Budapest University of Technology and Economics

Initial moment gradient load effect on the Vibration and Dynamic stability of a spatial thin walled structure

Abbas Talimian
PhD Candidate

Prof. Gábor. M. Vörös
Supervisor

Budapest
2013
Faculty of Transportation and Vehicle Engineering

Department of Vehicle Elements and Vehicle Structure Analysis
TO

My Parents
Acknowledgments

First of all, I would like to show appreciation to my supervisor, Prof. Gábor M. Vörös, who has leaded this research kindly. I should also gratitude to his patience and supports during my stay in Hungary.

I do like to express my special thanks to Prof. István Zobory, head of Kandó Kálmán Doctoral School. This thesis could not have been done without assistance of Prof. Péter Béda and Prof. János Tóth. I have to appreciate Prof. Dezső Szőke and Prof. Lajos Borbás who I used their consultation for editing the present dissertation. I also would like to express thanks to the Department of Vehicle Elements and Vehicle Structure Analysis staff for their sympathy among my studies.
Table of Contents

NOMENCLATURE 4

FIGURES TABLE 7

CHAPTER 1 11
1.1 History 11
1.2 State of art 13

CHAPTER 2 18
2.1 Introduction 18
2.2 Stability concept 19
2.3 Preliminaries 20

CHAPTER 3 27
3.1 Introduction 27
3.2 Total Potential Energy 28
3.3 Equations of motion 30
3.4 Natural frequencies and Critical Buckling Moment 35
  3.4.1 Natural frequencies 35
  3.4.2 Critical buckling moment 37
3.5 Forced Vibration 40
  3.5.1 Dominantly Bending mode $\omega_{b1} < \omega_{t1}$ 41
  3.5.2 Dominantly torsional mode $\omega_{t1} < \omega_{b1}$ 45
3.6 Summarize 50

CHAPTER 4 51
4.1 Introduction 51
4.2 Natural damped frequencies and Critical Buckling Moment 52
  4.2.1 Natural frequency 54
  4.2.2 Critical buckling moment 56
4.3 Forced Vibration 59
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.3.1</td>
<td>Dominantly bending mode $\omega_{b1} &lt; \omega_{t1}$</td>
<td>60</td>
</tr>
<tr>
<td>4.3.2</td>
<td>Dominantly torsional mode $\omega_{t2} &lt; \omega_{b2}$</td>
<td>68</td>
</tr>
<tr>
<td>4.4</td>
<td>Summarize</td>
<td>73</td>
</tr>
</tbody>
</table>

**CHAPTER 5**

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1</td>
<td>Introduction</td>
<td>74</td>
</tr>
<tr>
<td>5.2</td>
<td>Periodic excitation</td>
<td>75</td>
</tr>
<tr>
<td>5.3</td>
<td>Periodic solution</td>
<td>75</td>
</tr>
<tr>
<td>5.4</td>
<td>Instability graphs</td>
<td>78</td>
</tr>
<tr>
<td>5.5</td>
<td>Loading parameters effect on instability regions</td>
<td>81</td>
</tr>
<tr>
<td>5.6</td>
<td>Summarize</td>
<td>86</td>
</tr>
</tbody>
</table>

**CHAPTER 6**

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.1</td>
<td>Introduction</td>
<td>88</td>
</tr>
<tr>
<td>6.2</td>
<td>Periodic solution</td>
<td>89</td>
</tr>
<tr>
<td>6.3</td>
<td>Damping ratio effect on instability graphs</td>
<td>92</td>
</tr>
<tr>
<td>6.4</td>
<td>$\lambda$ effect on instability regions</td>
<td>94</td>
</tr>
<tr>
<td>6.5</td>
<td>Summarize</td>
<td>96</td>
</tr>
</tbody>
</table>

**CHAPTER 7**

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.1</td>
<td>Introduction</td>
<td>97</td>
</tr>
<tr>
<td>7.2</td>
<td>Assumptions</td>
<td>97</td>
</tr>
<tr>
<td>7.3</td>
<td>Coupled frequencies and critical buckling moment</td>
<td>97</td>
</tr>
<tr>
<td>7.3.1</td>
<td>Thesis I</td>
<td>98</td>
</tr>
<tr>
<td>7.4</td>
<td>Damped Coupled frequencies and critical buckling moment</td>
<td>98</td>
</tr>
<tr>
<td>7.4.1</td>
<td>Thesis II</td>
<td>99</td>
</tr>
<tr>
<td>7.5</td>
<td>Dynamic stability analysis (Un-damped)</td>
<td>99</td>
</tr>
<tr>
<td>7.5.1</td>
<td>Thesis III</td>
<td>99</td>
</tr>
<tr>
<td>7.6</td>
<td>Dynamic stability analysis (damped)</td>
<td>100</td>
</tr>
<tr>
<td>7.6.1</td>
<td>Thesis IV</td>
<td>100</td>
</tr>
</tbody>
</table>
7.7 Results utilization

8 APPENDIX

8.1 Cross section properties

8.2 Chapter three graphs and tables
   8.2.1 Critical length
   8.2.2 Frequencies and critical buckling moment
   8.2.3 Moment gradient coefficient $C(\psi)$
   8.2.4 Frequency and mixed mode ratio (CS3) (bending and torsional mode)
   8.2.5 Frequency and mixed mode ratio (CS2) (bending mode)

8.3 Chapter four graphs
   8.3.1 Frequency and mixed mode ratio (CS3) (bending mode)
   8.3.2 Frequency and mixed mode ratio (CS3) (torsional mode)
   8.3.3 Frequency and mixed mode ratio (CS2) (bending mode)

8.4 Chapter five figures
   8.4.1 Gradient moment factor effect on Instability regions
   8.4.2 Static buckling moment percentage effect on Instability regions

8.5 Chapter six figures
   8.5.1 Damping ratio and Gradient moment factor effect on Instability regions
   8.5.2 Static buckling moment percentage effect on damped Instability regions

BIBLIOGRAPHY
# Nomenclature

- \(x, y, z\) Cartesian coordinate system, principal inertia axes (global)
- \(r, s\) Coordinate axes parallel to principal inertia axes (local)
- \(C\) Area center
- \(S\) Shear center
- \(y_{NC}, z_{NC}\) Area center position
- \(y_{CS}, z_{CS}\) Shear center position
- \(u, v, w\) Rigid body translation in principal inertia axes direction
- \(\alpha, \beta, \gamma\) Rotations about the shear center axes parallel to principal inertia axes
- \(u_s\)
- \(R, R_s\)
- \(\Theta(x)\) (3-2-3)
- \(\varphi\) Warping function normalized with respect to shear center
- \(\phi\) Frequency ratio (5-2-13)
- \(N\) Axial force acting at the centeriod
- \(V_r, V_s\) Shear forces at the shear center
- \(M_r, M_s\) Bending moment with respect to \(r\) and \(s\) axes
- \(M_x\) Bending moment with respect to \(x\) axis
- \(B\) Bi-moment (2-3-6)
- \(T\) Total torsional moment respect to the shear center
- \(P\) Wagner coefficient (stress resultant)
- \(\sigma_x\) Normal stress distribution
- \(I_r, I_s\) Principal second moment
- \(I_{w}\) Warping constant respect to shear center
- \(I_{ps}\) Polar moment respect to shear center
- \(J\) St. Venant torsion constant
- \(A\) Section area
- \(L, L_c\) Beam length and critical length (3-4-5)
- \(\rho\) Density
- \(\beta_y\) Wagner’s coefficient
- \(\beta_r, \beta_\omega\) (2-3-8)
- \(C_i\) Latest calculated equilibrium state
- \(C_2\) Neighboring or desire state
- \(^2S_{ij}\) Cartesian component of the second Piola-Kirchhoff stress tensor
- \(^2\varepsilon_{ij}\) Green-Lagrange strain tensor
- \(^2U_i\) Total displacement
- \(^2R, R\) External virtual work (chapter 2)
- \(^2f_i\) Body force at desire state
- \(^2p_i\) Surface traction at desire state
- \(\tau_{ij}\) Cauchy stress tensor at \(C_i\) configuration
- \(U_i, U_i^*\) First and second order terms of displacement
\( \delta \) Increment symbol
\( e \) Geometrical eccentricity
\( e_{ij}, e^*_ij \)
\( \eta_{ij} \)
\( 1C_{ijmn} \)
\( \Pi_L \) Conventional strain energy
\( \Pi_{G1} \) Potential energy with respect to initial stresses
\( \Pi_{G2} \) Potential energy with respect to second order effects of eccentric initial loads
\( \Pi_{Ge} \) Potential energy with respect to initial concentrated forces
\( \Pi_M \) Potential energy with respect to inertia forces
\( W \) External loads work
\( E \) Elastic module
\( G \) Shear module
\( \dot{}, \ddot{} \) First and second derivative respect to displacement
\( \dddot{} \) second derivative respect to time
\( \Omega \) Input frequency (chapters 5 and 6)
\( \omega \) Circular velocity, frequency
\( \psi \) Gradient moment factor
\( M_{cr0} \) Critical moment load (3-4-8)
\( M_{cr} \) Moment critical value (3-4-10)
\( R \) Reaction force (chapter 3)
\( a(x,t) \) Lateral displacement
\( \nu(x,t) \) Twisting displacement
\( \mu \) steady state moment load factor (chapter 3)
\( F_i(t) \) Unknown time function coefficient for lateral displacement approximation
\( V_i(t) \) Unknown time function coefficient for warping approximation
\( A, B, C \)
\( D, E, E^* \) Numerical matrices (3-3-7)
\( H, H^* \)
\( M, S \) and \( K_e \) Mass, stability and stiffness matrices (3-3-16)
\( D_a \) Damping matrix (4-2-1)
\( C(\psi) \) gradient coefficient (3-4-12)
\( f_i(x) \) Trail function for lateral displacement
\( g_i(x) \) Trail function for warping displacement
\( \omega_{b1}, \omega_{b2}, \omega_{t1} \) First, second lateral bending and first torsional frequency (3-4-3)
\( Z_1 \) Ratio of natural frequency square over first bending one
\( Z_2 \) Ratio of natural frequency square over first torsional one
\( \eta \) Modal mixing factor
\( L_c \) Critical length (3-4-5)
\( \mu_c \) Critical steady state moment load factor
\( \zeta \) Rayleigh damping coefficient
\( \zeta \) Damping ratio
\( \omega_{b,1}, \omega_{b,2}, \omega_{t,1} \)  
Damped first, second lateral bending and first torsional frequency (4-2-10)

