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**Methodological improvements
in financial analyses**

Theses

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Table of contents

I.	Introduction	1
II.	Harmonic mean as an approximation for discounting intraperiod cash flows	5
III.	Present value under uncertain asset life: an evaluation of relative error	8
	III.1. Continuous case	8
	III.2. Discrete case	10
IV.	Cost of capital of energy efficiency projects: The case of space heating and cooling	12
V.	References	14
VI.	Publications related to the theses.....	15
VII.	Other related publications by the author.....	15

I. Introduction

A fundamental question in finance is how much an asset is worth. By “asset” a claim for funds is meant, such as a stock of a firm, which entitles its holder to monetary earnings resulting from operation of the firm. Over the past century, the notion of “present value” has been settled as the principal measure of worth, which is defined as the discounted sum of all future funds expected from the asset as of the time of valuation, including, of course, any funds available instantly upon acquisition of the asset. Instead of the term “fund,” the term “cash flow” is mostly used (and will be used hereinafter also), referring to the fundamentals of firm valuation. Discounting is the procedure which derives the value of a cash flow expected at some future point in time as of the time of valuation, i.e., at present. Technically, this is performed by using a discount rate accounting for the timing and the risk of the cash flow. Since value is derived by discounting future expected cash flows to present, the approach and the rules of such analyses are referred to as the discounted cash flow (DCF) framework.

Two main fields of finance are investments and corporate finance, and present value is a key concept in both. Contemporary corporate finance is concerned with maximization of stockholders’ wealth. Thus, managers of firms should acquire “valuable” assets, that is, assets which increase the value of stocks. The related decision-making process is called capital budgeting. Increasing the value of stocks requires identification of assets that are worth more than the price they can currently be purchased for, and it is the present value which is used to determine the worth of the assets. The difference between the present value and the current price is referred to as the net present value. Generally speaking, the field of investments is concerned with the behavior of asset prices in capital markets, which prices are obviously connected to the values of capital assets. And again, asset values are determined by their present values. Since our thesis is dedicated to aspects of present value computation, it is not “limited to” corporate finance; nevertheless, capital budgeting is where our findings could principally be applied, as the present value is rather merely an input or a reference point to the field of investments.

The typical textbook computation of the present value requires the following main inputs: 1) definition of the length of a time period (which is mostly a year) referred to as the “interest period;” 2) determination of the expected economic life of the asset under valuation, expressed in the same time unit as the interest period; 3) estimation of the aggregate expected cash flows of the asset for each of the periods during the expected life, assuming that they occur at the ends of the periods; 4) estimation of the discount rate for the interest period, which is identical across the periods. In this textbook approach, present value calculation is technically simple, however, simplicity, as usually, comes at a cost. This textbook approach may contain the following errors, for example: estimation errors regarding the expected cash flows and the discount rate, negligence of the fact that cash flows need not occur exclusively at the ends of the periods, and negligence of the uncertainty of the asset’s economic life, provided that, of course, the asset’s economic life is stochastic. Elimination of these errors inevitably calls for a toolbox of more complicated mathematical methods, making the computation of present value technically more involved. Therefore, it is particularly worth to explore the characteristics of these errors and to assess whether or not, or under what circumstances, these errors are so severe that the sacrifice of some simplicity is desirable.

Our research addresses three of the above-mentioned sources of error. First, we relax the assumption that all cash flows within an interest period occur at the end of the period. Second, we account for uncertainty related to the economic life of assets. Third, we strive to provide a more precise estimate of the discount rate applicable to energy efficiency projects related to space heating and cooling. We note that we tackle these three problem areas

separately; a thorough study of the aggregate effects of errors from various sources is left for future research.

As mentioned above, textbooks typically suggest calculating the present value of an asset by aggregating all cash flows within a given interest period to the end of that period, and then discounting them to the present. This is called “end-of-period” convention. Cash flows of an asset, in reality, may obviously occur throughout the interest period, so it is essential to assess the possible errors an analyst may make by approximating the present value by the end-of-period convention. In addition, our research was motivated by finding a relatively simple, easy-to-use formula which the end-of-period present value could be adjusted with to mitigate such errors. We developed a novel approximation formula called “harmonic” convention, which is based on the harmonic mean of the beginning-of-period and the end-of-period present values. Analogous to the end-of-period convention, the beginning-of-period convention aggregates all cash flows within a given interest period to the beginning of that period. We use the relative error (in fact, its absolute value) as the measure of error, defined as the approximative present value divided by the correct, actual present value of the asset, less one. We show that the harmonic convention minimizes the maximum possible error for the general case when the underlying cash flow pattern of the asset is not considered. Accordingly, our first thesis is the following:

Thesis 1 (Andor and Dülk, 2013): The harmonic mean of the end-of-period and the beginning-of-period present values minimizes the maximum possible error when approximating the present value of intraperiod cash flows. This harmonic convention is defined mathematically as

$$P_H = P_E \frac{1+i}{1+i/2}$$

where P_H and P_E are the harmonic and the end-of-period present values, respectively; and i is the discrete discount rate for the given interest period.

