are indices i, j such that e_iA is uniserial but e_jA is not, and $\operatorname{soc} e_iA \cong S(j)$.

Let γ denote the unique arrow from i to j. The module e_jA is not uniserial, so we can find distinct arrows $\alpha : j \to {}'j$ and $\beta : j \to j'$. Note that both $\gamma\alpha$ and $\gamma\beta$ are 0. Considering the indecomposable injective modules $I({}'j)$ and $I(j')$ corresponding to the vertices ${}'j$ and j', we see that none of them is equal to $P(j)$, so α and β cannot be maximal nonzero paths from the right. It means that they are not maximal nonzero paths from the left either. So there exist arrows α', β' such that $\alpha'\alpha$ and $\beta'\beta$ are not 0. The arrows α', β' are distinct from γ (since $\gamma\alpha = \gamma\beta = 0$) and are also distinct from each other by the SB property of A. So there would be three distinct arrows ending at j, which is impossible in the graph of a biserial algebra.

Finally, since A is a connected and self-injective algebra with only two-dimensional indecomposable projective modules, the graph of A is indeed a directed cycle. □

Remark 3.2.5. Note that in the second part of the proof, we have actually shown that if A does not have any uniserial projective module with dimension greater than 2, but possesses a uniserial projective module with dimension 2, then A must be an algebra described above.

We would like to show that a similar statement holds for standard Koszul self-injective algebras, namely, if the module e_iA with dimension at least 3 is uniserial, then A cannot be standard Koszul. That would mean – aside from the cases above – that it is enough to investigate algebras for which all the vertices in Γ have in- and out-degree 2.

Before we do that, we would remind the reader that the properties SB, self-injective and symmetric are "side-independent". That is, an algebra A is SB/self-injective/symmetric if and only if A° is SB/self-injective/symmetric.

Furthermore, as we noted before, A is Koszul if and only if A° is Koszul (cf. [20]). We will use these facts later on.

Lemma 3.2.6. *If A is a non-simple standard Koszul algebra having a uniserial projective module, then every indecomposable projective A-module is two-dimensional.*

Proof. Suppose that e_iA is uniserial with $\dim_{\mathrm{K}} e_iA \geq 3$ and $\operatorname{soc} e_iA \cong S(t)$. Now, A° is also self-injective SB, and Ae_t is a uniserial A°-module. Since $\Delta(i)$ and $\overline{\Delta}^\circ(t)$ are in $\mathcal{C}_A^1$ and $\mathcal{C}_{A^\circ}^1$, respectively, both of them have to be either simple or projective. If $\Delta(i)$ is projective, then $\overline{\Delta}^\circ(t)$ is not, hence it is simple. Both cases contradict Lemma 3.2.3.

If A is a non-local, then Remark 3.2.5 gives that $A \cong K\Gamma/I$, where Γ is a directed cycle on n vertices and I is generated by all the paths of length 2. So every indecomposable projective module is two-dimensional. □

In Proposition 3.2.7, we summarize the previous lemmas and observations.

Proposition 3.2.7. *If A is a standard Koszul self-injective SB algebra that has a uniserial projective module, then A is Koszul.*

Proof. The algebras K and $K[x]/(x^2)$ are Koszul, while the algebras described in Proposition 3.2.4. are quadratic and monomial, hence Koszul, cf. [21]. □

Now, we can move on, and may assume that Γ consists of vertices with in- and out-degree 2. We construct the minimal projective resolutions of simple modules over algebras of this type. Although our description uses Loewy diagrams – similarly to the diagrammatic method of [8] – we also give explicit calculations. These extend the result of Proposition 3.3 b) in [8].

As an aid for describing the projective resolutions of simple modules, we define a graph on $\mathbb{N} \times \mathbb{N}$ for each simple module $S(s)$ (Fig. 3.1).

The origin, $(0,0)$ represents the vertex $s \in \Gamma$. For all pairs (i,j), there are exactly two arrows starting at (i,j): $a_{i,j}$ to $(i+1,j)$ and $b_{i,j}$ to $(i,j+1)$. The arrows represent scalar multiples of nonzero paths in the following way. First, $a_{0,0}$ and $b_{0,0}$ are just the two arrows starting at s. The leftmost and the

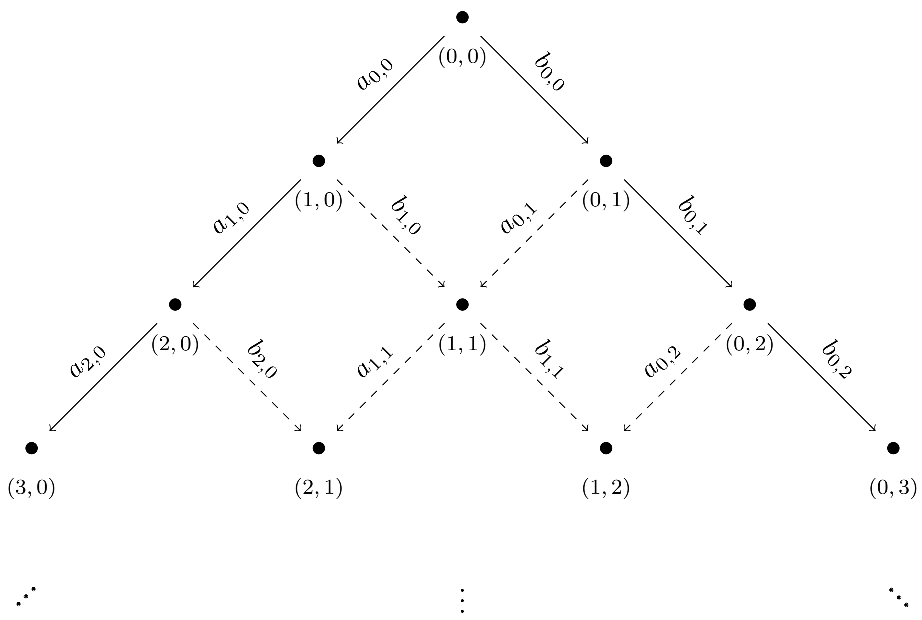
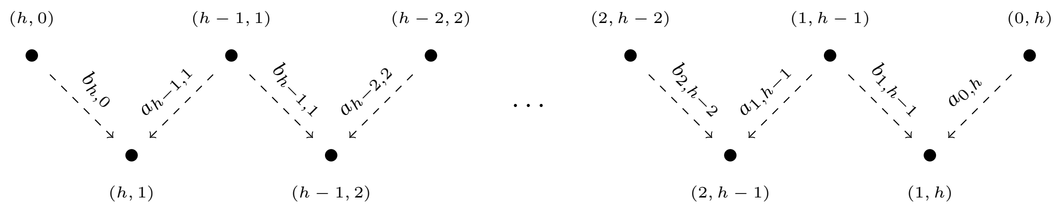