\( M_{cr}^{d} \)  
Critical moment load (4-2-16)

\( M_{cr}^{d} \)  
Moment critical value (4-2-19)

\( C(\psi, \zeta) \)  
Damped gradient coefficient

\( \mu_c^{d} \)  
Damped critical steady state moment load factor

\( M_s \)  
Static amplitude of time dependent moment

\( M_t \)  
Dynamic amplitude of time dependent moment

\( \lambda \)  
Static buckling moment percentage

\( \kappa \)  
dynamic buckling moment percentage

\( CS1, CS2 \)  
Cross sections (appendix)

\( CS3, CS4 \)
Figures table

Fig 1- 1 collapsed Tacoma Narrows Bridge (Left), Hartford Arena space frame dome (Right) ------- 12
Fig 1- 2 Integral bus body under frame--------------------------------------------------------------- 14
Fig 1- 3 dissertation description-------------------------------------------------------------------- 17

Fig 2- 1 Lyapunov stability-------------------------------------------------------------------------- 19
Fig 2- 2 Beam element local systems and eccentricities --------------------------------------------- 20
Fig 2- 3 Local displacement parameters and stress resultants-------------------------------------- 20

Fig 3- 1 Chapter 3 process ................................................................................................................ 27
Fig 3- 2 Displacement, Rotations and coordinates system (sample cross section)......................... 28
Fig 3- 3 $M_y$ for three different types of external loading............................................................. 29
Fig 3- 4 Boundary conditions ............................................................................................................ 30
Fig 3- 5 Minimum Torsional warping and Lateral Bending shape function (Uniform bending) ....... 31
Fig 3- 6 Minimum Torsional warping and Lateral Bending shape function (Asymmetric bending moment)................................................................. 31
Fig 3- 7 Ratio of First natural lateral bending frequency over first torsional one (CS1)................... 36
Fig 3- 8 Ratio of First natural lateral bending frequency over first torsional one (CS4)............... 37
Fig 3- 9 Variation of moment gradient coefficient C($\psi$) vs. moment gradient factor............... 39
Fig 3- 10 Variation of moment gradient coefficient C($\psi$) vs. moment gradient factor by different Ritz terms ........................................................................................................................................... 40
Fig 3- 11 Natural frequency changing vs. steady state moment load factor, $\psi$=-1 (uniform) (bending mode) (CS1).................................................................................................................................................. 42
Fig 3- 12 Natural frequency changing vs. steady state moment load factor, $\psi$=-1 (uniform) (bending mode) (CS4).................................................................................................................................................. 42
Fig 3- 13 Natural frequency changing vs. steady state moment load factor, $\psi$=1 (asymmetric) (bending mode) (CS1) .................................................................................................................................................. 43
Fig 3- 14 Natural frequency changing vs. steady state moment load factor, $\psi$=1 (asymmetric) (bending mode) (CS4) .................................................................................................................................................. 43
Fig 3- 15 Critical steady state moment load factor, $\psi$=1 (asymmetric) (bending mode) (CS1) ...... 44
Fig 3- 16 Critical steady state moment load factor, $\psi$=1 (asymmetric) (bending mode) (CS4) ...... 44
Fig 3- 17 Natural frequency changing vs. steady state moment load factor, $\psi$=-1 (uniform) (torsional mode) (CS4).................................................................................................................................................. 46
Fig 3- 18 Natural frequency changing vs. steady state moment load factor, $\psi$=1 (asymmetric) (torsional mode) (CS4).................................................................................................................................................. 47
Fig 3- 19 Change of frequency vs. moment load parameters, (bending mode) (CS1) ................. 48
Fig 3- 20 Change of mixed mode ratio vs. moment load parameters, (bending mode) (CS1) ....... 48
Fig 3- 21 Change of frequency vs. moment load parameters, (bending mode) (CS4).................... 49
Fig 3- 22 Change of mixed mode ratio vs. moment load parameters, (bending mode) (CS4) ....... 49
Fig 3- 23 Change of frequency vs. moment load parameters, (torsional mode) (CS4)................... 49
Fig 3-24 Change of mixed mode ratio vs. moment load parameters, (torsional mode) (CS4) ....... 50

Fig 4-1 Chapter 4 process .......................................................................................................................................... 51
Fig 4-2 Frequency vs. damping ratio for different damping coefficient................................................................. 53
Fig 4-3 Changing the natural frequency vs. damping ratio (CS1) (bending mode) .............................................. 55
Fig 4-4 Changing the natural frequency vs. damping ratio (CS4) (bending mode) .............................................. 55
Fig 4-5 Changing the natural frequency vs. damping ratio (CS4) (torsional mode) ............................................ 56
Fig 4-6 $M^d_{cr0}/M_{cr0}$ vs. damping ratio........................................................................................................... 57
Fig 4-7 Moment gradient coefficient $C(\psi, \zeta)$ vs $\psi$ (bending mode)......................................................... 58
Fig 4-8 Moment gradient coefficient $C(\psi, \zeta)$ vs $\psi$ (torsional mode) ..................................................... 59
Fig 4-9 Natural frequency changing vs. steady state moment load factor, $\psi=1$ (CS1) (bending mode) ............... 62
Fig 4-10 Natural frequency changing vs. steady state moment load factor, $\psi=1$ (CS4) (bending mode) ............... 62
Fig 4-11 Natural frequency changing vs. steady state moment load factor, $\psi=-1$ (CS1) (bending mode) ............... 63
Fig 4-12 Natural frequency changing vs. steady state moment load factor, $\psi=-1$ (CS4) (bending mode) ............... 63
Fig 4-13 Natural Frequency vs. moment load parameters (bending mode), $\zeta = 0.4$ (CS1) .................. 64
Fig 4-14 Mixed mode factor vs. moment load parameters (bending mode), $\zeta = 0.4$ (CS1) .................. 64
Fig 4-15 Natural Frequency vs. moment load parameters (bending mode), $\zeta = 0.6$ (CS1) .................. 65
Fig 4-16 Mixed mode factor vs. moment load parameters (bending mode), $\zeta = 0.6$ (CS1) .................. 65
Fig 4-17 Natural Frequency vs. moment load parameters (bending mode), $\zeta = 0.8$ (CS1) .................. 65
Fig 4-18 Mixed mode factor vs. moment load parameters (bending mode), $\zeta = 0.8$ (CS1) .................. 66
Fig 4-19 Natural Frequency vs. moment load parameters (bending mode), $\zeta = 0.4$ (CS4) .................. 66
Fig 4-20 Mixed mode factor vs. moment load parameters (bending mode), $\zeta = 0.4$ (CS4) .................. 66
Fig 4-21 Natural Frequency vs. moment load parameters (bending mode), $\zeta = 0.6$ (CS4) .................. 67
Fig 4-22 Mixed mode factor vs. moment load parameters (bending mode), $\zeta = 0.6$ (CS4) .................. 67
Fig 4-23 Natural Frequency vs. moment load parameters (bending mode), $\zeta = 0.8$ (CS4) .................. 67
Fig 4-24 Mixed mode factor vs. moment load parameters (bending mode), $\zeta = 0.8$ (CS4) .................. 68
Fig 4-25 Natural frequency changing vs. steady state moment load factor, $\psi=1$ (CS4) (torsional mode) .......................................................................................................................... 69
Fig 4-26 Natural frequency changing vs. steady state moment load factor, $\psi=-1$ (CS4) (torsional mode) .......................................................................................................................... 70
Fig 4-27 Natural Frequency vs. moment load parameters (torsional mode), $\zeta = 0.4$ (CS4) .......... 71
Fig 4-28 Mixed mode factor vs. moment load parameters (torsional mode), $\zeta = 0.8$ (CS4) .......... 71
Fig 4-29 Natural Frequency vs. moment load parameters (torsional mode), $\zeta = 0.6$ (CS4) .......... 71
Fig 4-30 Mixed mode factor vs. moment load parameters (torsional mode), $\zeta = 0.6$ (CS4) .......... 72
Fig 4-31 Natural Frequency vs. moment load parameters (torsional mode), $\zeta = 0.8$ (CS4) .......... 72
Fig 4-32 Mixed mode factor vs. moment load parameters (torsional mode), $\zeta = 0.8$ (CS4) .......... 72
Fig app -  5 Frequency and mixed mode ratio $\zeta = 0.4$ (bending mode) (CS3) .................................. 105
Fig app -  6 Frequency and mixed mode ratio $\zeta = 0.6$ (bending mode) (CS3) .................................. 106
Fig app -  7 Frequency and mixed mode ratio $\zeta = 0.8$ (bending mode) (CS3) .................................. 106
Fig app -  8 Frequency and mixed mode ratio $\zeta = 0.4$ (torsional mode) (CS3) .............................. 106
Fig app -  9 Frequency and mixed mode ratio $\zeta = 0.6$ (torsional mode) (CS3) .............................. 107
Fig app - 10 Frequency and mixed mode ratio $\zeta = 0.8$ (torsional mode) (CS3) .............................. 107
Fig app - 11 Frequency and mixed mode ratio $\zeta = 0.4$ (bending mode) (CS2) .............................. 107
Fig app - 12 Frequency and mixed mode ratio $\zeta = 0.6$ (bending mode) (CS2) .............................. 108
Fig app - 13 Frequency and mixed mode ratio $\zeta = 0.8$ (bending mode) (CS2) .............................. 108
Fig app - 14 Instability regions for $\psi$ boundary values, $\lambda = 0.5$ (CS2) (bending mode) ................. 108
Fig app - 15 Instability regions for different $\psi$, $\lambda = 0.5$ (CS2) (bending mode) ............................. 108
Fig app - 16 Instability regions for $\psi$ boundary values, $\lambda = 0.5$ (CS3) (bending mode) ................. 109
Fig app - 17 Instability regions for different $\psi$, $\lambda = 0.5$ (CS3) (bending mode) ............................. 109
Fig app - 18 Instability regions for $\psi$ boundary values, $\lambda = 0.5$ (CS3) (torsional mode) ................. 110
Fig app - 19 Instability regions for different $\psi$, $\lambda = 0.5$ (CS3) (torsional mode) ............................ 110
Fig app - 20 Instability regions for different $\lambda$, $\psi = -1$ (CS2) (bending mode) ............................. 110
Fig app - 21 Instability regions for different $\lambda$, $\psi = 1$ (CS2) (bending mode) ............................. 111
Fig app - 22 Instability regions for different $\lambda$, $\psi = -1$ (CS3) (bending mode) ............................ 111
Fig app - 23 Instability regions for different $\lambda$, $\psi = -1$ (CS3) (torsional mode) .......................... 111
Fig app - 24 Instability regions for different $\lambda$, $\psi = 1$ (CS3) (torsional mode) .......................... 112
Fig app - 25 Instability regions for different $\zeta$, $\psi = -1$ (CS1) (bending mode) ............................ 112
Fig app - 26 Instability regions for different $\zeta$, $\psi = -1$ (CS3) (bending mode) ............................ 112
Fig app - 27 Instability regions for different $\zeta$, $\psi = 1$ (CS3) (bending mode) ............................ 113
Fig app - 28 Instability regions for different $\zeta$, $\psi = -1$ (CS3) (torsional mode) .......................... 113
Fig app - 29 Instability regions for different $\zeta$, $\psi = 1$ (CS3) (torsional mode) .......................... 113
Fig app - 30 Instability regions for different $\lambda$ values, $\psi = -1$, $\zeta = 0.1$ (CS2) (dominantly bending) .. 114
Fig app - 31 Instability regions for different $\lambda$ values, $\psi = -1$, $\zeta = 0.1$ (CS3) (dominantly bending) .. 114
Fig app - 32 Instability regions for different $\lambda$ values, $\psi = -1$, $\zeta = 0.1$ (CS3) (dominantly torsional) 114
Fig app - 33 Instability regions for different $\lambda$ values, $\psi = 1$, $\zeta = 0.02$ (CS3) (dominantly torsional) 115
Chapter 1