We extend the evaluation of errors to specific cash flow pattern assumptions, and compare the performance of the harmonic convention with alternative conventions given in the relevant literature. These alternative conventions can all be stated as adjustments to the end-of-period convention, and are the beginning-of-period and mid-period conventions, in particular. The mid-period convention, as its name suggests, aggregates all cash flows within a given interest period exactly to the mid-point of the period. The cash flow pattern assumptions we examine apply to a single period and are defined by a single parameter; in particular, cash flows are described by triangular, PERT, and intermittently uniform (termed “semester estimate”) distributions. These distributions provide a reasonable coverage of realistic cases. We find that the harmonic convention, even though not being the best choice for all the scenarios, has acceptable errors in the majority of cases, and thus can be a good tool for practitioners. Thus, our second thesis is:

Thesis 2 (Andor and Dülk, 2013): For realistic cash flow patterns the approximation error of the harmonic convention is usually small (less than 5%) and smaller than that of other conventions, especially when the majority of cash flows fall into the second half of the period.

Our second research endeavor is the thorough mathematical analysis of present value calculation when the cessation point of an asset’s cash flow pattern is uncertain. Textbooks typically suggest computing present value according only to the expected life of an asset,

without further consideration of the probability distribution of the asset life. This is, in effect, equivalent to treating the asset life as deterministic. In reality, the life of, e.g., equipment is stochastic (cf. reliability engineering); thus, such uncertainties should be involved in financial analyses. To do this, the expected present value should be calculated to base investment decisions on, instead of the present value according to the expected life. We examine both the case of continuous exponential cash flow pattern combined with exponentially distributed life, and their discrete equivalent of geometric gradient series pattern with geometrically distributed life. We present in detail the mathematical framework for obtaining closed form solutions for the present values with cessation point uncertainty in general, and the formulas particularly for the two cases just highlighted. We evaluate possible errors, by using the same relative error measure (in fact, its absolute value) as previously, defined as the present value according to the conventional approach divided by the correct, expected present value, less one. We generally find that if the discount rate is equal to the growth rate, the error is always zero. If the growth rate exceeds the discount rate, then the error can reach the theoretical maximum of 100%. However, if the growth rate is less than the discount rate, which can be considered to be the usual case, the error cannot exceed 30%. This is actually the value of a local maximum of the error function, which local maximum exists for every expected life. The main difference between the continuous and the discrete case is that the value of this local maximum is invariant regarding the expected life and it is (approximately) 30% in the continuous case, but varies with expected life in the discrete case, being smaller for shorter expected lives but being at least 12.5%. We formulate the following general rule of thumb: for a given discount rate – growth rate combination, the longer the expected life, or, alternatively, for a given expected life, the larger the difference in absolute value between the two rates, the larger is the error in magnitude. In particular, we find that even a small percentage point difference between the discount rate and the growth rate can lead to considerable errors. For typical expected lives of 10 to 20 periods, a rate difference of 2% to 1% gives a non-negligible error of approximately 10%. These results also call attention to the importance of precision in discount rate and growth rate estimation. Our third thesis concludes these:

Thesis 3 (Andor and Dülk, 2014*): In typical cases, neglecting the uncertainty of asset life may often lead to considerable error (above 10%) in the present value but may be regarded as tolerable for a rough estimation as the error cannot exceed 30%.

Finally, the third part of our research is somewhat distinct from the technicalities of the previous two. We aim at estimating the cost of capital, the relevant risk in particular, of energy efficiency projects related to space heating and cooling. We conduct the analysis in the Capital Asset Pricing Model (CAPM) framework, and strive to estimate empirically the beta parameter of such projects, which parameter expresses the relevant risk of an asset in the CAPM. We do the estimation from historical energy price and weather data, as these two factors determine fundamentally the risk of energy-saving projects. This follows from the cash flows of savings in energy bills taking up the vast majority of the project's (risky) cash flows, and the savings flows being the product of the unit price of energy and the amount of consumption, which latter is assumed to depend exclusively on the weather. Our research is motivated by providing a more precise assessment of project risk than those provided by the industry betas, which serve obviously just as rough estimates. We face the apparent problem of lack of relevant historical stock price data on energy efficiency projects, from which beta could be directly estimated. Therefore, we take the indirect approach of estimating from historical data on the factors affecting the project's cash flows, i.e., energy price and weather, which are available. We test the hypothesis whether the project beta is statistically significantly different from zero. Since beta is determined by the correlation of returns on the

asset with returns on the market portfolio in a multiplicative way, the beta is zero if this correlation is zero. Because asset returns are not observed, as noted above, we compute every pair-wise correlation between the cash flow risk factors and market portfolio returns. We argue that if none of these pair-wise correlations is different from zero, then the correlation of asset returns with market portfolio returns should not be different from zero either. In our case of energy efficiency projects, this means testing the null hypothesis of zero correlation in the following three instances: 1) energy unit price measure and market portfolio returns; 2) weather measure and market portfolio returns; 3) energy unit price measure and weather measure. We follow the approach, found also in related studies, of treating these factors as investments, and calculate historical percentage changes on them as their measure. Thus, the energy unit price uncertainty is proxied by historical percentage changes in the unit price of energy, and weather uncertainty is proxied by historical percentage changes in heating degree-days. We examine natural gas and electricity as two energy carriers on which a project might save on, in several European countries, both for households and businesses. None of the cases is found to have correlation significantly different from zero; therefore, the hypothesis that energy efficiency projects have a zero beta and consequently their cost of capital is the risk-free rate, cannot be rejected. We remark that our results are constrained in the sense that we do not quantify the beta, which would obviously require a solid theoretical framework linking the evolution of cash flows and stock prices. An attempt regarding this, which also challenges the results of previous related studies and accords with our results reported here, is made in a recent yet unpublished draft of ours (Andor and Dülk, 2012). Our fourth thesis summarizes the above:

Thesis 4 (Dülk, 2012a, 2012b): The hypothesis that the cost of capital is the risk-free rate for space heating or cooling energy efficiency projects saving on electricity or natural gas cannot be rejected.

