

METHODS IN QUANTUM MECHANICAL
INVERSE SCATTERING THEORY
AT FIXED ENERGY

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Résumé

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Background

The development of inverse scattering theory of quantum mechanics is a remarkable result of the 20th century mathematical physics. The inverse scattering problem consists of the recovery of the potential in the Schrödinger equation using scattering data accessible by measurement. The solution can be found in the framework of inverse spectral theory of Schrödinger equation. The physical interest in inverse scattering theoretical methods is traditionally originates from nuclear physics. The forces relevant in nuclear physics are less understood still today and inverse scattering provides a way to explore them in a model independent way.

A simplified description of the two-particle quantum mechanical scattering phenomena involves the identification of the interaction with a local and central potential. The method of partial waves gives an infinity of uncoupled one dimensional (radial) Schrödinger equations describing the scattering event. For central potentials the scattering problem is overdetermined: it can be shown that when fixing either the scattering energy or the angular momentum quantum number the data determine the potential. Thus we have fixed energy and fixed angular momentum inverse scattering methods (and potentials).

The case of fixed angular momentum can be treated using the inverse spectral theory of the one-dimensional Schrödinger equation. A more interesting problem is the fixed energy case: although it has been examined since the sixties there is no generally accepted inverse scattering method at fixed energy. The most obvious approach developed in the sixties uses the inverse spectral theory of the one-dimensional Schrödinger equation in analogy to the fixed angular momentum case proved to be useless in the practice as it requires nonphysical input data.

Later, the so-called matrix methods were developed, which seek the solution approximately through an ansatz, which is characterized by a finite number of parameters. Such methods are the Newton-Sabatier and Cox-Thompson procedures. The former was

made usable for practical calculations in the eighties, while the latter only in 2003. It turned out that the Newton-Sabatier method boils down to the solution of a system of linear equations, the Cox-Thompson procedure involves a highly nonlinear problem. With the Newton-Sabatier method long-range and multichannel scattering problems were treated also. Although the matrix methods proved to be efficient in practice recently some doubts were raised about their consistency.

A few years ago Horváth and Apagyi introduced a conceptionally new method, that proposes to recover the potential with the help of a Liouville transformation via the fixed angular momentum formalism and an inverse momentum problem. There exist further approximative inverse methods (e.g. WKB, iterative-perturbative, Bargmann potential methods) which however are out of the scope of the thesis.

Aims

The aim of the research was to study, develop and apply inverse scattering theoretical methods.

In the first part of the thesis the aim was the development of the Cox-Thompson method. Three paths were pursued here: i) simplification of the system of nonlinear equations, ii) the extension to efficiently treat long-range potentials, iii) repair of the mathematical foundations.

The second part of the thesis is concerned with the method proposed by Horváth and Apagyi. My aim was here to further develop the original method to be applicable for experimental data. Another aim was to find an alternative formalism based on the Liouville transformation which would avoid the disadvantages of the original method (such as the pole at the origin).

New scientific results

- I. I managed to simplify the system of nonlinear equations in the Cox-Thompson method for the case of identical particle scattering when partial waves of the same parity contribute to the scattering amplitude. The original system of equations involved explicit matrix inversion, while the simplified one avoids that. Furthermore, I proposed approximate procedures for the fixed energy inverse scattering problem through approximative solutions of the system of nonlinear equations in the Cox-Thompson method. These procedures either employ the simplified equations or do not require the solution of a system of equations at all. [1, 2]
- II. I developed extensions of the Cox-Thompson method, that enable the reconstruction of the inner (or nuclear) potential in the presence of a known long-range interaction. In order to do this the approximative ansatz used in the Cox-Thompson method needed modification. A detailed study of the case of charged particle scattering when the relevant asymptotics is Coulombic is presented. [3]
- III. I gave a consistency criterion for the Cox-Thompson method. It turned out that when employing a finite set of phase shifts it is possible that the solution of the underlying integral equation, expressing completeness, becomes non-unique. This result in singularities for the potential and the procedure breaks down. I gave the criteria to avoid the encounter of singularities, whose fulfillment can be numerically checked for the solutions of the system of nonlinear equations. In the case of a single input phase shift I proved uniqueness of the consistent solution. [4, 5]
- IV. For the proof in III. previously unknown inequalities involving the zeros of the Bessel functions were required. Such inequalities lead to the development of generally usable math-

ematical tools for the study of interlacing properties. Using them I managed to give necessary and sufficient conditions for the interlacing of Bessel functions, their derivatives and linear combinations. I succeeded in unifying three distinct inequality sequences known for at least a hundred years, also given in "Abramowitz". [6, 7]