1.1 History

In engineering science, the stability of structures is one of the researchers desire. Many structures failures, either static or dynamic, have been attributed because of structural instability. The loading nature defines the problem’s nature to be solved. Structures may be subjected to static, dynamic or even stochastic load what will be dealt with in this dissertation is the dynamically loaded structure [1] [2] [3] [4].

In mathematical point of view, a fundamental property of a dynamical system, which is the qualitative behavior of trajectories that is unaffected by $C^1$- small perturbations is called structural stability. Although a well-known theory in stability field, Lyapunov stability\footnote{It is named after Aleksandr Lyapunov (1857-1918), a Russian mathematician, mechanician and physicist, who published his book “The General Problem of Stability of Motion” in 1892 [53]. Lyapunov was the first to consider the modifications necessary in nonlinear systems to the linear theory of stability based on linearizing near a point of equilibrium. His work, initially published in Russian and then translated to French, received little attention for many years. Interest in it started suddenly during the Cold War (1953-1962) period when the so-called “Second Method of Lyapunov” was found to be applicable to the stability of aerospace guidance systems which typically contain strong nonlinearities not treatable by other methods.}, speaks about considering perturbations of initial conditions for a fixed system.

Among the history, neglecting this important field or having not so clear mind about it have caused several tragic happenings. Tacoma Narrows Bridge was collapsed in 1940 because of aerodynamic instability. In 1978 Hartford Arena space frame and reticulated dome past College Theater collapsed. What have been mentioned already were only some examples in this case but unfortunately these types of accidents had been repeated more and more.

Speaking about stability returns to almost two hundred years ago where stability analysis was sparked in Euler’s mind in the forth decade of eighteen centuries. He advised a solution for buckling of an elastic column. Until the end of the 19th century several types of problem who’s related to the fundamental linear elastic were solved.

In 1807 Young, realized that imperfections such as initial curvature, initial bending moments or load eccentricity are playing an important role, and derived a magnification factor for deflections and moments in columns due to axial load [5]. Euler gave some solutions for critical loads of columns that are having different restraints, but the experimental results did not confirm them [6]. In 1859 geometrically non-linear large deflections theory has extended by Kirchhoff, he also suggested a solution for the deflection curve [7].
Von Mises and Ratzersdorfer [8] and Chwalla [9] worked on flexibility method for analyzing the frames and calculating the critical load by formulating the dependence of the flexibility matrix of a column and its axial force. After a short period in 1935 their jobs was continued by James [10], Livesley and Chandler [11]. The matrix stiffness method in the form of beam’s finite element has been going to be familiar for engineers because of its simplicity in going with computer software analysis.

A structure’s stability can be in trouble if it is under periodic load. In 1893 Lyapunov worked on the behavior of dynamic instabilities for having general stability. In other words, it means a structure motion can be stable if any possible small changes being in initial condition, or better to say that there shall be a few changes in the response [12].

In the middle of the century columns were subjected to various idealized non-conservative loads for instance follower forces instability was taken into account. One of the most important instability analyses is related to the foundation of rotating machinery as well as bridges, chimneys or guyed masts. In these types of structures the axial displacement doubled the frequency of the lateral vibration which causes to resonance. Mathieu differential equation is an ideal form in this field of studying, which was solved by Rayleigh in 1894, approximately [13].

Conservative systems are special types of stability problems. They are stable if potential energy is positive definite. This method can help in finding the optimum shape or dynamic loads [14] [15]. Using Lagrange-Dirichlet theorem causes structure stability, can be checked only by checking the positive definiteness of the tangential stiffness matrix of the structure. While twentieth century began, the engineering science world opened its eyes to nonlinear behavior stability analysis. Researcher’s attempt, made the dynamic stability especially non-conservative systems, understandable.
Snap through is another important class of elastic systems stability. It is related to nonlinear systems, while bifurcation with deflections does not exist, for instance in flat arches or shallow shells. The potential energy is one of the useful techniques that based on approximated solutions e.g. Rayleigh’s method. This thesis uses potential energy method for evaluating the stability of a structure.

In the case of thin walled cross section structures, a deformation mode that should be noted is the warping of the cross section, with the bi-moment being the associated force variable. The torsional warping depends on lateral axial buckling that can be seen as a coupled equation of motion.

Numerical techniques (procedures) have been suggested widely for seeking the solution in later mentioned problems. One of these numerical methods whose has been applied for several types of structures e.g. circular, rectangular plates or beams and in this text as well is series expansions.

While a structure is under a dynamic load(s), the system is called parametrically excited system in brief. The parametric instability problems are not limited to the above mentioned situation. The steady- state motions of nonlinear dynamical systems can be categorized in this field also. The main aim for analyzing the parametrically excited systems is to find the regions in parametric space which the system is going to be unstable. The later domains have been known as dynamic instability regions. The curve which is separating stable and instable regions is named stability boundary. Illustrating these boundaries on the parameter space is called stability diagrams [16].

In 1924 N.M.Beliaev opened the mathematical analysis field for problems in dynamic stability [17]. He also extended the dynamic stability solutions to both linear and non-linear bars as well as did for plates. Bolotin, who was well known for his literatures in stability field, did not consider the inertia forces associated with the rotation of the cross-sections of rods, with respects to its own principle axes [18]. In 1968 a research was published by Brown et.al which shows the usage of finite element method for studying an Euler-Bernoulli beam dynamic stability behavior [19].

### 1.2 State of art

Beams are widely being used in public vehicles, for instance buses frame, trains and etc. as a fundamental element in chassis structure. These elements are carrying different type of loading i.e. static and dynamic. Static loads are related to the weight of the structure, engine, and passengers. Dynamic loads, which are much more important than the previous one, are related to the external excitation effect on beams, such as rotary parts or road irregularities [20] [21] [22] [23] [24]. Many machines and structural members can be modeled, as beams with different geometries, for example beams with uniform cross-section, tapered and twisted beams. These components may have different boundary conditions depending on their applications.
Some researchers put their interest on seeking the stability regions of a structure in recent years as like following were selected from a mass of literatures. In general some of them focused on radial, axial and stochastic loading, whereas the other decided to work on different material type such as functional materials.