The rest of our discussion is divided into three main chapters corresponding to the above-summarized three fields of research. First, we present the harmonic convention in more detail as an improvement over the end-of-period convention. Second, we provide a more thorough overview of our results regarding uncertainty as to asset life. Third, the main elements of cost of capital estimation for energy efficiency projects related to space heating and cooling close the discussion.

II. Harmonic mean as an approximation for discounting intraperiod cash flows

An intraperiod cash flow is a cash flow that occurs some time during a given interest period. The present value of an asset with multiple (intra)period cash flows (Figure 1) is derived as (1) (for a proof, see, e.g., Fleischer, 1986):

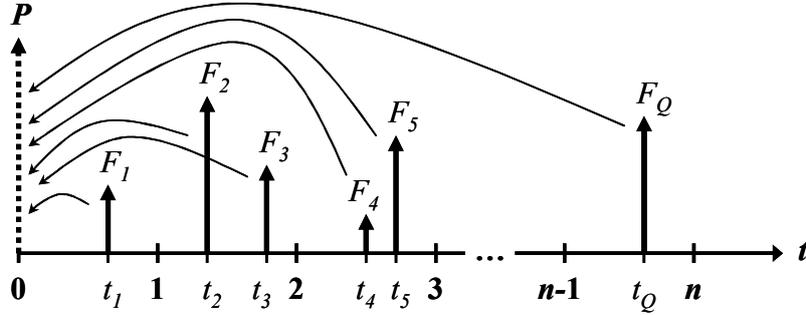


Figure 1: Discounting multiple cash flows of an asset.

$$P = \sum_{q=1}^Q F_q (1+i)^{-t_q} \quad (1)$$

where F is a cash flow, P is the present value, i is the discrete discount rate for the given interest period, t is cash flow timing (expressed in the unit of the interest period), Q is the total number of cash flows of the asset, q is the index of cash flows and their timing, and n is the index of interest periods in Figure 1.

Employing the procedure suggested by (1) is quite cumbersome, especially when there are a large number of cash flows. Therefore, it is reasonable to use approximations that may not be completely accurate but that are simpler and easier to use.

The typical approach is simplifying the present value calculation by aggregating all cash flows within an interest period to a specific time point in that period and then applying (1). The most notable and widely used of them is the so-called end-of-period convention, which assumes that all cash flows in a given interest period occur at the end of the period and can be calculated by the following formula:

$$P_E = \sum_{n=1}^N A_n (1+i)^{-n} \quad (2)$$

where index E of P refers to “end-of-period”, N is the total number of interest periods during the asset’s life, and A_n is the total amount of cash flows in interest period n .

Another notable method that builds similarly on timing adjustment is the mid-period convention (indexed by M), in which all cash flows are aggregated to the midpoint of their interest periods. This method uses the following formula:

$$P_M = \sum_{n=1}^N A_n (1+i)^{-n+\frac{1}{2}}, \text{ which is actually } P_M = P_E \sqrt{1+i} \quad (3)$$

As (3) shows, the mid-period present value is easily derived from the end-of-period present value (and vice versa).

A third method, not frequently used but a logical alternative, is the beginning-of-period convention (indexed by B), which is also related to the end-of-period present value:

$$P_B = \sum_{n=1}^N A_n (1+i)^{-n+1}, \text{ which is actually } P_B = P_E (1+i) \quad (4)$$

As seen in (2)-(4), these approximations can be written in the next general form, where the approximative present value is obtained via an adjustment to the most widely used end-of-period present value:

$$P_{approx} = P_E k(i), \text{ according to which } k_E(i) = 1, k_M(i) = \sqrt{1+i}, \text{ and } k_B(i) = 1+i \quad (5)$$

where P_{approx} is the approximative present value of the given convention, and $k(i)$ is an adjustment function of the discount rate i .

We introduce a new adjustment function that is superior to all of the above mentioned conventions in many ways. We refer to this as the “harmonic” approximation (indexed by H) because it actually yields the harmonic mean of the end-of-period and beginning-of-period present values:

$$P_H = \frac{2}{1/P_E + 1/P_B}, \text{ from which } P_H = P_E \frac{1+i}{1+i/2}, \text{ and } k_H(i) = \frac{1+i}{1+i/2} \quad (6)$$

We apply the relative error as the measure of error, as defined in the study by Lohmann and Oakford (1984):

$$\varepsilon = \frac{P_{approx}}{P_{actual}} - 1, \text{ which is, in our discussion, equivalent to } \varepsilon = \frac{P_E k(i)}{P_{actual}} - 1 \quad (7)$$

where ε is the relative error of an approximation, P_{approx} is the approximative present value according to the approximation, and P_{actual} is the actual present value of the given cash flow pattern.

