Quantum entanglement in finite-dimensional Hilbert spaces

Ph.D. thesis

by

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Preliminaries

The laws of quantum mechanics proved to be very successful in the description and prediction of the behaviour of the microworld. Among these predictions, however, there were some very surprising ones which are in connection with the description of composite quantum systems. In the formalism of quantum mechanics, the so called *entangled (or inseparable) states* of composite systems appear naturally, while the understanding of the correlations of the physical qantities measured on the subsystems of a system being in an entangled state is a challenge for the mind. Namely, these correlations arise from the quantum mechanical interactions between the subsystems, and they can not be modelled classically, these are the manifestations of the entirely quantum behaviour of the nature. Entanglement theory is therefore a deep and fundamental field of central importance, lying in the very basics of the understanding of the physical world [HHHH09].

An interesting twist of the story is that these nonclassical correlations can be used for nonclassical solutions of classical, moreover, of nonclassical tasks, leading to the idea of quantum computation [Fey82]. These nonclassical computational and information theoretical methods are the subject of the emerging field of quantum information theory, which is the extension of the classical information theory for quantum systems, dealing with these quantum correlations [NC00]. The significance of this relatively new field of science is hallmarked, among other things, by the Wolf Prize in Physics in this year.

In the scope of quantum information theory, there are entirely nonclassical, information theoretical tasks (such as quantum communication with super-dense coding, quantum teleportation, quantum cryptography, quantum key distribution, quantum error correction) and also classical computational tasks (such as quantum algorithms for factoring numbers, for quantum search, and for further tasks.) What is really fascinating is that quantum algorithms significantly outperform the best known classical algorithms for the same task, moreover, they are able to solve some problems in polynomial time, which problems can not be solved in polynomial time by the known classical algorithms.

During the run of all the above quantum protocols, the basic resource expended is entanglement, that is, composite quantum systems being in entangled states. A fundamental need is then the studying of the characterization of entanglement, which is the main concern of my work. Although the entanglement which is used for quantum information processing tasks is presented mostly in maximally entangled Bell pairs of two qubits, but the structure of entanglement is far richer than that of these two-qubit pure states. We will consider some aspects of this issue in the present dissertation, here and now we just want to emphasize that the rich structure of multipartite entanglement might provide a lot of opportunities, which are still far from being explored and utilized.

The utilization of even the bipartite entanglement is by no means an easy job. Quantum mechanics works in microscopic scales, and, due to the environmental decoherence, the manifestations of this particular behaviour are hard to reach. Effects of entanglement are studied in many-body systems as well, but an important color in the picture is that the experimental manipulation of individual quantum objects is not out of reach, as is also illustrated by the Nobel Prize in Physics in last year.

The main reference on quantum entanglement is [HHHH09].

Objectives

As quantum entanglement has been recognized to be the basic resource in quantum information theory, a fundamental need is the understanding of its *qualification* and its *quantification*: Is the state entangled, and if it is, then how much entanglement is carried by it? These questions introduce the topics of *separability criteria* and *entanglement measures*, both of which are based on the problem of *classification* of entanglement.

All of these three issues are far more complicated for the case of mixed states than for that of pure ones. On the other hand, bipartite states, either pure or mixed, can be either entangled or separable, but this situation gets involved in multipartite systems, where many different kinds of entanglement arise. During the investigations concerning quantum states, a geometric "insight" turns out to be very useful [BŻ06].

The objective, carried out in the years of research, the results of which are published in the present dissertation, is threefold. *First,* gaining comprehensive knowledge about the entanglement of mixed states, especially about the aforementioned three fundamental issues. *Second,* gaining experiences by demonstrating formerly known methods and by performing the calculations of the related quantities. *Third,* giving some kind of sollutions for these three issues.

New scientific results

In the following thesis statements, I present my contribution to the three fundamental issues outlined in the previous section.

I. I study a 12-parameter family of two-qubit mixed states, arising from a special class of two-fermion systems with four single particle states or alternatively from a four-qubit state vector with amplitudes arranged in an antisymmetric matrix. I obtain a local unitary canonical form for those states. By the use of this I calculate two famous *entanglement measures* which are the Wooters concurrence and the negativity in a closed form. I obtain bounds on the negativity for given Wootters concurrence, which are strictly stronger than those for general two-qubit states. I show that the relevant entanglement measures satisfy the generalized Coffman-Kundu-Wootters formula of distributed entanglement. I give an explicit formula for the residual tangle as well.