Öztürk and Sabuncu on 2005 worked on static and dynamic stabilities of laminated composite cantilever beam having linear translation and torsional spring as elastic supports. The beam was subjected to periodic axial loading. They applied Euler beam and finite element method and used energy expressions in conjunction with Bolotin approach. In the study they obtained various disturbing frequency ranges in which the beam being unstable [25].

Liew et.al applied periodic axial force on rotating cylindrical shells and by the help of Ritz method and Bolotin’s first approximation, the dynamic stability was investigated. Mathieu-Hill equation is obtained for Ritz energy minimization procedure [26].

Kumar and Mohammed in 2007 studied on dynamic stability of columns and frames subjected to axially periodic loads. They used finite element method with two nodes beam element for their analysis. Stability regions of columns were determined by decomposing the periodic loading into various harmonics by use of Fourier series expansion. They also applied direct integration of motion equations for discrete system. Newmark method was carried out to verify the results [27].

In 2008, Bazhenov et.al worked n the possibility of stabilizing dynamic states, which occurred by deterministic periodic parametric loading [4]. Sakar and Sabuncu in 2008 published their research results which were dealt with applying a finite element model for static and dynamics stability analysis of airfoil which is pre-twisted and it is rotating [28].

Saravia et.al studied dynamic stability behavior of a thin-walled composite beam which is rotating in 2011. They applied finite element method in their research. The investigation was based on Bolotin method for an axial periodic load. The regions of instability are obtained and expressed in non-dimensional terms [29].
Sabuncu and his colleagues in 2011 used a finite element method to obtain out-of-plane stability analysis of thin curved beams with tapered cross section. Beams were subjected to uniformly distribute radial loading. They used Bolotin approach to analyze dynamic stability and speak about first unstable regions. In this study out-of-plane vibration and buckling had discussed as well [30].

In 2012 Yang and Fang, investigated the stability of an axially moving beam constituted by fractional order material subjected to parametric resonances. They applied Newton’s second law and the fractional derivative Kelvin constitutive relationship. It was assumed that time dependent axial speed is changing harmonically, about a constant mean velocity [31].

In 2007 Vörös introduced seven degrees of freedom for finding the warping effect on buckling and vibration of stiffened plates [32]. He followed his research in 2009 to concentrate on the warping effect on the coupling of torsional bending coupling while a beam is initially loaded by uniform bending moment at its ends [33]. Questions which had been opened up to the present research were; firstly if a beam is initially loaded by linear gradient moment at the ends and warping is considering as well, what would happen for natural frequencies. Second question related to the effect of this type of loading on the structure dynamic stability regions. According to questions which this survey is going to answer them, Initial moment gradient load effect on the Vibration and Dynamic stability of a spatial thin walled structure, was selected as the title of the research.

The first step was to assume a thin walled prismatic beam with a constant cross section area along the length, as like what my supervisor has assumed in his latest research [34] for keeping the continuity of the research. Using virtual work principal whilst the warping is taken into account for deriving motion equations leads to three differential equations. One of these equations is about the deflection of the beam and it is the uncoupled one. In this research the other two equations, which are coupled, were considered. The coupling is between lateral motion and the axial warping.

This research can be chopped into two different parts. The first part goes to find the answer of first question which was mentioned earlier and tried to elaborate the effect of initial moment loading when it is changing between uniform and asymmetric one on natural frequencies. Critical buckling moment equation for a prismatic thin walled beam was also given. Then by using these values and helping of Mathieu-Hill equation the stability regions were obtained as the second part of the survey which was tried to answer the second question. In brief, it means that second part of the studying shows this type of loading effect on instability curves.

For plotting graphs, computer programs by Maple software were applied. Some thin walled cross sections were selected as examples for preparing graphs. They were named by CS1, CS2, CS3 and

---

1 There is no clear distinction between thin and thick sections, sometimes the rule \((t_{max}/d) < 0.1\) is used, where \(d\) is some other overall sectional dimension. The cross section of thin walled beams can be divided into two main groups; closed or open cross sections. Typical closed sections are round, square, and rectangular tubes. Open sections are I-beams, T-beams, L-beams, and etc. Thin walled beams are useful in engineering because of their bending stiffness per unit cross sectional area is much higher than that for solid cross sections such a rod or bar. In this way, stiff beams can be achieved with minimum weight.

2 One of these examples (CS1) was selected as same as [34] for keeping the research continuity.
CS4 in abbreviation in present text and their geometrical details and cross sectional properties are given in the appendix. CS2 and CS4 cross sections properties were used for evaluating derived equations and plotting graphs in the text and CS1 and CS3 cross sections plots were given in the appendix for representing the solution’s generality. Due to this point that the load type effect was the present research interest, graphs were given in dimensionless format. Here it has worth to be noted that in for using these graphs in practice, material limitations, loading type and supports conditions should be considered.

As a quick glance, the following procedure was used for achieving main goals of the current survey:

- Discuss about the coupled bending torsional frequency of a beam that is subjected by a gradient moment load.
- Finding the effect of Reylaigh damping on the coupled bending torsional frequency while this type of loading is taken into account.
- Illustrating the stability boundaries for a beam while excited by parametric gradient moment.
- Investigating the approximated stability boundaries for an asymmetric moment in space and periodic in time while the damping effect is taken into account.

The first two steps are covering the first basic question which was mentioned earlier, and expanded in chapters three and four. For dealing with second fundamental question the third and forth steps were applied in chapter five and six respectively.

In chapter two, fundamental and preliminary equations will have been derived. This chapter shall show how coupled motion equations are adopted for a beam which subjected to initial moment. Vlasov theory has been used.

Chapter three covered the first aim of the research, and goes to express equations for the coupled bending torsional frequency of the beam. Summarized results will be represented as a group Thesis I.

In chapter four the Rayleigh damping effect was added to equations. The later results have been given as group Thesis II. Outcomes of chapters three and four were used in subsequent sections for illustrating the instability regions.

The instability regions and discussing on the effect of different loading parameters on them for un-damped and damped cases were done in chapter five and six respectively (group Thesis III and IV). The last but not least section (chapter seven) relates to new scientific results and their utilization.
Initial moment gradient load effect on the vibration and dynamic stability of a spatial thin walled structure

Fig 1- 3 dissertation description
Chapter 2

2.1 Introduction

As it has been mentioned previously on state of art, most of the mechanical elements can be simulated by a beam which has different boundary conditions due to its usage in the system. Many machines and structural members can be modeled as beams with different geometries as like beams with uniform cross sections. This chapter’s aim is, deriving the essential equations which will be brought into play in subsequent chapters.

According to Saint-Venant’s theory of free torsion, the cross-section will not be remain as a plain generally and points can move freely in the direction of the rod and the torsion’s angle changes linearly with constant rate. By restricting the torsional warping with internal or external constraints, torsion rate will also being changed along the beam. The impeded torsion was developed by Valsov [35]. The Bernoulli-Vlasov theory was implemented because of the main objective of this research which is constraint torsion effect.

In the updated Lagrangian (UL) approach applied in this research, system quantities are referred to the last known equilibrium configuration, here to be called as initial state. In this work, basic assumptions are as follows: the beam member is straight and prismatic, the cross-section is rigid in its plane but is subjected to torsional warping, rotations are large but strains are small, the material is homogeneous, isotropic and linearly elastic.
2.2 Stability concept

Taking into account a system by the following characteristic equation [15],

\[ \dot{x} = f(x, t) \quad f(0, t) = 0 \]  

(2-2-1)

Here \( x \) nominate the state vector of the system, and \( f(0,t) \) represents the initial condition. The goal is to find steady state solutions while \( \dot{x} = 0 \). If one assumes that (2-2-1) is a linear system, this linearity causes the steady state solution being unique. When \( \dot{x} = 0 \), it can be seen that the steady states of the system are not being change regarding to the time, while there will be no excitations.

In sense of Lyapunov theory, (2-2-1) stability solution should be defined as follows; if \( x(t;x_0) \) is a solution for dynamic system (2-2-1), with initial condition \( x(t)|_{t=0} = x_0 \), the solution will be stable if for a given \( \varepsilon>0 \) there exists such \( \delta>0 \) where,

\[
\|x(t;x_0)\| < \varepsilon \quad \forall t > 0 \quad \text{if} \quad \|x_0\| < \delta(\varepsilon)
\]  

(2-2-2)

It has to be mentioned here that \( \|\cdot\| \) is a proper vector norm for the given dynamical system as like (2-2-1).