It can be shown that the harmonic convention minimizes the maximum possible error, i.e., the largest error that may possibly occur regardless of the actual cash flow pattern, which is defined as

$$\varepsilon_{max} = \max \left\{ \left| \frac{P_{approx}}{P_E} - 1 \right|, \left| \frac{P_{approx}}{P_B} - 1 \right| \right\} \quad (8)$$

which, for the harmonic convention, is precisely

$$\varepsilon_{H,max} = \max \left\{ \left| \frac{P_E k_H(i)}{P_E} - 1 \right|, \left| \frac{P_E k_H(i)}{P_E (1+i)} - 1 \right| \right\} = \max \left\{ \left| \frac{i}{2+i} \right|, \left| \frac{-i}{2+i} \right| \right\} = \frac{i}{2+i} \quad (9)$$

The same properties of the end-of-period, mid-period, and beginning-of-period conventions can be found in the study of Lohmann and Oakford (1984). Figure 2 exhibits the maximum possible errors as functions of the discount rate. The results indicate that it is quite worthwhile to adjust the end-of-period present value with the harmonic approximation; however, it is only slightly better than the mid-period, even at the extreme.

We also evaluate the relative errors defined in (7) for several actual cash flow patterns. One of such patterns is based on the PERT distribution, which describes cash flows within one period, and its “peak” (denoted by c) may be varied. Convenient nomograms (two-dimensional diagrams) can be generated to illustrate the approximation errors by method, and regions of preference diagrams can be drawn to highlight which method is the best choice over the others for what combination of parameters (see Figure 3 for an illustration for the PERT-style patterns).

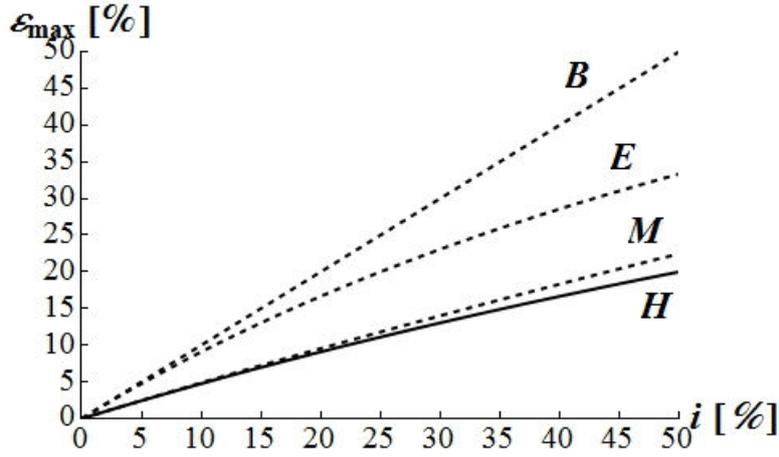


Figure 2: Comparison of maximum possible errors of the beginning-of-period (*B*), end-of-period (*E*), mid-period (*M*), and harmonic (*H*) conventions.

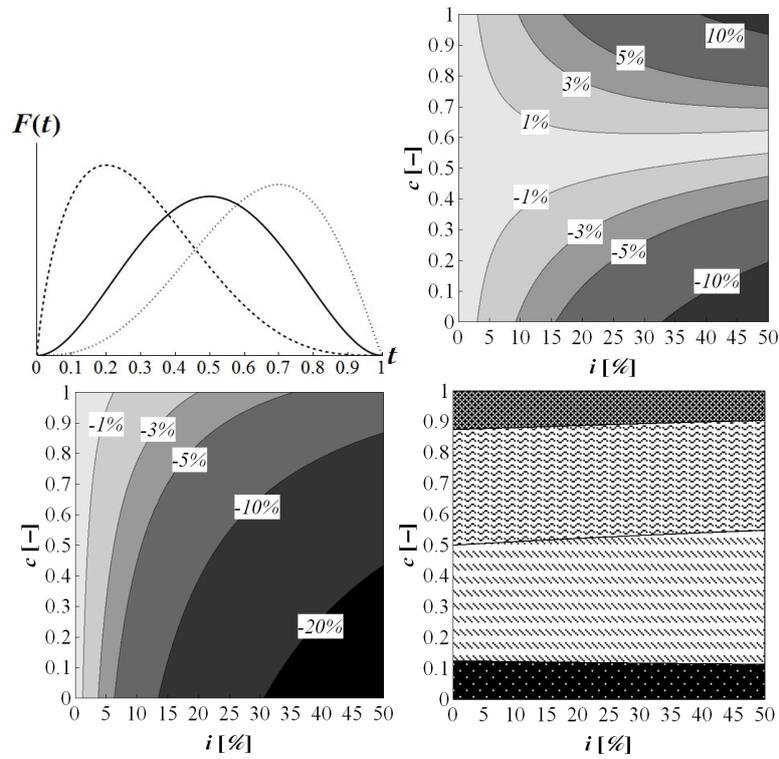


Figure 3: PERT-style cash flow patterns (upper left), approximation errors of the end-of-period convention (lower left) and the harmonic convention (upper right), and regions of preference (lower right, tildes denoting the region where the harmonic is best) in case of PERT-style patterns.