Figure 3.1: the graph corresponding to the resolution of $S(s)$

rightmost arrows are defined recursively such that $a_{i+1,0}$ is the unique arrow of Γ for which $a_{i,0}a_{i+1,0}$ is a minimal zero path. The definition of $b_{0,i+1}$ is analogous. The other arrows are defined via the function δ. Let $a_{i,j}$ be $\delta(b_{i,j-1})$, for $j \geq 1$, and similarly, $b_{i,j} = \delta(a_{i-1,j})$ for $j \geq 1$. Let the common endpoint of the paths determined by $a_{i-1,j}$ and $b_{i,j-1}$ correspond to (i, j). We will denote the indecomposable projective module corresponding to the vertex assigned to (i, j) by $P(i, j)$.

Proposition 3.2.8 links this graph of s with the projective resolution of the simple module $S(s)$, namely, it will turn out that the hth term in the projective resolution of $S(s)$ is isomorphic to the direct sum of the indecomposable projective modules corresponding to the vertices contained in the hth level. Moreover, the syzygies can be tracked by stepping downwards on the levels of the graph.

Proposition 3.2.8. *Let s be a vertex in the graph Γ of A, where Γ consists of vertices with in- and out-degree 2. The Loewy-diagram of the hth syzygy of $S(s)$ follows the row*



of Fig. 3.1. In fact, the hth term of the minimal projective resolution of the simple module $S(s)$ is

$$\mathcal{P}_h = \bigoplus_{i=0}^{h} P(i, h-i), \tag{3.2}$$

while the corresponding kernel is

$$\Omega_{h+1} = \widetilde{a}_{h,0}A + \sum_{i=0}^{h-1} \left(b_{h-i,i} + \widetilde{a}_{h-i-1,i+1}\right) A + b_{0,h}A, \tag{3.3}$$

where $\widetilde{a}_{h-i,i} = (-1)^h a_{h-i,i}$.

Proof. We prove the statement by induction on h. For $h = 0$, it is obvious that $\mathcal{P}_0 = P(0,0)$ and $\Omega_1 = a_{0,0}A + b_{0,0}A$. Let us assume that our statement holds for some h.

Let $d_{h+1} : \bigoplus_{i=0}^{h+1} P(i, h+1-i) \to \Omega_{h+1}$ be described by the matrix

$$d_{h+1} = \begin{bmatrix} \widetilde{a}_{h,0} & b_{h,0} & & & & & & \\ & \widetilde{a}_{h-1,1} & b_{h-1,1} & & & & & \\ & & \ddots & \ddots & & & & \\ & & & \widetilde{a}_{h-i,i} & b_{h-i,i} & & & \\ & & & & \ddots & \ddots & & \\ & & & & & \widetilde{a}_{1,h-1} & b_{1,h-1} & \\ & & & & & & \widetilde{a}_{0,h} & b_{0,h} \end{bmatrix}$$

acting from the left. The map d_{h+1} is clearly surjective, and it is easy to check from Fig. 3.1 that

$$\widetilde{a}_{h+1,0}A + \sum_{i=0}^{h} \left(b_{h+1-i,i} + \widetilde{a}_{h-i,i+1}\right) A + b_{0,h+1}A$$

is in the kernel $\ker d_{h+1}$. On the other hand, for an arbitrary $x \in \ker d_{h+1}$, we have

$$\widetilde{a}_{h-i,i}[x]_i + b_{h-i,i}[x]_{i+1} = 0$$

for all i, so

$$\widetilde{a}_{h-i,i}[x]_i = -b_{h-i,i}[x]_{i+1} \in \operatorname{soc} P(h-i,i),$$

thus $[x]_i \in \operatorname{rad} P(h+1-i,i)$ for all i. This implies that d_{h+1} is the projective cover of Ω_{h+1}. Furthermore,

$$[x]_i = \alpha_{h-i+1,i} r_i + \beta_{h-i+1,i} s_i, \quad i = 0, \dots, h+1,$$

for some $s_i, r_i \in A$, where $\alpha_{h-i+1,i}$ and $\beta_{h-i+1,i}$ are the two distinct arrows (starting at the vertex $(h-i+1,i)$) in the ith summand of $\mathcal{P}_{h+1}$. Note that $\alpha_{h-i+1,i}$ and $\beta_{h-i+1,i}$ are the starting segments of $a_{h-i+1,i}$ and $b_{h-i+1,i}$, respectively, so both $\widetilde{a}_{h-i,i}\alpha_{h-i+1,i}$ and $b_{h-i,i}\beta_{h-i,i+1}$ are zero according to the graph defined in Fig. 3.1. Let us compute $[d_{h+1}x]_i$.

$$[d_{h+1}x]_i = \widetilde{a}_{h-i,i}\beta_{h-i+1,i}s_i + b_{h-i,i}\alpha_{h-i,i+1}r_{i+1} = 0. \tag{3.4}$$