The system will be called asymptotically stable if,

\[
\|x(t;x_0)\| \to 0 \quad \text{while} \quad t \to \infty
\]  

(2-2-3)

If the start point for the system being in \( \delta(\varepsilon) \) region, \( x_0 \), and subsequently the previously mentioned solution \( x(t;x_0) \) remains inside \( \varepsilon \) region for any time, the system will be called stable. Meanwhile \( x(t;x_0) \) goes to the space origin by increasing the time, \( t \to \infty \), the system shall be known as an asymptotically one. While speaking about finite dimensional system, all norms will be equivalent. It means that when a system such (2-2-1) is stable in an apt norm, it is stable in any other norms as well. In infinite dimension systems, the stability must be defined regarding to a specific norm. For instance in an elastic continuum, the energy norm is the appropriate one,
\[ \|u\|^2 = \frac{1}{V} \int v u dV \quad \|\dot{u}\|^2 = \frac{1}{M} \int \rho \dot{u} \dot{u} dV \quad (2-2-4) \]

Here \( u(x,t) \) is the displacement field and \( \dot{u}(x,t) \) is the velocity. \( \rho, V \) and \( M \) are density, volume, and mass respectively. Due to Koiter, the elastic continuum is stable if given \( \epsilon \) and \( \epsilon' \), there exist such \( \delta(\epsilon,\epsilon') \) and \( \delta'(\epsilon,\epsilon') \) which,

\[ \|u\| < \epsilon \quad \text{and} \quad \|\dot{u}\| < \epsilon' \quad \forall t > 0 \quad \text{then} \quad \|u(x,0)\| \leq \delta \quad \text{and} \quad \|\dot{u}(x,0)\| \leq \delta' \quad (2-2-5) \]

### 2.3 Preliminaries

As shown in (fig 2-1) for a beam structure with an asymmetric cross section in general, the local axis \( x \) of the Cartesian system is parallel to the assumed straight beam element. The other coordinate axes, \( y \) and \( z \) are parallel to the principal inertia axes named \( r \) and \( s \). the element’s center of area, \( C \), and the shear center, \( S \), in the plane of cross section are measured by \( y_{NC}, z_{NC} \) and \( y_{CS}, z_{CS} \) respectively. The notations are selected as same as in Refs. [36] [32] [37].

![Beam element local systems and eccentricities](image1)

By these assumptions, the beam behaves as like a rigid body in a plane normal to the axis. Based on what have been mentioned, rigid body translations in \( x, y \) and \( z \) directions of \( S \) are \( u, v \) and \( w \). Similarly the relevant rotations about the shear center axes parallel to \( x, y \) and \( z \) are given by \( \alpha, \beta \) and \( \gamma \). The axial displacement is the sum of axial displacement \( u \), effect of rotations \( \beta \) and \( \gamma \) of the planar section and the out of plane torsion warping displacement. All local displacement parameters are designated in (fig 2-2).

![Local displacement parameters and stress resultants](image2)
According to the theory of large rotation, the displacement vector includes the translational and rotational deformation [38].

\[ u = u_i + \left( \Omega + \frac{1}{2} \Omega \Omega \right) (R - R_i) = U + U^* \]  \hspace{1cm} (2-3-1)

where,

\[ R = \begin{bmatrix} 0 & r & s \end{bmatrix}^T, R_i = \begin{bmatrix} 0 & y_{CS} & z_{CS} \end{bmatrix}^T \] and \[ \Omega = \begin{bmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{bmatrix} \]  \hspace{1cm} (2-3-2)

\( \Omega \) is known as the small rotation matrix. By expanding (2-3-1) with the help of (2-3-2), the linear and nonlinear part of (2-3-1) can be rewritten as (2-3-3) and (2-3-4) respectively.

\[ U = [U_i] = u_i + \Omega (R - R_i) = \begin{bmatrix} u + \varphi \beta \\ v \\ w \end{bmatrix} + \begin{bmatrix} \beta (s - z_{CS}) - \gamma (r - y_{CS}) \\ -\alpha (s - z_{CS}) \\ \alpha (r - y_{CS}) \end{bmatrix} \]  \hspace{1cm} (2-3-3)

\[ U^* = [U_i^*] = \frac{1}{2} \Omega \Omega (R - R_i) = \frac{1}{2} \begin{bmatrix} \alpha \beta (r - y_{CS}) + \alpha \gamma (s - z_{CS}) \\ -\left( \alpha^2 + \gamma^2 \right) (r - y_{CS}) + \beta \gamma (s - z_{CS}) \\ \beta \gamma (r - y_{CS}) - \left( \alpha^2 + \beta^2 \right) (s - z_{CS}) \end{bmatrix} \]  \hspace{1cm} (2-3-4)

\( \varphi(x) \) warping parameter, defines the out-of-plane torsional warping displacement and \( \varphi(r,s) \) has used for warping function normalized with respect to the shear center.

\[ \int_A \varphi dA = 0 \quad \int_A r \varphi dA = \int_A s \varphi dA = 0 \]

\[ \int_A \frac{\partial \varphi}{\partial r} dA = -A z_{CS} \quad \int_A \frac{\partial \varphi}{\partial s} dA = A y_{CS} \]

\[ \int_A \left[ \left( \frac{\partial \varphi}{\partial r} \right)^2 + \left( \frac{\partial \varphi}{\partial s} \right)^2 \right] dA = A y_{CS}^2 + A z_{CS}^2 + \int_A \left( s \frac{\partial \varphi}{\partial r} - r \frac{\partial \varphi}{\partial s} \right) dA \]  \hspace{1cm} (2-3-5)

Here \( A \) is the cross-section area and in thin-walled sections \( \varphi = -\omega [35] \), the sector area coordinate introduced in Vlasov model. In these equations the warping function \( \varphi(r,s) \) and the shear center location are the same as in the case of free torsion. Having rigid in plane deformation assumption, the stress resultants can be introduced as:
where, $N$ represents the axial force acting at the Centeroid. Shear forces at the shear center are $V_r$ and $V_s$. $M_r$ and $M_s$ are the bending moment with respect to $r$ and $s$ axes. $T$ gives the total torsional moment respect to the shear center and $B$ is called bi-moment $^1$ [39]. The stress resultant known as the wagner coefficient has given by $P$.

The normal stress distribution can be written by the help of linear engineering theory:

$$\sigma_x = \frac{N}{A} + \frac{M_r}{I_r}s - \frac{M_s}{I_s}r + \frac{B}{I_{w\phi}} \varphi$$  \hspace{1cm} (2-3-7)

Cross-sectional geometric properties in previously defined system (figures 2-1 and 2-2) are:

$$I_r = \int_A s^2 dA \hspace{1cm} I_s = \int_A r^2 dA$$

$$J = I_r + I_s - \int_A \left( s \frac{\partial \varphi}{\partial r} - r \frac{\partial \varphi}{\partial s} \right) dA \hspace{1cm} I_{w\phi} = \int_A \varphi^2 dA$$

$$\beta_r = \frac{1}{I_r} \int_A s \left( r^2 + s^2 \right) dA - 2z_{CS} \hspace{1cm} i^2 = \frac{I_r + I_s}{A} + y_{CS}^2 + z_{CS}^2 \hspace{1cm} (2-3-8)$$

$$\beta_s = \frac{1}{I_s} \int_A \varphi \left( r^2 + s^2 \right) dA \hspace{1cm} \beta = \frac{1}{I_s} \int_A r \left( r^2 + s^2 \right) dA - 2y_{CS}$$

$$I_{ps} = I_s + I_r + A \left( y_{CS}^2 + z_{CS}^2 \right) \hspace{1cm} i_{ps}^2 = \frac{I_{ps}}{A}$$

While using Euler-Bernoulli, Vlasov theory $^2$ and shear deformation effects are not taken into account,

---

$^1$ Bi-moment is using in beam analysis and related to warping and torsion. It gives the warping stress distribution at a cross-section for the cases of torsional warping and distortional warping respectively. In general bi-moment can be given by a pair of equal and opposite bending moments.

$^2$ Vlasov’s theory includes extension, bending and torsion deformations of isotropic homogeneous beams and describes the torsional shear stress flow as a superposition of two parts. The Saint-Venant flow (pure torsion) and shear stresses induced by the un-uniform warping of the cross-section.
\[ \beta = -\frac{dv}{dx} = -w' \quad \gamma = \frac{dv}{dx} = v' \quad \delta(x,t) = \frac{d\alpha}{dx} = \alpha' \]  

(2-3-9)

Subdividing the deformation path into tiny parts or increments, an incremental analysis can be used. There exist two consecutive configurations \( C_1 \) and \( C_2 \). Last calculated equilibrium state shall be shown with \( C_1 \) and neighboring or desired state will represented by \( C_2 \). Updated Lagrangian formulation, general continuum virtual work principle can be written as:

\[
\int_V 2S_{ij} \delta \left( \varepsilon_{ij} \right) dV = \int_V 2R \delta \left( u_i \right) dV + \int_A 2p_i \delta \left( u_i \right) dA
\]

(2-3-10)

\( S_{ij} \) is the Cartesian component of the second Piola-Kirchhoff stress tensor and \( \varepsilon_{ij} \) is used as the Green-Lagrange strain tensor which has been measured respect to \( C_1 \). Total displacement is given by \( u_i \). \( 2R \) is selected for representing the external virtual work whose been done by body force \( f_i \) and surface traction \( p_i \) at desired state. The left superscript points out the configuration, that quantity has been occurred. The stress, displacement and external loads can be expanded as follow for deriving the incremental form of the (2-3-10).

\[
\begin{align*}
2S_{ij} &= \tau_{ij} + S_{ij} \\
2f_i &= f_i + f_i \\
2p_i &= p_i + p_i \\
2u_i &= u_i + u_i = u_i + U_i + U_i^* \rightarrow \delta \left( u_i \right) = \delta u_i = \delta \left( U_i + U_i^* \right)
\end{align*}
\]

(2-3-11)