In conclusion of our analysis, we establish that for small discount rates, i.e., below 7%, it does not matter which approximation is used, or, alternatively, whether the end-of-period present value is adjusted or not. However, the use of the harmonic is suggested because it has the smallest maximum possible error. For rates above 7%, if most of the cash flows are expected to occur in the first half of the period, then the mid-period is recommended; however, if they are expected to occur in the second half, then the harmonic approximation should be preferred. The errors of the harmonic and the mid-period formulas hardly ever exceed the 10% level. With the help of the nomograms, the approximative present values can also be corrected to arrive at the close-to-actual present value. We note also that in case of varying patterns over multiple periods, we may still rely on some of the single-period findings, e.g., by grouping periods identical in some respect and working with these groups separately.

III. Present value under uncertain asset life: an evaluation of relative error

As described in related textbooks (e.g., Park and Sharp-Bette, 1990) the computation of the present value can typically take one of two forms: 1) discounting discrete cash flows at a discrete discount rate, or 2) discounting a continuous cash flow stream at a continuous discount rate. In such approaches, essentially all variables – such as the cash flows, discount rates, and asset lives – may be uncertain; that is, they may be random variables. Therefore, valuation and investment decisions should be based on the *expected* present value (see, e.g., Park and Sharp-Bette, 1990; Tufekci and Young, 1987 and references therein). Although textbooks usually recognize the stochastic nature of these variables, for computational simplicity, they substitute the expected values of the random variables into the present value formula instead of taking the expectation of the whole expression. These are obvious sources of computational errors, leading to biased present value and, possibly, to incorrect investment decision.

Here, we are concerned with error attributable to the uncertainty of the economic life of an asset, N or T (henceforth usually simply “life”), that is, the moment in time when the series of cash flows terminates. We assume non-stochastic discount rates and cash flows stochastically independent of asset life. (Based on this latter assumption, for notational parsimony, we omit the expectations operator for cash flows, so that F itself denotes an expected cash flow.) The error clearly depends on the cash flow pattern and the distribution of the asset life. Here, we examine the geometric gradient series pattern and geometrically distributed asset life for the discrete case, and the exponential pattern and exponentially distributed life for the continuous case. We assume that growth rates are non-stochastic.

We evaluate the relative error defined as

$$\varepsilon = \frac{\hat{P}}{E(P)} - 1 \quad (10)$$

where ε denotes the relative error; \hat{P} denotes the present value according the “conventional” approach, i.e., using expected life; $E(P)$ denotes the correct expected present value.

We are the first to look at the relative error, which we deem a better descriptor of computational error than the absolute error, i.e., $\hat{P} - E(P)$, which was examined by Chen and Manes (1986). Using relative error, we gain new insight which cannot be done with absolute error which depends on actual dollar amounts. We explicitly extend the analysis to the negative domain of growth rates and discount rates, and provide a thorough comparison of the discrete and the continuous case.

III.1. Continuous case

For a continuous exponential cash flow pattern, under the conventional approach, the present value is computed as (e.g., Remer et al., 1984)

$$\hat{P} = \frac{C}{r - j} (1 - e^{-(r-j)E(T)}) \quad (11)$$

where j is the continuous growth rate, r is the continuous discount rate, C is a constant cash flow parameter, $E(T)$ is the expected value of asset life T , and e is the base of the natural logarithm.

The correct calculation, by contrast, is – also substituting the moment-generating function of the exponential distribution to obtain the closed form solution for the expected present value (see, e.g., Zinn et al., 1977):

$$E(P) = E\left(\frac{C}{r-j}(1 - e^{-(r-j)T})\right) = \frac{C}{r-j} \left(\frac{\theta(r-j)}{1 + \theta(r-j)}\right) \quad (12)$$

where λ is the parameter of the exponential distribution, and θ denotes the mean of the exponential distribution, that is, the expected life of the asset (in other words, $E(T) = \theta$), which is related to the distribution parameter via $\theta = 1/\lambda$.

Verifying the convergence criteria for the moment-generating function to exist, we find that, assuming positive expected asset life, $\theta(r-j) > -1$ must hold in order for $E(P)$ to exist; otherwise, $E(P) = \infty$. (If θ is zero, then $E(P)$ is also zero, and the relative error is undetermined due to division by zero.) Note that the direction of error, i.e., over- or understatement, can be determined from Jensen's inequality, independently of the actual probability distribution of T . Additionally, because C is a constant, $\hat{P} = E(P)$ if and only if $r = j$. We note also that the correct expected present value can be approximated by a Taylor series.

Dividing (11) by (12) and subtracting 1, we obtain the relative error, which can be expressed as a function of a single variable, defining $x = \theta(r-j)$, as follows

$$\varepsilon = \left(1 - e^{-x}\right) \left(1 + \frac{1}{x}\right) - 1 \quad (13)$$

Analyzing the absolute value of the error function in (13), we find that it has a local maximum. Differentiating (13) with respect to x and solving for zero produces only one real root, at $x \approx 1.79$, where $\varepsilon \approx 29.84\%$. Because we take the absolute value of the error, the global maximum is 100%. In the negative domain, the error is found to be very sensitive to x . Further examining the composition of the variable x and knowing that, as θ is positive, x is positive if and only if $r > j$, and x is negative if and only if $r < j$, the results can be interpreted as follows. If the discount rate exceeds the growth rate, then the error cannot exceed 30%, but this maximum can be attained for every expected life θ , as there exist several (in fact, infinitely many) $r - j$ combinations for which the extremum condition is satisfied. If the discount rate is smaller than the growth rate, then the upper limit of the error is 100%, which again, for the reason just mentioned, can be attained for every expected life. When the two rates are equal, the error is zero, as pointed out earlier. Figure 4 plots error functions for specific expected lives, to better illustrate the above points, and also shows a nomogram for $\theta = 5$ to exhibit the severity of error for various rate combinations.