The publication belonging to this thesis statement is [1] of the list on page 12.

The main references belonging to this thesis statement are [LNP05, VADM01, CKW00, OV06].

II. Local unitary invariance is a fundamental property of all *entanglement measures*. I study quantities having this property for general multipartite systems. In particular, I give explicit index-free formulas for all the algebraically independent local unitary invariant polynomials up to degree six, for finite dimensional multipartite pure and mixed quantum states. I carry out this task by the use of graph-technical methods, which provide illustrations for this rather abstract topic.

The publication belonging to this thesis statement is [3] of the list on page 12.

The main references belonging to this thesis statement are [HW09, HWW09, Vra11a, Vra11b].

III. I study the noisy GHZ-W mixture and demonstrate some *necessary but not sufficient criteria* for different *classes of separability* of these states. I find that the partial transposition criterion of Peres and the criteria of Gühne and Seevinck dealing directly with matrix elements are the strongest ones for different separability classes of this two-parameter state. I determine a set of entangled states of positive partial transpose. I also give constraints on three-qubit entanglement classes related to the pure SLOCC-classes, and I calculate the Wootters concurrences of the two-qubit subsystems.

The publication belonging to this thesis statement is [2] of the list on page 12.

The main references belonging to this thesis statement are [Per96, GS10].

IV. I elaborate the *partial separability classification* of mixed states of quantum systems composed of arbitrary number of subsystems of Hilbert spaces of arbitrary dimensions. This extended classification is complete in the sense of partial separability and gives $1 + 18 + 1$ partial separability classes in the tripartite case contrary to the formerly known $1+8+1$. I also give *necessary and sufficient criteria* for the classes by the use of convex roof extensions of functions defined on pure states. I show that these functions can be defined so as to be entanglement-monotones, which is another fundamental property of all *entanglement measures*.

The publication belonging to this thesis statement is [4] of the list on page 12.

The main references belonging to this thesis statement are [DCT99, DC00, SU08].

V. For the case of three-qubit systems, by the use of the Freudenthal triple system approach of three-qubit pure state entanglement, I obtain a set of functions on pure states, whose convex roof extensions give *necessary and sufficient criteria* for the partial separability classification. These functions have some advantages over the ones defined in the general construction, which is given in the previous thesis statement. Moreover, these functions fit naturally for a special three-qubit classification which arises as the combination of the partial separability classification with the classification obtained by Acín et. al. for three-qubit mixed states.

The publication belonging to this thesis statement is [4] of the list on page 12.

The main references belonging to this thesis statement are [BDD+09, DCT99, DC00, ABLS01, SU08].

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[Vra11b] Péter Vrana, *On the algebra of local unitary invariants of pure and mixed quantum states*, J. Phys. A **44** (2011), no. 22, 225304.

Publications belonging to thesis statements

The following research articles are covered by this thesis. The publications are listed in chronological order.

[1] **Szilárd Szalay,** Péter Lévay, Szilvia Nagy, János Pipek, *A study of two-qubit density matrices with fermionic purifications,* J. Phys. A **41**, 505304 (2008) (arXiv: 0807.1804 [quant-ph])

[2] **Szilárd Szalay,** *Separability criteria for mixed three-qubit states,* Phys. Rev. A **83**, 062337 (2011) (arXiv: 1101.3256 [quant-ph])

[3] **Szilárd Szalay,** *All degree 6 local unitary invariants of k qudits,* J. Phys. A **45**, 065302 (2012) (arXiv: 1105.3086 [quant-ph])

[4] **Szilárd Szalay,** Zoltán Kökényesi *Partial separability revisited: Necessary and sufficient criteria,* Phys. Rev. A **86**, 032341 (2012) (arXiv: 1206.6253 [quant-ph])

Further publications

The following research articles are the results of another research project of the author, done in the related field of Black Hole / Qubit correspondence. The publications are listed in chronological order.

[5] Péter Lévay, **Szilárd Szalay,**
Attractor mechanism as a distillation procedure,
Phys. Rev. D **82**, 026002 (2010)
(arXiv: 1004.2346 [hep-th])

[6] Péter Lévay, **Szilárd Szalay,**
STU attractors from vanishing concurrence,
Phys. Rev. D **84**, 045005 (2011)
(arXiv: 1011.4180 [hep-th])