Here the Cauchy stress tensor at Last calculated equilibrium state configuration \( (C_1) \) has been represented by \( \tau_{ij} \) while \( U_i \) and \( U_i^* \) are the first and second order terms of displacement parameters as defined in (2-3-3) and (2-3-4) by putting the incremental displacements into the Green-Lagrange strain’s increment and omitting the higher order terms an incremental strain can be written such as:

\[
\delta \left( \varepsilon_{ij} \right) = \delta \varepsilon_{ij} = \delta \frac{1}{2} \left( u_{i,j} + u_{j,i} + u_{k,j} u_{k,j} \right) \approx \delta \left( \varepsilon_{ij} + \eta_{ij} + \varepsilon_{ij}^* \right)
\]

(2-3-12)

where,

\[
\begin{align*}
e_{ij} &= \frac{1}{2} \left( U_{i,j} + U_{j,i} \right) & \eta_{ij} &= \frac{1}{2} U_{k,j} U_{k,j} & \varepsilon_{ij}^* &= \frac{1}{2} \left( U_{i,j}^* + U_{j,j}^* \right)
\end{align*}
\]

(2-3-13)

According to the linear constitutive law that related the incremental stress to the incremental strain, it can be written:

\[
S_{ij} \delta \left( \varepsilon_{ij} \right) = S_{ij} \delta \left( \varepsilon_{ij} \right) = \varepsilon_{ij}^{\prime} \epsilon_{mn}^{\prime} \delta \left( \varepsilon_{ij} \right) \approx \varepsilon_{ij}^{\prime} \epsilon_{mn}^{\prime} \delta \left( \varepsilon_{ij} \right)
\]

(2-3-14)

Substituting equation (2-3-11) to (2-3-13) into the general continuum virtual work principle (2-3-10), incremental principle will be given as a new form,
\[
\int_V^{1} C_{jmn}^{e_{mn}} \delta e_{ij}^{*} d^{*}V + \int_V^{1} \tau_{ij}^{e} \delta \left( \eta_{ij}^{e} + e_{ij}^{*} \right) d^{*}V - \int_V^{1} f_{j}^{e} \delta U_{j}^{e} d^{*}V \\
- \int_A^{1} p_{j} \delta U_{j}^{e} d^{*}A - \int_V^{1} f_{j} \delta \left( U_{j}^{e} + U_{j}^{*} \right) d^{*}V - \int_A^{1} p_{j} \delta \left( U_{j}^{e} + U_{j}^{*} \right) d^{*}A \\
= \int_V^{1} f_{j} \delta U_{j}^{e} d^{*}V + \int_A^{1} p_{j} \delta U_{j}^{e} d^{*}A - \int_V^{1} \tau_{ij}^{e} \delta e_{ij}^{*} d^{*}V
\]

\[(2-3-15)\]

In the last calculated equilibrium state \( C_1 \) initial stress, body and surface forces can be given such as:

\[
\int_V^{1} \tau_{ij}^{e} \delta \left( e_{ij}^{*} \right) d^{*}V = R = \int_V^{1} f_{j}^{e} \delta \left( u_{j}^{e} \right) d^{*}V + \int_A^{1} p_{j} \delta \left( u_{j}^{e} \right) d^{*}A
\]

\[(2-3-16)\]

In (2-3-16), \( \delta \left( u_{j}^{e} \right) \) and \( \delta \left( e_{ij}^{*} \right) \) are replaced by \( \delta U \) and \( \delta e \) respectively. For deriving the final form of the linearized virtual work principal for beam structure subjected to initial stress, (2-3-15) has to subtract from (2-3-16) while deformation-independent loads are considered only:

\[
\delta \left[ \frac{1}{2} \int_V^{1} C_{jmn}^{e_{mn}} e_{ij}^{*} d^{*}V + \int_V^{1} \tau_{ij}^{e} \eta_{ij}^{e} d^{*}V \\
+ \int_V^{1} \sigma_{ij}^{e} \delta e_{ij}^{*} d^{*}V - \int_V^{1} f_{j}^{e} U_{j}^{e} d^{*}V - \int_A^{1} p_{j} \delta U_{j}^{e} d^{*}A - R \right] = 0
\]

\[(2-3-17)\]

The above equation can be expressed in short form,

\[
\delta \left( \Pi_L + \Pi_{G1} + \Pi_{G2} + \Pi_{Ge} \right) - \partial \Pi_M - \partial W = 0
\]

\[(2-3-18)\]

\( \Pi_L \) denotes the conventional elastic strain energy. Changing in potential energy with respect to initial stresses, and the second order effects of eccentric initial loads were indicated by summation of \( \Pi_{G1} \) and \( \Pi_{G2} \) terms and \( \Pi_{Ge} \) respectively. \( \Pi_M \) is called the virtual work of inertia loads. \( W \) is the work of external loads. By replacing the displacement terms (2-3-3) and (2-3-4), strain displacement relation (2-3-13) and internal constraints (2-3-9) into (2-3-18), and using the notation that was mentioned in (2-3-6) after integrating over the cross-section, the later mentioned energy terms can be expressed one by one in details. The \( \Pi_L \) in (2-3-18) is:

\[
\Pi_L = \frac{1}{2} \int_V^{1} C_{jmn}^{e_{mn}} e_{ij}^{*} d^{*}V \\
= \frac{1}{2} \int_V^{1} \left[ E U_{s,s}^{2} + G \left( U_{s,s}^{2} + U_{s,s}^{2} \right) \right] d^{*}V \\
= \frac{1}{2} \int_0^L \left[ E A u^{2} + E I w^{2} + E I v^{2} + E I \omega^{2} + G I \alpha^{2} \\
+ A y_{cs} v^{2} + A z_{cs} w^{2} + 2 A y_{cs} u w^{*} + 2 A z_{cs} u w^{*} + 2 A y_{cs} z_{cs} \nu w^{*} \right] dx
\]

\[(2-3-19)\]
$E$ and $G$ are selected as the symbol of elastic and shear modules which are the $i^{th} C_{ijma}$ components respectively. The second order effects of eccentric initial loads shall be,

$$
\Pi_{Gi} = \Pi_{G1} + \Pi_{G2} = \int_V \tau_{ij} e_{ij}^* \, dV + \int_V \tau_{ij} e_{ij}^* \, dV
$$

$$
= \int_V \frac{1}{2} \sigma_{ij} \left( U_{i,x}^2 + U_{y,x}^2 + U_{z,x}^2 \right) + \tau_{ij} \left( U_{i,x} U_{j,x} + U_{i,z} U_{j,z} \right)
+ \int_V \frac{1}{2} \sigma_{ij} U_{i,x} U_{j,x} + \tau_{ij} \left( U_{i,x}^* U_{j,x} + U_{i,z}^* U_{j,z} \right) \right] \, dV
+ \tau_{ij} \left( U_{i,x} U_{j,x} + U_{i,y} U_{j,y} \right) \right] \, dV
$$

(2-3-20)

For having a clear form of (2-3-20), noting here defined in (2-3-6) should be adopted. Omitting the square of derivative of axial displacement $u$ in the first term leads:

$$
\Pi_{Gi} = \frac{1}{2} \int_0^L \left[ N \left( v'^2 + w'^2 \right) + P \alpha'^2 + \left( M_r - z_{CS} N \right) \left( v'^* \alpha' - v^{*'} \alpha' \right) \right.
+ \left. \left( M_r + y_{CS} N \right) \left( w'^* \alpha - w^{*'} \alpha' \right) + V \left( w'^* - 2u'^* \right) \right.
- V \left( v'^* + 2u'^* \right) \right] \, dx
$$

(2-3-21)

With returns to the fig(2-1) and distributed line load which were given at point $P$, the second order effects of eccentric initial loads in the equation (2-3-18), $\Pi_{Ge}$ will be:

$$
\Pi_{Ge} = -\int_A p \, dA
$$

$$
= -\frac{1}{2} \int_0^L \left[ p_x \left( -y_{SP} w' + z_{SP} v' \right) \alpha - p_y \left( y_{SP} \left( v'^2 + \alpha^2 \right) + z_{SP} v' w' \right) \right.
- p_z \left( z_{SP} \left( w'^2 + \alpha^2 \right) + y_{SP} v' w' \right) \right] \, dx
$$

(2-3-22)

The other part of the $\Pi_{Ge}$ is related to the initial concentrated forces whose acting at point $P$, in this sense the subsequent term should be added to (2-3-22) for having a complete initial forces potential effect.

$$
\Pi_{Ge} = -\frac{1}{2} \left[ F_x \left( y_{SP} \beta + z_{SP} \gamma \right) \alpha + F_y \left( z_{SP} \beta \gamma - y_{SP} \left( \gamma^2 + \alpha^2 \right) \right) \right.
+ F_z \left( y_{SP} \beta \gamma - z_{SP} \left( \beta^2 + \alpha^2 \right) \right)
$$

(2-3-23)

The virtual work of inertia forces, $\Pi_{Mb}$ and load increment, $\delta W$, are
\begin{align*}
\delta \Pi_M &= -\int_0^L \rho \left[ A (\ddot{u} + y_{cs} \ddot{v} + z_{cs} \ddot{w}) (\delta u + y_{cs} \delta v' + z_{cs} \delta w') \\
&\quad + A (\ddot{v} + z_{cs} \ddot{\alpha}) \delta v + A (\ddot{w} - y_{cs} \ddot{\alpha}) \delta w + A \left( z_{cs} \ddot{v} - y_{cs} \ddot{w} + \ddot{\alpha} + I_{ps}^2 \right) \delta \alpha \\
&\quad + I_{s} \delta v' + I_{w} \delta w + I_{s} \ddot{\alpha} \delta \alpha' \right] dx
\end{align*}
(2-3-24)

\begin{align*}
\delta W &= p_x (\delta u - y_{sp} \delta v' - z_{sp} \delta w' + \varphi_{p} \delta \alpha') + p_y (\delta v - z_{sp} \delta \alpha) + p_z (\delta w + y_{sp} \delta \alpha) \tag{2-3-25}
\end{align*}
Chapter 3

3.1 Introduction

Calculation of natural frequencies and critical – buckling – loads are among the basic problems in the engineering sciences, and lots of researchers have dealt with this category up to now. Different methods have been suggested for calculation. If the cross section is symmetric in two perpendicular dimensions then the bending and torsional free vibration frequencies, also their mode shapes can be separated easily (there will be no coupled frequency and shape mode). Otherwise, if the shear centre and centroid are not coincided, the frequencies and shape modes will be coupled. The following investigations are selected to be mentioned among several literatures are in this field.