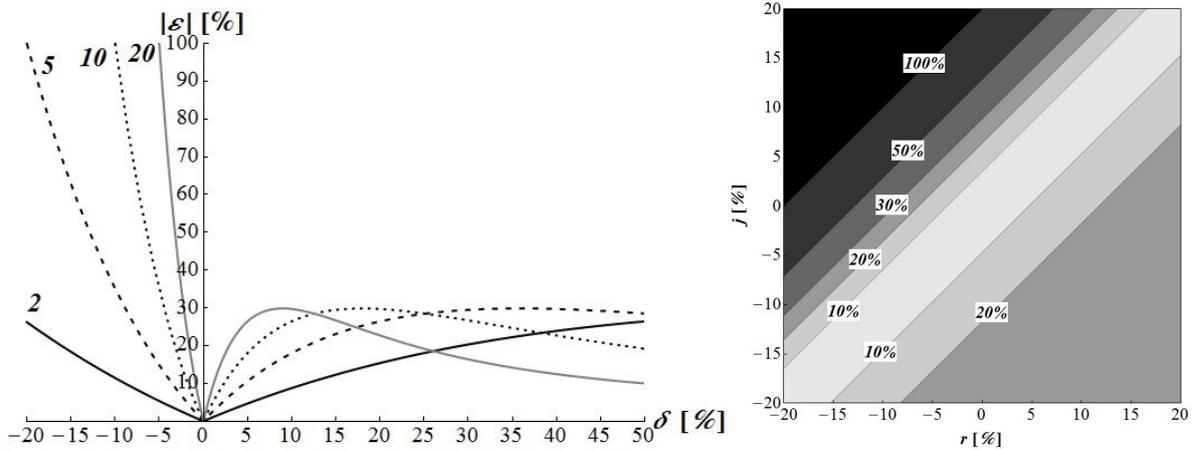


Figure 4: Absolute value of the relative error as a function of the rate difference $\delta = r - j$ for expected lives of $\theta = 2$ (solid), 5 (dashed), 10 (dotted), and 20 (gray) (to the left); severity of relative error for an expected life of $\theta = 5$, darker colors indicate larger errors (to the right).

It is alarming to observe that the error may easily exceed 10%, which can be considered significant, and for that to happen, the rate difference need only be a few percentage points. For example, for $\theta = 10$, rate differences of 2% and -2% produce errors of 8.8% and -11.4% , respectively. For $\theta = 20$, rate differences of 1% and -1% produce the same error percentages.

III.2. Discrete case

For a geometric gradient series cash flow pattern, under the conventional approach, the present value is computed as (e.g., Remer et al., 1984)

$$\hat{P} = \frac{F_1}{i - g} \left(1 - \left(\frac{1+g}{1+i} \right)^{E(N)} \right) \quad (14)$$

where g is the discrete growth rate, i is the discrete discount rate, F_1 is the cash flow parameter, $E(N)$ is the expected value of asset life N .

The correct calculation, by contrast, is – also substituting the probability-generating function of the geometric distribution to obtain the closed form solution for the expected present value (see, e.g., Gerchak and Åstebro, 2000):

$$E(P) = E \left(\frac{F_1}{i - g} \left(1 - \left(\frac{1+g}{1+i} \right)^N \right) \right) = \frac{F_1}{i - g} \left(\frac{1}{1 + \frac{1+g}{\eta(i-g)}} \right) \quad (15)$$

where α is the parameter of the geometric distribution, and η denotes the mean of the geometric distribution, that is, the expected life of the asset (in other words $E(N) = \eta$), which is related to the distribution parameter via $\eta = 1/\alpha$. There are actually two versions of the geometric distribution – we use the one for the domain of positive integers; that is, asset life cannot be zero (see Gerchak and Åstebro, 2000).

Verifying the convergence criteria for the power series of the probability-generating function, we find, exploiting the fact that expected asset life must be positive, that $\left| \frac{1+g}{1+i} \right| < \frac{\eta}{\eta-1}$ must hold in order for $E(P)$ to exist; otherwise, $E(P) = \infty$. Both g and i are assumed to be greater than -1 , therefore, the absolute value sign can be omitted, and the ratio is always positive. Note also that η cannot be zero, as asset life is assumed to be at least 1 period. Note that the direction of error, i.e., over- or understatement, can be determined from Jensen's inequality, independently of the actual probability distribution of N . Additionally, similarly to the continuous case, $\hat{P} = E(P)$ if and only if $i = g$. The correct expected present value can, again, be approximated by a Taylor series.

Dividing (14) by (15) and subtracting 1, we obtain the relative error (after considerable algebraic manipulation), which can be rewritten at best as a function of two variables, defining $y = (1+g)/(1+i)$, as follows

$$\varepsilon = (1 - y^\eta) \left(1 + \frac{1}{\eta \left(\frac{1}{y} - 1 \right)} \right) - 1 \quad (16)$$

It follows from the previous discussion that if $y \geq \eta/(\eta-1)$, then, because $E(P)$ equals infinity, the error is -100% ; and if $y = 1$, which occurs when $i = g$, the error is zero. Note also

that for $\eta = 1$, the error is also always zero (in this case, we have a degenerate distribution similar to $\theta = 0$ in the continuous case). Figure 5 shows the plot of the absolute value of the error function of two variables, given in (16), for specific values of η , and also shows a nomogram for $\eta = 5$ to exhibit the severity of error for various rate combinations. Again, we take the absolute value of the error to express the magnitude.