The two terms in (3.4) have their images in distinct components of the factor $\operatorname{rad} \mathcal{P}_h / \operatorname{soc} \mathcal{P}_h$, so both terms must be in the socle of $\mathcal{P}_h$. Hence,

$$\beta_{h-i+1,i}s_i = b_{h-i+1,i}s'_i \quad \text{and} \quad \alpha_{h-i,i+1}r_{i+1} = a_{h-i,i+1}r'_{i+1}.$$

By the definitons of the monomials a, b and the function δ, we can write

$$\widetilde{a}_{h-i,i}b_{h-i+1,i}s'_i = (-1)^h b_{h-i,i}a_{h-i,i+1}s'_i.$$

Therefore, after rewriting (3.4),

$$b_{h-i,i}a_{h-i,i+1}((-1)^h s'_i + r'_{i+1}) = 0,$$

that is, $(-1)^h s'_i + r'_{i+1} = j \in \operatorname{Ann}_r(b_{h-i,i}a_{h-i,i+1})$. Let us express j as a linear combination of paths u_i of A, and separate the terms:

$$j = \sum \lambda_i u_i = \underbrace{\sum_{a_{h-i,i+1}u_i = 0} \lambda_i u_i}_{j'} + \underbrace{\sum_{a_{h-i,i+1}u_i \neq 0} \lambda_i u_i}_{j''}.$$

Note that $b_{h-i+1,i}j'' = 0$ by the SB property, so we get

$$[x]_i = \alpha_{h-i+1,i} r_i + b_{h-i+1,i}(s'_i + (-1)^{h+1} j'')$$

and

$$[x]_{i+1} = a_{h-i,i+1}((-1)^{h+1} s'_i + j'') + \beta_{h-i+2,i} s_{i+1},$$

and on the right-hand side of the latter, the first term is $\widetilde{a}_{h-i,i+1}(s'_i + (-1)^{h+1} j'')$. Applying this for every index i, we see that

$$\ker d_{h+1} \leq \widetilde{a}_{h+1,0} A + \sum_{i=0}^{h} (b_{h-i+1,i} + \widetilde{a}_{h-i,i+1}) A + b_{0,h+1} A$$

because at both ends, $a_{h+1,0}$ and $b_{0,h+1}$ are just the arrows $\alpha_{h+1,0}$ and $\beta_{0,h+1}$. $\square$

Using the graph defined above, we can now give a characterization of Koszul self-injective SB algebras.

Proposition 3.2.9. *With notations as above, A is Koszul if and only if for all vertices s in Γ, one of the elements $a_{0,1}$ and $b_{1,0}$ is a scalar multiple of an arrow.*

Proof. First, it is easy to see that if for a vertex $s \in \Gamma$, both of $a_{0,1}$ and $b_{1,0}$ are in $\mathrm{rad}^2\, A$, then $\Omega_2(s) \overset{t}{\nleq} \mathrm{rad}\, P_1(s)$, hence the simple module $S(s) \notin \mathcal{C}_A^2$.

Conversely, fix an arbitrary vertex s. It is enough to show that

$$a_{0,1} \text{ or } b_{1,0} \text{ is an arrow} \Rightarrow (\forall h : \mathrm{rad}^2\, \mathcal{P}_h \cap \Omega_{h+1} \leq \mathrm{rad}\, \Omega_{h+1}).$$

Note that on the right-hand side of (3.3), each generating element (except $a_{h,0}$ and $b_{0,h}$) is a linear combination of two paths. One term of the combination is a scalar multiple of an arrow, otherwise these two paths would be the end segments of two distinct nonzero paths with length at least 3, ending at the same vertex. Then we could find an appropriate vertex s' for wich the condition of the statement fails.

Let h and $x \in \Omega_{h+1}$ be arbitrary. By using the form (3.3) of Ω_{h+1} and rearranging, we can express x as

$$x = \sum_{i=0}^{h} \widetilde{a}_{h-i,i} r_i + b_{h-i,i} r_{i+1} \tag{3.5}$$

with appropriate elements $r_i \in A$. In (3.5), the members are sorted so that for each i, the term $\widetilde{a}_{h-i,i}r_i + b_{h-i,i}r_{i+1}$ belongs to the same direct component of $\mathcal{P}_h$, therefore, $x \in \operatorname{rad}^2 \mathcal{P}_h$ means that all these terms are in the radical-square of the component they are contained in. Focusing on one particular term, we observe that its image in $\operatorname{rad} \mathcal{P}_h / \operatorname{soc} \mathcal{P}_h$ is a combination of two elements from distinct components, so each member of this sum has to be in the radical-square, hence $r_i \in \operatorname{rad} A$ for all $i \geq 1$. □

Corollary 3.2.10. *A is Koszul if and only if all of its simple modules are in $\mathcal{C}_A^2$.*

Proof. One direction is obvious. For the converse, we observe that if both elements $a_{0,1}$ and $b_{1,0}$ are in $\operatorname{rad}^2 A$, then $S(s) \notin \mathcal{C}_A^2$. □

In the remaining part of the section, we will prove that a standard Koszul self-injective SB algebra satisfies the conditions of Proposition 3.2.9 for being Koszul. Without loss of generality we may assume that there are no uniserial projective modules.

Definition 3.2.11. Let $u = s_1 \to s_2 \to \ldots \to s_m$ be a path in Γ. We say that u contains a *valley* if there exist indices $j < k < \ell$ for which $s_j \geq s_k$ and $s_k < s_\ell$.