This chapter’s goal as mentioned earlier, chapter one, is to investigate the coupling existence for a double axes symmetric structure which is loaded by non-uniform initial moment while the torsional warping theory is taken into account. The current part basic steps can be categorized in the following order:

1. The total potential energy of a beam subjected to initial moment and stress resultant based on warping effect should be dealt with, firstly. Motion’s equation and its boundary condition will have been derived and by the help of a numerical solution method, natural frequencies and critical buckling moments shall be given. Finally by selecting CS1, CS4 cross sections that their properties are given the appendix equations were evaluated and plotted.

![Fig 3-1 Chapter 3 process](image-url)
3.2 Total Potential Energy

In (Fig 3-2) a cross section of a prismatic beam is shown with the relevant coordinates. The axis $x$ is the beam axis and $y, z$ axes are in the section plane and coincide with principal axes. The centroid and shear centre are signed by $C$ and $S$ respectively and the external loads are applied at point $P$.

For the above mentioned problem’s sketch, the stationary condition of the total potential energy states is:

$$\delta (\Pi_L + \Pi_{Gi} + \Pi_{Ge}) - \delta \Pi_M - \delta W = 0$$

(3-2-1)

Here $\delta \Pi_L$, $\delta \Pi_{Gi}$ and $\delta \Pi_{Ge}$ are elastic strain energy with respect to initial stresses, changing in potential energy with respect to initial stresses, the second order effects of eccentric initial loads respectively. $\delta \Pi_M$ is the virtual work of inertia forces and $\delta W$ gives the virtual work of external loads\(^1\) [34] [33].

A mono symmetric cross section has been considered and $\psi(x,t)$ and $\alpha(x,t)$ where chosen for the lateral displacement and twisting, as given in (Fig 3-1). Here it has worth to mention that, subsequent equations will be valid for a double symmetric cross section also. The difference is, $y_{CS}$ and $z_{CS}$ will be zero due to the symmetry, in consequence the polar moment respect to shear center, $I_{ps}$, (2-3-8) will needed to be modified. Moreover the initial, static bending moment distribution over beam length is:

$$0 M_s(x) = -M \left(1 - x \frac{1 + \psi}{L}\right) = M \Theta(x)$$

(3-2-2)

$$\Theta(x) = \left(x \frac{1 + \psi}{L} - 1\right)$$

(3-2-3)

\(^1\) More details are in chapter 2, Equs. (2-3-18) to (2-3-23)
The shear force distribution can be expressed as,

\[ 0V_z = M \frac{1 + \psi}{L} \tag{3-2-4} \]

A point, that must be noted here is, in this text it was supposed that the initial gradient moment is acting on the strongest principle axes \((I_z < I_y)\). According to this assumption some terms were omitted while supplementary equations were deriving in chapter two. If the problem needs to be extended for any arbitrary cross section which the initial moment acts on the section without later restriction that limited the acting moment effect axes to the strongest one, two method of solutions can be discussed. First the cross section can be rotated around the centriod till the acting load fixed on the strongest principle axes. The other solution is to re derive the equations in chapter two with new format. In this way it must put an emphasis on this point that the current literature’s equation is valid only for either symmetric or mono symmetric cross sections whose carrying an initial gradient moment on their strong axes. From now on the example cross sections will be selected, respect to this note. \(\psi\) (gradient moment factor) is changing between minus one (uniform bending) and one (asymmetric bending). With returns to what have been mentioned up to now (3-2-1) can be re-expressed in(3-2-5), by using the chapter 2 equations,

\[
\delta \frac{1}{2} \int_0^L \left[ E I_z \alpha'' + EI_{x_0} \alpha'' + \left( GJ + 0M \beta_y \right) \alpha'' + 2^0M \psi \alpha \right] dx + \int_0^L \left[ A(\dot{\psi} + e \dot{\alpha}) \delta \nu + \left( A e \nu + I_{x_0} \dot{\alpha} \right) \delta \alpha + I_z \nu' \delta \nu' + I_{x_0} \alpha \delta \alpha' \right] dx \tag{3-2-5}
\]

\[-M \delta \left[ \nu' \alpha \right]_{x=0} - M \psi \delta \left[ \nu' \alpha \right]_{x=L} + R t \delta \left[ \frac{\alpha^2}{2} \right]_{x=0} = 0\]

The prime and the dot are denoting differentiation with respect to \(x\) (the beam axes) and \(t\) time variable, respectively. \(A, I_z, J, I_{x_0}\) and \(I_{x_0}\) are the area of section, principal second moment,
St. Venant torsion constant, warping constant and polar moment with respect to the shear centre, respectively\(^1\) [34] [33]. \(\beta_y\) is named Wagner’s coefficient.

### 3.3 Equations of motion

By taking the first variations with respect to \(v\) (lateral displacement) and \(\alpha\) (twist), the equations of motion and the boundary conditions can be derived from (3-2-5), the motion equation and its relevant boundary conditions will be,

\[
\begin{align*}
EI_z v'' + \left(0, M_y, \alpha'\right)'' + \rho \left(A \ddot{v} + A \ddot{e}e - I_z \ddot{v}\right) &= 0 \\
EI_o \alpha'' - GJ \alpha' - \beta_y \left(0, M_y, \alpha'\right)' + 0, M_y, v'' + \rho \left(I_p \ddot{\alpha} + A e \ddot{v} - I_o \ddot{\alpha}\right) &= 0
\end{align*}
\]

(3-3-1)

\[
\left[ \begin{array}{c} (EI_z v') \delta v \end{array} \right]_0^L = 0, \quad \left[ \begin{array}{c} (EI_o \alpha') \delta \alpha \end{array} \right]_0^L = 0
\]

\[
\left[ \begin{array}{c} EI_z v'' + \left(0, M_y, \alpha'\right)'' + \rho I_z \ddot{v} \end{array} \right]_0^L = 0
\]

\[
\left[ \begin{array}{c} -EI_o \alpha'' + \left(GJ + 0, M_y, \beta_y\right) \alpha' - 0, M_y, v'' + R t \alpha + \rho I_o \ddot{\alpha} \end{array} \right]_0^L = 0
\]

(3-3-2)

There is impossible to find an expression which can satisfy above coupled equations (3-3-1). Even if eccentricities of section and load, rotary and warping acceleration terms are omitted \((e = 0, \beta_y = 0, t = 0, \ddot{v} = 0, \ddot{\alpha} = 0)\). For this reason a numerical procedure has to be applied. In the case of simply supports at each end\(^2\). The boundary conditions are:

\[
\begin{align*}
&x = 0 \\
&v = 0 \quad \alpha = 0 \quad \ddot{v} = 0 \quad \ddot{\alpha} = 0
\end{align*}
\]

(3-3-3)

\[\text{Fig 3-4 Boundary conditions}\]

\(^1\) (2-3-8)

\(^2\) Fork like support, which prevents torsional rotation and allows free warping.
The second derivative with respect to displacement in (3-3-3) means there will be no resistant moment at the hinges ($v'' = 0$) and the beam can be warped at the end supports freely ($\alpha'' = 0$). Whereas $v = 0$ and $\alpha = 0$ express that there exists no lateral movement and torsional rotation. The main step in dealing with the free harmonic vibration is to pick up a trail solution such as:

$$v(x,t) = \sum_{i=1}^{n} F_i(t) f_i(x) \quad , \quad \alpha(x,t) = \sum_{i=1}^{m} V_i(t) g_i(x)$$

(3-3-4)

Due to Ritz-Galerkin approximation solution the geometrical boundary condition, where mentioned in (3-3-3), has to be satisfied completely by selecting apt functions in (3-3-4). A usual possible assumption for a beam with simply supports at each end as shown in (fig 3-2) can be:

$$f_i(x) = \sin\left(\frac{i\pi x}{L}\right) \quad , \quad g_i(x) = \sin\left(\frac{i\pi x}{L}\right)$$

(3-3-5)

Here it should be mentioned that for uniform bending, $\psi = -1$, the beam buckles with the half-sine wave length, while for asymmetric bending, $\psi = 1$, the lateral deflected shape is an anti-symmetrical full sine and the torsional shape $\alpha$ is a symmetrical half-sine form. The minimum number of terms must be set to $n=2, m=1$ [40].