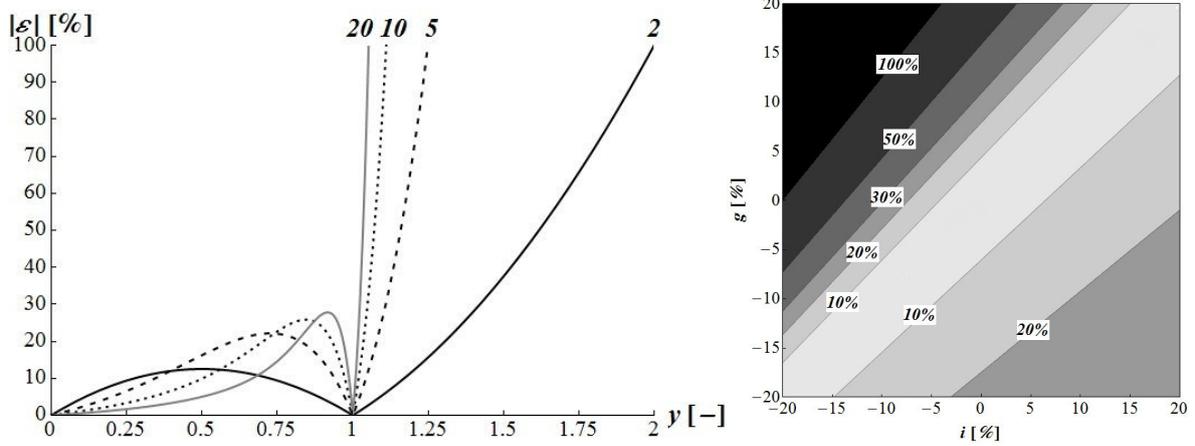


Figure 5: Function of the absolute value of the relative error in variable $y = (1 + g)/(1 + i)$ for expected lives of $\eta = 2$ (solid), 5 (dashed), 10 (dotted), and 20 (gray) (to the left); severity of relative error for an expected life of $\eta = 5$, darker colors indicate larger errors (to the right).

As Figure 5 also shows, for every η , for $y < 1$, the relative error has a local maximum, the value of which is conditional on η . Because we take the absolute value of the error, the global maximum is 100% for every η . In the domain where $y > 1$, the error is very sensitive to y . These observations closely accord with those for the continuous case, with the difference that, in the discrete case, the value of the local maximum varies with η . The main characteristics, however, are the same – that is, for a given expected life, the discrete and continuous errors behave identically. This is confirmed when we examine more closely the composition of the variable y . If $y < 1$, which occurs if and only if $i > g$, there exists a local error maximum. The error is very sensitive to values of $y > 1$, which occur if and only if $i < g$. The error is zero when $y = 1$, which occurs if and only if $i = g$. (Our assumptions do not allow y to equal 0.)

The main difference between the discrete and the continuous case is that, in the discrete case, the value of the local error maximum is not the same for every expected life, as was the case in the continuous case. For example, for an expected life of 5 periods, the value of the local error maximum is approximately 22%, in contrast to the approximately 30% attainable in the continuous case. Figure 5 also shows that (and this can be verified by examining the partial derivatives) the location and the value of the local maximum are both increasing in η . That is, they are lowest for $\eta = 2$, when the value of the local error maximum is 12.5%, obtained at $y = 0.5$, and highest as η approaches infinity. For example, for $\eta = 1,000$, the value of the local error maximum is 29.8%, which is the same maximum value we found in the continuous case and is obtained at $y = 0.998$. It is intuitive that the largest possible value of the local error maximum in the discrete case should be the same as that in the continuous case because, from the “view of infinity,” the length of the discrete periods is infinitesimal, that is, the discrete case looks to be continuous. Therefore, we can establish that the error in the discrete case cannot exceed 30% if $i > g$. Last but not least, it is important to recognize that because both i and g are typically small (i.e., close to zero), the ratio y behaves quite analogously to the rate difference $i - g$.

In summary, we can draw conclusions very similar to those we drew in the continuous case. If the discount rate exceeds the growth rate, then the error cannot exceed 30%, but a

maximum of at least 12.5% can be attained for every expected life η (except for 1), as there exist several $i - g$ combinations for which the extremum condition is satisfied (in fact, infinitely many because both i and g can be real numbers). If the discount rate is smaller than the growth rate, then the upper limit of the error is 100%, which can be attained for every expected life. When the two rates are equal, the error is zero, as pointed out earlier.

IV. Cost of capital of energy efficiency projects: The case of space heating and cooling

One particular area of possible energy saving is space heating and cooling. Space heating typically accounts for about 70% of a household's energy consumption in the EU (Odyssee-Mure project, 2009); cooling is rather negligible accounting for less than 1% (Atanasiu and Bertoldi, 2007). Studies have shown considerable energy-saving potential related to space heating and cooling in both the residential and commercial sectors globally (Novikova and Ürge-Vorsatz, 2008).