Lemma 3.2.12. *If $u = s_1 \to s_2 \to \ldots \to s_m$ contains a valley, then it also contains a short valley, that is, there exists an index i such that $s_{i-1} \geq s_i$ and $s_i < s_{i+1}$.*

Proof. Let $u' : s_j \rightsquigarrow s_k \rightsquigarrow s_m$ be the subpath of u with $j < k < m$ such that $s_j \geq s_k$ and $s_k < s_m$. Let t denote the largest index between j and m for which s_t is minimal. Then $s_{t-1} \to s_t \to s_{t+1}$ is a short valley. □

Lemma 3.2.13. *Let u be a nonzero path such that $|u|$ (the length of u) is at least 3. If u contains a valley, then A is not standard Koszul.*

Proof. Suppose that u starts at the vertex i. We may assume that e_iA is not uniserial (see Lemma 3.2.6). By the previous lemma, there is a subpath $u' : j \to k \to m$ of u, which is a short valley. If $j > m$, then $u'A$ is a top submodule of $\Omega(\Delta(j))$ but it is not a top submodule of rad e_jA, so $\Delta(j) \notin \mathcal{C}_A^1$. On the other hand, if $j \leq m$, then a similar argument shows that $\overline{\Delta}^\circ(m) \notin \mathcal{C}_{A^\circ}^1$. □

Proposition 3.2.14. *Let A be standard Koszul. For all i, at least one of the maximal nonzero paths in e_iA has length 2.*

Proof. Assume - on the contrary - that e_iA contains two distinct paths u, v with length at least 3. Let t denote the common endpoint of u and v:

$$u : i \to {}'i \to {}''i \to \ldots \to {}^{(b)}i \to t$$
$$v : i \to i' \to i'' \to \ldots \to i^{(r)} \to t,$$

where $r, b \geq 3$.

Recall from the proof of Proposition 3.2.9 that if for some s, both $a_{0,1}$ and $b_{1,0}$ are in $\mathrm{rad}^2 A$, then the simple module $S(s)$ fails to be in $\mathcal{C}_A^2$. Therefore, $S(i) \notin \mathcal{C}_A^2$ and $S^\circ(t) \notin \mathcal{C}_{A^\circ}^2$, so $\Delta(i) \neq S(i)$ and $\overline{\Delta}^\circ(t) \neq S^\circ(t)$. These also imply that none of $\Delta(i)$ and $\overline{\Delta}^\circ(t)$ can be projective. For example, if $\Delta(i)$ is projective, then ${}^{(b)}i, i^{(r)} \leq i$, and since $\overline{\Delta}^\circ(t) \neq S^\circ(t)$, one of $i^{(r)}$ and ${}^{(b)}i$ – let us say ${}^{(b)}i$ – must be less than t. But then u contains a valley, contradicting Lemma 3.2.13. Similarly, if $\overline{\Delta}^\circ(t) = Ae_t$, then u or v contains a valley.

So neither $\Delta(i)$ nor $\overline{\Delta}^\circ(t)$ is simple or projective, hence exactly one of i' and ${}'i$ is greater than i, and exactly one of ${}^{(b)}i$ and $i^{(r)}$ is less than t.

Since none of the paths u and v contains a valley, it can be assumed that the indices are increasing along u and decreasing along v. That would mean that both $i < t$ and $t < i$, a contradiction. □

Theorem 3.2.15. *If A is a self-injective standard Koszul SB algebra, then A is Koszul.*

Proof. The theorem is an easy consequence of Propositions 3.2.7, 3.2.9 and 3.2.14. □

3.2.2 Standard Koszul symmetric special biserial algebras

We now turn our attention to SSB algebras. In this section, let $A = K\Gamma/I$ be a standard Koszul SSB algebra. Recall that the existence of the symmetric form φ implies that $\operatorname{soc} A_A$ is generated by cycles. Moreover, if the path $u = \alpha_1, \alpha_2 \ldots, \alpha_k$ is in $\operatorname{soc} A_A$, then the paths $\alpha_i, \alpha_{i+1}, \ldots, \alpha_{i-1}$ are in the socle of A_A for all i.

Observe that if e_iA is an indecomposable projective A-module, and all composition factors of e_iA are isomorphic to $S(i)$, then the fact that A is connected implies that $A_A = e_iA$, i.e. A is local.

First, we handle non-local algebras; local algebras will be discussed later. So from now on, A possesses at least two non-isomorphic simple modules, and for every i, there exists $j \neq i$ such that $S(j)$ is a composition factor of e_iA. Thus $\dim_{\mathrm{K}} e_iA \geq 3$ for every i. This condition, along with the former results of the paper, implies the next statement.

Proposition 3.2.16. *If A is non-local, then no e_iA is uniserial.*

We can use combinatorial arguments again to characterize the graphs Γ for which $K\Gamma/I$ can be standard Koszul. To obtain the generating relations of I for such graphs, we have to use only that A is symmetric (i.e. the existence of some symmetric form φ).

Proposition 3.2.17. *If A is standard Koszul, and u is a maximal nonzero path in A with $|u| \geq 3$, then u is a power of a loop.*

Proof. Let $u : u_1 \to u_2 \to \ldots \to u_m \to u_1$ be a maximal nonzero path with $m \geq 3$. Suppose that u passes through at least two distinct vertices. Let u_k be a maximal vertex in u. Now, $u' : u_k \to u_{k+1} \to \ldots \to u_{k-1} \to u_k$ is also a maximal nonzero path. Since u_k is maximal, the path u' contains a valley, contradicting Lemma 3.2.13. □

Corollary 3.2.18. *If A is non-local, then all the vertices in Γ are contained in exactly two maximal nonzero cycles. If one of these cycles has length greater than 2, then it is a power of a loop.*

Lemma 3.2.19. *If $\alpha : a \to b$ is an arrow in Γ with $a \neq b$, then there must also exist a unique arrow $\beta : b \to a$. Moreover, both $\alpha\beta$ and $\beta\alpha$ are maximal nonzero paths.*