- V. I extended the applicability and successfully applied for the recovery of test potentials and the inversion of experimental scattering data the method proposed by Horváth and Apagyi. This was made possible by the introduction and optimal choice of two free parameters and the treatment of an arbitrary number of bound states in the spectrum of the transformed Schrödinger equation. [8]
- VI. I developed a new inverse scattering method at fixed energy, that uses the Liouville transformed Schrödinger equation similarly to the Horváth-Apagyi procedure but instead of solving a moment problem the potential is determined via the scattering data of the transformed equation. The method is closely related to the fixed energy formalism analog to the inverse spectral theory of the one dimensional Schrödinger equation, however it only requires physically accessible input data. The procedure was applied to reconstruct test potentials. [9]
- VII. I applied the Newton-Sabatier, Cox-Thompson and Horváth-Apagyi method to determine effective central potentials governing the electron-argon and nucleon-alpha scattering. For the electron-argon scattering I obtained results in agreement with previous calculations exhibiting a short-ranged repulsion originating from the nonlocality of the interaction. Similar potential were obtained from neutron-alpha and proton-alpha data, which describe an interaction with short-range repulsive and medium-range attraction. Reassuringly, the results are approximately independent of the isospin. [3]
- VIII. From pion-pion phase shifts I derived effective quasi poten-

tials (of Logunov-Tavkhelidze type) using inverse scattering methods. The potentials at low energy are similar to that obtained by a rudimentary lattice field theoretical calculation.

Publications connected to the thesis

- [1] T. Pálmai, M. Horváth, and B. Apagyi, “Simplified solutions of the Cox–Thompson inverse scattering method at fixed energy,” *Journal of Physics A: Mathematical and Theoretical* **41** (2008) 235305.
- [2] T. Pálmai, M. Horváth, and B. Apagyi, “Semi-analytic equations to the Cox–Thompson inverse scattering method at fixed energy for special cases,” *Modern Physics Letters B* **22** no. 23, (2008) 2191–2199.
- [3] T. Pálmai, B. Apagyi, and W. Scheid, “Development of a Cox–Thompson inverse scattering method to charged particles,” *Journal of Physics G: Nuclear and Particle Physics* **37** (2010) 025101.
- [4] T. Pálmai and B. Apagyi, “On nonsingular potentials of Cox–Thompson inversion scheme,” *Journal of Mathematical Physics* **51** (2010) 022114.
- [5] T. Pálmai and B. Apagyi, “Quantum Mechanical Inverse Scattering Problem at Fixed Energy: A Constructive Method,” *Methods and Applications of Analysis* **18** no. 1, (2011) 93–104.
- [6] T. Pálmai and B. Apagyi, “Interlacing of positive real zeros of Bessel functions,” *Journal of Mathematical Analysis and Applications* **375** no. 1, (2011) 320–322.
- [7] T. Pálmai, “On the zeros of the cylinder functions,” *Mathematical Inequalities and Applications* (2012) . (to appear).

- [8] T. Palmai and B. Apagyi, “Fixed energy potentials through an auxiliary inverse eigenvalue problem,” *submitted to Inverse Problems*, *arXiv:1202.1931 [math-ph]* (2012) .
- [9] T. Palmai and B. Apagyi, “The inverse scattering problem at fixed energy based on the Marchenko equation for an auxiliary Sturm-Liouville operator,” *submitted to Physics Letters A*, *arXiv:1205.0460 [math-ph]* (2012) .

Further publications

- [10] D. Hilk, W. Scheid, T. Palmai, and B. Apagyi, “Study of the Cox–Thompson inverse scattering method with a Coulomb potential,” *Journal of Physics A: Mathematical and Theoretical* **43** (2010) 225302.
- [11] G. Feher, T. Palmai, and G. Takacs, “Sine-Gordon multisoliton form factors in finite volume,” *Physical Review D* **85** (2012) 085005.
- [12] T. Palmai, “Regularization of multi-soliton form factors in sine-Gordon model,” *Computer Physics Communications* **183** (2012) 1813–1821.