Again, we conduct the analysis in the DCF framework, focusing on the estimation of the cost of capital. The cost of capital adjusts to the (relevant) risk of the project, which stems from the uncertainty regarding the project's cash flows. Among the relevant cash flows of energy efficiency projects, the savings in energy bills are the most determining as pertains to project risk; the other cash flows can either be assumed to be certain or negligible in magnitude. Savings in energy bills depend on the amount of consumption on the one hand, which is a function of the weather, in our case, outdoor temperature in particular. On the other hand, savings depend on the unit price of energy, which may change considerably over time. We examine natural gas (henceforth simply: gas) and electricity end-use prices, for both households and businesses. We emphasize that end-use prices should be used as these are what consumers actually pay, and not, e.g., the energy price as a commodity. Admittedly, the actual end-use prices may vary by consumers; therefore, we take average "typical" prices charged to a representative group of consumers, as collected by Eurostat.

The cost of capital (r_{alt}) in the CAPM is determined as

$$r_{alt} = r_f + \beta_{project} (E(r_M) - r_f) \quad (17)$$

where r_f is the risk-free rate, $E(r_M)$ is the expected return on the market portfolio, and $\beta_{project}$ is the relevant risk parameter of the project.

In financial practice, there is good consensus on the values of r_f and $E(r_M)$, which are annually about 2% and 8%, respectively, in real terms (Andor and Tóth, 2009). Thus, practically, the beta of a project is the only parameter that needs to be determined to determine the cost of capital.

Mathematically, the project beta can be written as

$$\beta_{project} = k_{project,M} \frac{\sigma(r_{project})}{\sigma(r_M)}, \text{ the relevant risk being } \sigma(r_{project})_{relevant} = \beta_{project} \sigma(r_M) \quad (18)$$

where r denotes return, $\sigma(\cdot)$ denotes standard deviation, and k denotes correlation coefficient.

The parameters composing beta, theoretically, reflect future expectations, but in practice, the betas of securities are calculated from historical data, with the assumption that the past is a good predictor for the future. Calculating the beta of public companies is a relatively simple task because there is usually a sufficiently long time series of past stock returns. Projects, however, are not traded publicly, and therefore there are no stock price data available from which returns could be calculated. In financial practice, a project's beta is

often estimated by taking an average of betas of public firms with risk characteristics similar to the given project, which typically means firms in industries similar to the project. In our case, this suggests taking the average beta of firms in the energy sector or, more narrowly, that of electricity or gas companies. We argue that this approach yields a distorted estimate in our case because energy companies are more exposed to some risks to which our project is less exposed (e.g., from human resources, suppliers, and prices of other resources) and conversely may be less exposed to other risks that affect our project more (e.g., due to protection by regulation and governmental aid).

We focus on the correlation coefficient in (18), and test if it differs statistically significantly from zero. (If the correlation coefficient is zero, then the beta of the project is also zero.) Due to the lack of historical return data on the project itself, we use, analogously to investments, historical returns (i.e., percentage changes from period to period) on some measures of the relevant sources of uncertainty. This approach can be found in several related papers, e.g., Metcalf (1994), Awerbuch and Deehan (1995), Bolinger et al. (2006). We can state that if we find no correlation between any two risk factors (including market portfolio returns) the overall correlation must also be zero. In our case, this requires the examination of three correlations: between gas/electricity consumption and the market portfolio, between the price of gas/electricity and the market portfolio, and between gas/electricity consumption and the price of gas/electricity. As there are no data available regarding individual consumption in the past, but detailed records on temperature are available, we assume that energy consumption is determined solely by the weather.

It seems plausible that financial processes have no impact on outdoor temperature and conversely, that weather in any country has no impact on the world economy. Therefore, no correlation can be assumed without any calculation.

To determine the correlation of gas and electricity prices with the market, we use the MSCI World Index in U.S. dollars as the proxy for a global market portfolio. The end-user energy prices are taken from the Eurostat database, which provides a collection of semi-annual prices. The method of data collection was changed in 2007; therefore, we use data for the 1991–2007 period to have a consistent time series. For households, we take the electricity prices of group “Dc” and gas prices of group “D3” in euros, with all taxes included. For businesses, we take the electricity prices of group “Ie” and gas prices of group “I3-1” in euros, excluding VAT. The following nine countries are examined: Germany, France, United Kingdom, Italy, Belgium, Spain, Luxembourg, the Netherlands, and Hungary. We convert prices to U.S. dollars and calculate the real log returns on them as the measure for the price factor.

As mentioned above, air conditioning is responsible for only a negligible share in energy consumption. Thus, an increase in the demand for cooling is unlikely to have any effect on energy prices. Prices clearly have no impact on the weather; therefore, in the case of cooling, no correlation can be assumed without any calculation. Heating, however, accounts for a vast share in consumption. Therefore, we test if any reasonable correlation can be found between gas or electricity prices and the coldness of the weather. We use the heating degree-days as the proxy for the demand for heating, data on which we obtain from Eurostat (noted as ADD in the database), and again, compute log differences of them as the measure for the weather factor.

Summarizing our results, we find no statistically significant correlation between gas or electricity prices, the weather, and the market portfolio. Consequently, the hypothesis of the beta being zero for space heating/cooling energy efficiency projects cannot be rejected, in case of both households and businesses.

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VI. Publications related to the theses

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VII. Other related publications by the author

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