Proof. The existence of such β is just a reformulation of Proposition 3.2.17 and Corollary 3.2.18. For the uniqueness, suppose that there are two arrows $\beta_{1,2} : b \to a$. Using the connectedness and the SSB property of A, along with Corollary 3.2.18, we see that Γ consists of two vertices and the four arrows $\alpha_{1,2} : a \to b$ and $\beta_{1,2} : b \to a$. It follows from Corollary 3.2.18 that $\overline{\Delta}^\circ(2) = Ae_2/\operatorname{soc} Ae_2$ but this is not in $\mathcal{C}^1_{A^\circ}$. □

Lemma 3.2.20. *Γ contains at most 2 vertices that are endpoints of loops.*

Proof. Let us assume that there are (at least) three vertices in Γ that are endpoints of loops. Let them be s_1, s_2, s_3. Since A is connected, there are directed paths from s_1 to s_2 and from s_1 to s_3. Let u_1 and u_2 be two such paths with minimal length. Let t be the vertex where u_1 and u_2 differ for the first time. Using Lemma 3.2.19, it is easy to check that both the in- and out-degree of t is at least 3, and this contradicts the SB property of A. □

Lemma 3.2.21. *If A is non-local, then Γ contains exactly 2 vertices that are endpoints of loops.*

Proof. First, suppose that there are no loops in Γ. We take an arbitrary vertex s_0. Since A is connected, s_0 has a neighbour, i.e. there exists an arrow $\alpha_1 : s_0 \to s_1$ (and by Lemma 3.2.19 an arrow $\beta_1 : s_1 \to s_0$, too). The vertex s_1 also has out-degree 2, so there must be an arrow $\alpha_2 : s_1 \to s_2 \neq s_0$, and so on. At some point, s_n coincides with s_0. This means that the arrows $\alpha_0, \alpha_1, \ldots, \alpha_n$ and $\beta_0, \beta_1, \ldots, \beta_n$ in Γ form two disjoint, parallel and oppositely directed cycles on n vertices, and the maximal nonzero paths of A are of the form $\alpha_i\beta_i$ and $\beta_i\alpha_i$. One can check that $\overline{\Delta}^\circ(n) = P^\circ(n)/\operatorname{soc} P^\circ(n) \notin \mathcal{C}^1_{A^\circ}$.

We can repeat the first part of the previous argument in the situation where Γ contains exactly 1 loop. Now, starting with the vertex s_0 (which belongs to the loop) and running over the vertices $s_1, s_2, \ldots$ will lead us to some vertex s_n that has to coincide with a former vertex. But that vertex would have in- and out-degree (at least) 3. □

Corollary 3.2.22. *If Γ has at least 2 vertices (i.e. A is not local), then Γ has the shape shown in Fig. 3.2.*

Figure 3.2: the graph of a non-local standard Koszul SSB algebra

Proposition 3.2.23. *Let $A = K\Gamma/I$ be a non-local standard Koszul SSB algebra. Then Γ has the shape shown in Fig. 3.2, and I is generated by the relations*

$$\begin{aligned} I = \big(&\alpha_i\alpha_{i+1},\ \beta_{i+1}\beta_i,\ \gamma\alpha_1,\ \alpha_{n-1}\delta,\ \beta_1\gamma,\ \delta\beta_{n-1},\ \alpha_{i+1}\beta_{i+1} - \beta_i\alpha_i, \\ &\gamma^k - \alpha_1\beta_1,\ \lambda\delta^m - \beta_{n-1}\alpha_{n-1} \mid i = 1, \ldots, n-2\big),\ \lambda \in K \setminus \{0\},\ k, m \geq 2. \end{aligned} \tag{3.6}$$

The vertices are indexed so that the indices are increasing from the vertex with the smallest index towards either end according to Fig 3.2.

Proof. We may assume that I is generated by paths and differences of paths except the term $\lambda\delta^m - \beta_{n-1}\alpha_{n-1}$. Otherwise, we can exchange the arrows for their scalar multiples repeatedly (let us say moving from left according to the graph in Fig. 3.2). At the last vertex, we might not be able to do this if K is not algebraically closed.

If the ordering differs from the one we stated, then there is a vertex k having neighbours only with lower indices. In this situation $\overline{\Delta}^\circ(k) \notin \mathcal{C}^1_{A^\circ}$. □

Let us investigate now the local algebras. There are two cases depending on the degree of the single vertex $s \in \Gamma$. If there is only one arrow in Γ, then A is monomial and commutative. Therefore A is standard Koszul if and only if $\overline{\Delta}(1) = S(1) \in \mathcal{C}_A$. Hence $A \cong K$ or $K[x]/(x^2)$.

For the other case, suppose that $s \in \Gamma$ has degree 2, and let the two arrows be x and y. If $xy \neq 0$, then $x^2 = 0$ and $y^2 = 0$ by the SB property. We show that $xyx = 0$. Assume – on the contrary – that $xyx \neq 0$. If $xyxy = 0$, then $\varphi(xyxy) = \varphi(yxyx) = 0$, and $AxyxA$ would be a proper ideal in $\ker\varphi$ because we have $\varphi(xyx) = \varphi(x^2y) = 0$. Otherwise, if $xyxy \neq 0$, then both $\dim_K yA$ and $\dim_K xA \geq 3$, hence A is not standard Koszul (cf. Lemma 3.2.14). Therefore if $xy \neq 0$, then $xyx = 0$, and similarly, $yxy = 0$. Since A is symmetric, $\varphi(xy - yx) = 0$, and $x^2 = y^2 = xyx = yxy = 0$ implies that the ideal generated by $xy - yx$ is in $\ker \varphi$, so $xy = yx$. Consequently, $A \cong K[x, y]/\left(x^2, y^2\right)$.

Suppose that $xy = 0$. Then $yx = 0$ (otherwise $AyxA$ is a proper ideal in $\ker \varphi$), but since the socle is simple, $x^2, y^2 \neq 0$. Besides, if k, m are the smallest integers such that x^k and y^m are in the socle, then $I = \left(xy, yx, x^k - \lambda y^m\right)$, where $\lambda \in K \setminus \{0\}$. Since A is standard Koszul, we may assume, for example, that $k = 2$.

Theorem 3.2.24. *$A = K\Gamma/I$ is a standard Koszul SSB algebra if and only if either*

(a) A is isomorphic to one of the K-algebras: K, $K[x]/\left(x^2\right)$, $K[x, y]/\left(x^2, y^2\right)$, $K\langle x, y\rangle/\left(xy, yx, x^2 - \lambda y^m\right)$, or

(b) Γ has the shape shown in Fig. 3.2 and I is generated by the relations of (3.6).

Proof. In the local case, we have shown that the conditions of (a) are necessary. Note that if A is local, then it is standard Koszul if and only if it is Koszul. One may apply our former observations or Proposition 3.2.9 to the algebras described in (a), and see that they are standard Koszul. In the non-local case, we have seen that the conditions of (b) are necessary.

Suppose now that the condition (b) holds for A. We show that both $\Delta(i)$ and $\overline{\Delta}^{\circ}(i)$ are Koszul for all i. We may assume that none of them is projective or simple. (Projective modules are in $\mathcal{C}_A$, and if A satisfies the conditions of (b), then it is Koszul by Proposition 3.2.9, so all its simple left and right modules are Koszul.) If $\Delta(i)$ (or $\overline{\Delta}^{\circ}(i)$) is neither simple nor projective, then we can see from the induction step in the proof of Lemma 3.2.2 that each of the syzygies $\Omega_h(\Delta(i))$ (or $\Omega_h(\overline{\Delta}^{\circ}(i))$) is generated by a respective arrow for all h. According to Lemma 3.2.1, these submodules are top submodules of the radical of the projective module that they are contained in.

The only thing to check is whether these algebras are symmetric, since they are obviously SB. For the algebras in (a), let φ be 1 on an arbitrary basis element of $\operatorname{soc} A_A$ and 0 on all the other subspaces generated by paths. For the algebras in (b), define φ to be 1 on all the maximal nonzero paths of A except that $\varphi(\delta^m) = 1/\lambda$. Let φ vanish on all the other paths. It is easy to check that these functions can be extended to symmetric forms for the given algebras. $\square$

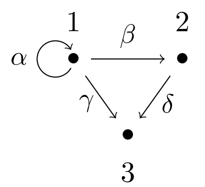
Appendix A

Examples

We conclude the work with a few examples. Some of them point out differences between the behaviour of quasi-hereditary algebras and standardly stratified algebras with respect to the machinery developed in Chapter 2, while others show why some of the results presented here can not be strengthened.

Example A.1. Let us start our examples with a standard Koszul standardly stratified algebra together with its extension algebra. The purpose of this example is to demonstrate the results of Chapter 2. The algebra $A = K\Gamma/I$ is given by

Γ :



$$I = \big(\alpha^2, \alpha\beta, \alpha\gamma - \beta\delta\big)$$

We give their right and left regular representations by their Loewy diagrams.

$$A_A = \begin{array}{ccccc} & & 1 & & \\ & / & \mid & \backslash & \\ 3 & & 1 & & 2 \\ & & \backslash & / & \\ & & & 3 & \end{array} \oplus \begin{array}{c} 2 \\ \mid \\ 3 \end{array} \oplus 3 \quad \text{and} \quad {}_AA = \begin{array}{c} 1 \\ \mid \\ 1 \end{array} \oplus \begin{array}{c} 2 \\ \mid \\ 1 \end{array} \oplus \begin{array}{ccc} & 3 & \\ / & & \backslash \\ 1 & & 2 \\ \backslash & & / \\ & 1 & \end{array}$$

It is easy to check that the standard modules $\Delta(1) = \begin{smallmatrix} 1 \\ \mid \\ 1 \end{smallmatrix}$, $\Delta(2) = S(2)$ and $\Delta(3) = P(3) = S(3)$ are Koszul. Similarly, $\overline{\Delta}^\circ(1) = S^\circ(1)$, $\overline{\Delta}^\circ(2) = P^\circ(2)$ and

$\overline{\Delta}^{\circ}(3) = P^{\circ}(3)$ are Koszul. Therefore, we should see that the extension algebra ${}_{A^*}A^*$ is standardly stratified. Without giving the details of construction, we provide the Loewy diagram of ${}_{A^*}A^*$, along with the left proper standard A^* modules.

$$
{}_{A^*}A^* = \begin{array}{ccccc} & & 1 & & \\ & / & | & \backslash & \\ 3 & & 1 & & 2 \\ & / & | & \backslash & | \\ 2 & & 1 & & 3 \\ | & / & | & \backslash & \\ 3 & & 1 & & 2 \\ & / & | & \backslash & | \\ 2 & & 1 & & 3 \\ & & | & & \\ & & \vdots & & \end{array} \oplus \begin{array}{c} 2 \\ | \\ 3 \end{array} \oplus 3 \qquad \text{and} \qquad \overline{\Delta}_{A^*}(1) = \begin{array}{ccc} & 1 & \\ / & & \backslash \\ 3 & & 2 \end{array}
$$

while $\overline{\Delta}_{A^*}(2) = P^{\circ}_{A^*}(2)$ and $\overline{\Delta}_{A^*}(3) = S^{\circ}_{A^*}(3)$. Looking at the diagrams, one can easily confirm that A^* is standardly stratified.

Example A.2. This simple example shows that Koszul algebras are not standard Koszul in general. Moreover, quasi-hereditary Koszul algebras do not need to be standard Koszul.

$$
\Gamma : \quad \overset{2}{\bullet} \xrightarrow{\ \alpha\ } \overset{1}{\bullet} \xrightarrow{\ \beta\ } \overset{3}{\bullet} \qquad\qquad I = 0
$$

with the regular representation

$$
A_A = \begin{array}{c} 1 \\ | \\ 3 \end{array} \oplus \begin{array}{c} 2 \\ | \\ 1 \\ | \\ 3 \end{array} \oplus 3, \quad \text{where} \quad \Delta(2) = \begin{array}{c} 2 \\ | \\ 1 \end{array} \quad \text{is not in } \mathcal{C}^1_A.
$$

Example A.3. As it was shown in [12], $A_A \in \mathcal{F}(\Delta)$ is equivalent to ${}_AA \in \mathcal{F}(\overline{\Delta}^{\circ})$. That is to check whether A is standardly stratified, it suffices to consider only one side. This is not true for the standard Koszul property. Consider the following graph algebra A.

$$
\Gamma : \quad \overset{1}{\bullet} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \overset{2}{\bullet} \qquad\qquad I = (\beta\alpha\beta)
$$

This right and left regular representations of A have the Loewy diagrams as below.

$$A_A = \begin{array}{c} 1 \\ | \\ 2 \\ | \\ 1 \\ | \\ 2 \end{array} \oplus \begin{array}{c} 2 \\ | \\ 1 \\ | \\ 2 \end{array} \qquad {}_AA = \begin{array}{c} 1 \\ | \\ 2 \\ | \\ 1 \end{array} \oplus \begin{array}{c} 2 \\ | \\ 1 \\ | \\ 2 \\ | \\ 1 \end{array}$$

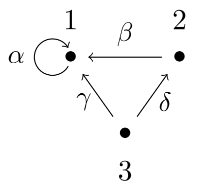
This algebra is a monomial standardly stratified algebra whose standard modules $\Delta(2) = P(2)$ and $\Delta(1) = S(1)$ are Koszul. But since A contains a valley $\beta\alpha : 2 \to 1 \to 2$, it is not standard Koszul (see Proposition 3.1.7). Indeed,

$$\overline{\Delta}^\circ(2) = \begin{array}{c} 2 \\ | \\ 1 \end{array} \text{ is not in } \mathcal{C}^1_{A^\circ}.$$

Besides, the example also shows that standardly stratified algebras whose standard modules are Koszul are not necessarily Koszul.

Example A.4. In [4], it was shown that the classes $\mathcal{K}_2$ and $\mathcal{K}$ coincide when A is standard Koszul and quasi-hereditary. It was also shown that, in that context, the class $\mathcal{K}$ is closed under the operation ω. In our case, both properties fail. In the example below, $A = K\Gamma/I$ is standard Koszul and standardly stratified,

$\Gamma :$

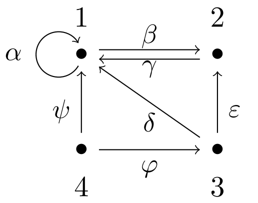


$$I = \left(\alpha^2, \gamma\alpha - \delta\beta\right)$$

$X = P(2)/\operatorname{soc} P(2)$ belongs to $\mathcal{K}_2$ but it is not Koszul. It is also easy to check that $Y = (e_1 + e_3)A/(\alpha - \gamma + \delta)A \in \mathcal{K}$ but $\omega(Y) = X \notin \mathcal{K}$. With Loewy diagrams:

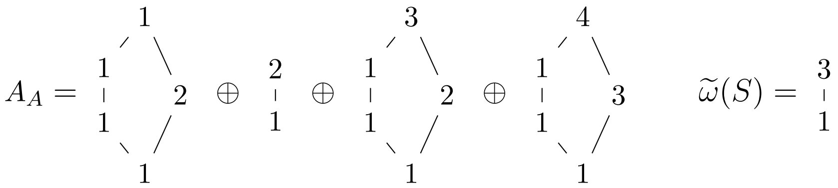
$$A_A = \begin{array}{c} 1 \\ | \\ 1 \end{array} \oplus \begin{array}{c} 2 \\ | \\ 1 \\ | \\ 1 \end{array} \oplus \begin{array}{ccc} & 3 & \\ / & & \backslash \\ 1 & & 2 \\ \backslash & & / \\ & 1 & \end{array} \qquad X = \begin{array}{c} 2 \\ | \\ 1 \end{array} \qquad Y = \begin{array}{ccc} 1 & & 3 \\ \backslash & & / \\ & 1 & \end{array}$$

Example A.5. This example shows that on the Δ-filtered side, the simple modules over a standard Koszul standardly stratified algebra do not have to be in $\mathcal{K}^+$, even $\widetilde{\omega}(S)$ does not have to be Koszul for each simple module S (see Theorem 2.3.42). Let $A = K\Gamma/I$ be the following graph algebra,

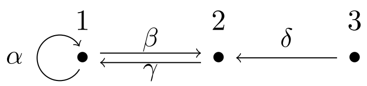
Γ : 

$$I = \big(\alpha\beta, \gamma\beta, \delta\beta, \psi\beta, \gamma\alpha, \varphi\varepsilon, \alpha^3 - \beta\gamma, \delta\alpha^2 - \varepsilon\gamma, \psi\alpha^2 - \varphi\delta\big)$$

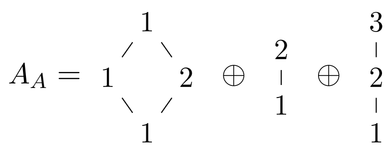
and let $S = S(4)$. Then both $S(4)$ and $\widetilde{\omega}(S) = P(3)/(\varepsilon A + \delta\alpha A)$ are not in $\mathcal{C}_A$.



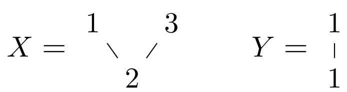
Example A.6. None of the defining conditions of the class $\mathcal{K}^+$ can be omitted in Proposition 2.3.34. Consider the algebra $A = K\Gamma/I$,

Γ :  $\qquad I = \big(\alpha^2 - \beta\gamma, \gamma\alpha, \gamma\beta, \alpha\beta\big)$

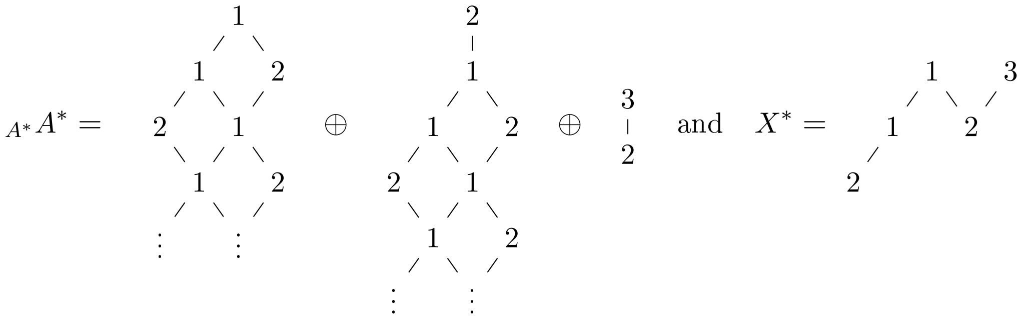
whose regular representation is the following,



and consider modules $X = P(1) \oplus P(3)/\alpha A + (\beta - \delta)A$ and $Y = P(1)/\beta A$

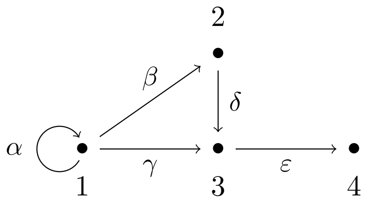


Here, A° is standard Koszul and standardly stratified, $X \in \mathcal{K}$, and $\overline{\omega}_k(X)$ is semisimple for all k but $\widetilde{X} \notin \mathcal{C}_A$. The A^*-module $A^* f_1 X^*$ is not projective:



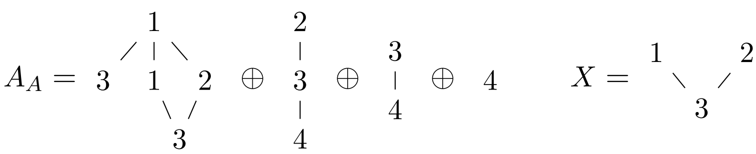
On the other hand, Y is not semisimple but satisfies all the other conditions prescribed by the definition of $\mathcal{K}^+$, and $Y^* \cong \overline{\Delta}^\circ_{A^*}(1) \neq P^\circ_{A^*}(1)$.

Example A.7. The map q defined in Section 2.3 does not have to be an epimorphism if $X \notin \mathcal{K}_2$. Let $A = K\Gamma/I$ be defined by the following graph and relations.

$\Gamma :$ 

$$I = \big(\alpha^2, \alpha\beta, \gamma\varepsilon, \alpha\gamma - \beta\delta\big)$$

Let $X = e_1 J(e_1 + e_2)A$



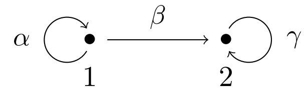
Here X fails to be in $\mathcal{K}_2$ because $X\varepsilon_2 \notin \mathcal{C}_{C_2}$, while $\mathrm{Ext}^h_A(X, S(4)) = 0$ for all h but $\mathrm{Ext}^1_{C_2}(X\varepsilon_2, S(4)\varepsilon_2) \neq 0$.

To see that the other defining condition of $\mathcal{K}_2$ is also necessary consider the (hereditary) algebra $A = K\Gamma$ with $\Gamma : 1 \xrightarrow{\alpha} 2$, whose regular representation is

$$A_A = \begin{smallmatrix} 1 \\ | \\ 2 \end{smallmatrix} \oplus 2.$$

Here $P(1)\varepsilon_2 \in \mathcal{C}_{C_2}$ but $P(1)\varepsilon_2 A \overset{t}{\nleq} P(1)$, so $P(1) \notin \mathcal{K}_2$. It is easy to check that $\mathrm{Ext}^*_A(P(1)) = S^\circ_{A^*}(1)$ and $\mathrm{Ext}^*_{C_2}(P(1)\varepsilon_2) \neq 0$.

Example A.8. Our last counter-example shows that it is not true in general that $\ker q_X = A^* f_1 X^*$, even if A is monomial and satisfies $\varepsilon_i J^2 \varepsilon_i = \varepsilon_i J \varepsilon_i J \varepsilon_i$ for all i and $X \in \mathcal{K}$ (see Proposition 2.3.13). We take the algebra $A = K\Gamma/I$, where

$\Gamma :$ 

$$I = \left(\alpha^2, \gamma^2\right)$$

and the A-module $X = A/\left((\alpha\beta - \gamma)A + \beta A\right)$, i.e.

$$A_A = \begin{array}{ccc} & 1 & \\ / & & \backslash \\ 1 & & 2 \\ | & & | \\ 2 & & 2 \\ | & & \\ 2 & & \end{array} \oplus \begin{array}{c} 2 \\ | \\ 2 \end{array} \qquad X = \begin{array}{ccc} & & 1 \\ & & | \\ 2 & & 1 \\ \backslash & & / \\ & 2 & \end{array}$$

Here A° is standard Koszul and standardly stratified. The A-module X is in $\mathcal{K}$ but $A^* f_1 X^* \neq \ker q_X$ as

$${}_{A^*}A^* = \begin{array}{ccc} & 1 & \\ / & & \backslash \\ 1 & & 2 \\ | & & \\ 1 & & \\ | & & \\ \vdots & & \end{array} \oplus \begin{array}{c} 2 \\ | \\ 2 \\ | \\ \vdots \end{array} \qquad X^* = \begin{array}{c} 1 \\ | \\ 2 \end{array} \oplus \begin{array}{c} 2 \\ | \\ 2 \end{array} \quad \text{and} \quad q_X(X^*) = S(2).$$

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