---

**Torsional warping $\alpha \ Sin(\pi x/L)$**  
**Lateral Bending $v \ Sin(\pi x/L)$**  

![Fig 3-5 Minimum Torsional warping and Lateral Bending shape function (Uniform bending)](image)

**Torsional warping $\alpha \ Sin(\pi x/L)$**  
**Lateral Bending $v \ Sin(2\pi x/L)$**  

![Fig 3-6 Minimum Torsional warping and Lateral Bending shape function (Asymmetric bending moment)](image)
Substituting (3-3-5) in (3-2-5) will generate the following equation,

\[ \delta \frac{1}{2} \int_{0}^{L} \left( EI \sum_{i=1}^{n} \sum_{j=1}^{n} F_i(t) f_j''(x) f_j''(x) + EI_\omega \sum_{i=1}^{m} \sum_{j=1}^{m} V_i(t) V_j(t) g_i''(x) g_j''(x) \right) \, dx \]

\[ + GJ \sum_{i=1}^{m} \sum_{j=1}^{m} V_i(t) V_j(t) g_i'(x) g_j'(x) + 2^6M_y(x) \sum_{i=1}^{n} \sum_{j=1}^{m} F_i(t) f_j''(x) V_i(t) g_j(x) \right) \, dx \]

\[ + \int_{0}^{L} \rho \left( A \sum_{i=1}^{n} \ddot{F}_i(t) f_j(x) \sum_{i=1}^{n} \delta F_i(t) f_j(x) + I_{pe} \sum_{i=1}^{m} \ddot{V}_i(t) g_i(x) \sum_{j=1}^{m} \delta V_j(t) g_j(x) \right) \, dx \]

\[ + \int_{0}^{L} \rho A e \left( \sum_{i=1}^{m} \ddot{V}_i(t) g_i(x) \sum_{j=1}^{m} \delta F_j(t) f_j(x) + \sum_{i=1}^{n} \ddot{F}_i(t) f_i(x) \sum_{j=1}^{m} \delta V_j(t) g_j(x) \right) \, dx = 0 \]

(3-3-6)

By defining the numerical matrices (3-3-7),

\[ A_{ij} = \int_{0}^{L} f_i''(x) f_j''(x) \, dx \quad B_{ij} = \int_{0}^{L} g_i''(x) g_j''(x) \, dx \quad C_{ij} = \int_{0}^{L} g_i'(x) g_j'(x) \, dx \]

\[ D_{ij} = \int_{0}^{L} \Theta(x) f_i''(x) g_j(x) \, dx \quad E_{ij} = \int_{0}^{L} f_i'(x) f_j'(x) \, dx \quad H_{ij} = \int_{0}^{L} g_i'(x) g_j(x) \, dx \]

\[ E_{ij}^* = \int_{0}^{L} g_i'(x) f_j'(x) \, dx \quad H_{ij} = \int_{0}^{L} f_i'(x) g_j(x) \, dx \]

(3-3-7)

(3-3-6) is expressed in a close matrix format, where the matrices dimensions are defined by the number of terms in (3-3-4) and (3-3-5). In (3-3-7) \( \Theta(x) \) is as same as what defined erstwhile in (3-2-3).

\[ \delta \frac{1}{2} \int_{0}^{L} \left( EI \sum_{i=1}^{n} \sum_{j=1}^{n} F_i(t) F_j(t) A_{ij} + EI_\omega \sum_{i=1}^{m} \sum_{j=1}^{m} V_i(t) V_j(t) B_{ij} \right) \]

\[ + GJ \sum_{i=1}^{m} \sum_{j=1}^{m} V_i(t) V_j(t) C_{ij} + 2^6M_y(x) \sum_{i=1}^{n} \sum_{j=1}^{m} F_i(t) V_j(t) D_{ij} \right) \]

\[ + \rho A \sum_{i=1}^{n} \sum_{j=1}^{n} \ddot{F}_i(t) E_{ij} + \rho I_{pe} \sum_{i=1}^{s} \sum_{j=1}^{m} \ddot{V}_i(t) \delta V_j(t) H_{ij} \]

\[ + \rho A e \sum_{i=1}^{m} \sum_{j=1}^{m} \ddot{V}_i(t) \delta F_j(t) E_{ij} + \rho A e \sum_{i=1}^{n} \sum_{j=1}^{m} \ddot{F}_i(t) \delta V_j(t) H_{ij} = 0 \]

(3-3-8)
By collecting similar terms and doing some simplification on (3-3-8):

\[
\sum_{i=1}^{n} \sum_{j=1}^{m} \left( EI_i F_i(t) A_i \delta F_j(t) + \rho A F_i(t) E_i \delta F_j(t) \right) \\
+ \sum_{i=1}^{n} \sum_{j=1}^{m} \left( I_i \omega_i \delta V_j(t) + GJ V_j(t) C_i \delta V_j(t) + \rho I_m \dot{V}_i(t) H_0 \delta V_j(t) \right) \\
+ \sum_{i=1}^{n} \sum_{j=1}^{m} \left( M \delta F_i(t) D_i \delta V_j(t) + M F_i(t) D_i \delta V_j(t) + \rho A F_i(t) H_0 \delta \nu_j(t) \right) \\
+ \sum_{i=1}^{n} \sum_{j=1}^{m} \rho A e \ddot{V}_i(t) E_i \delta F_j(t) = 0
\]  
(3-3-9)

Let’s define two vectors whose their elements are the unknown time function coefficients in (3-3-4) as follows,

\[
F(t) = \left[ F_1(t) \ldots F_n(t) \right]^T \quad \nu(t) = \left[ V_1(t) \ldots V_m(t) \right]^T
\]  
(3-3-10)

The last defined coefficient vectors (3-3-10), help to rearrange (3-3-9), in a compressed shape without needing to use the summation symbols anymore.

\[
EI_i F_i^T A_i \delta F +EI_i \nu_i V_j^T B_i \delta V + GJ V_j^T C_i \delta V + M \delta F_i D_i \delta V + M F_i^T D_i \delta V \\
+ \rho A F_i^T E_i \delta F + \rho A e \dot{V}_i(t) H_0 \delta V + \rho A e \dot{V}_i^T H_0 \delta V = 0
\]  
(3-3-11)

Incorporating (3-3-10) unknown time function coefficients makes a new vector that its components are given by (3-3-12). Appyling the increment symbol on (3-3-12), leads to (3-3-13).

\[
X = \left[ F_1(t) \ldots F_n(t) \nu_1(t) \ldots \nu_m(t) \right]^T = \left[ x_1 \ldots x_{n+m} \right]^T
\]  
(3-3-12)

\[
\delta X = \left[ \delta F_1(t) \ldots \delta F_n(t) \delta \nu_1(t) \ldots \delta \nu_m(t) \right]^T = \left[ \delta x_1 \ldots \delta x_{n+m} \right]^T
\]  
(3-3-13)
Substituting (3-3-12) and (3-3-13) in (3-3-11), will give the matrix form of the motion equation. This matrix form shall be used later for frequency and stability analysis in the current and further chapters (four, five and six).

\[
\begin{bmatrix}
\delta F_1(t) & \ldots & \delta F_n(t) & \delta V_1(t) & \ldots & \delta V_m(t)
\end{bmatrix}
\begin{bmatrix}
\rho AE & \rho AeH \\
\rho AE & \rho l_{ps}H
\end{bmatrix}
\begin{bmatrix}
\ddot{F}_1(t) \\
\ddot{F}_n(t) \\
\ddot{V}_1(t) \\
\ddot{V}_m(t)
\end{bmatrix}
\]

\[
+ M \begin{bmatrix}
\delta F_1(t) & \ldots & \delta F_n(t) & \delta V_1(t) & \ldots & \delta V_m(t)
\end{bmatrix}
\begin{bmatrix}
EI_A & 0 & 0 \\
0 & GJ_C + EI_{aB} & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{F}_1(t) \\
\dot{F}_n(t) \\
\dot{V}_1(t) \\
\dot{V}_m(t)
\end{bmatrix}
= 0
\]

Or by defining, three new numerical matrices to make it as simple as possible, the motion equation in the absence of the damping is,

\[
M \ddot{X} + \left( K_c + MS \right) X = 0
\]

(3-3-15)

\( M, S \) and \( K_c \) are mass, stability and stiffness matrices in equation of motion, were defined such as,

\[
M = \begin{bmatrix}
\rho AE & \rho AeH \\
\rho AE & \rho l_{ps}H
\end{bmatrix}
, \quad S = \begin{bmatrix}
0 & D \\
D^T & 0
\end{bmatrix}
, \quad K_c = \begin{bmatrix}
EI_A & 0 \\
0 & EI_{aB} + GJ_C
\end{bmatrix}
\]

(3-3-16)

Solution of (3-3-15) can be found in the modal Eigen-value problem section of any vibration analysis standard literature [e.g. [41]]. In this way (3-3-15) should be rearranged as follow, and the final form expresses such as,(3-3-16)
\[
\left(-\omega^2 M + K + MS\right)X = 0
\]  
(3-3-17)

### 3.4 Natural frequencies and Critical Buckling Moment

As it discussed in the last subsection, \( n \) and \( m \) in (3-3-4) are needed to be set to minimal numbers for the first solution approximation [40], whose are 2 and 1 respectively. In this way the motion equation (3-3-15) dimension is three by three.

\[
\begin{bmatrix}
EI \frac{\pi^4}{L^4} - \omega^2 \rho A & 0 & -\rho Ae\omega^2 - \frac{M(\psi - 1)}{2} \left(\frac{\pi}{2}\right)^2 \\
16EI \frac{\pi^4}{L^4} - \omega^2 \rho A & \frac{64M}{9L^2}(\psi + 1) & 0 \\
\text{symmetric} & \frac{\pi^2}{L^2} \left(\frac{EI_w}{L^2} + GJ\right) - \omega^2 \rho I_{ps} & 0
\end{bmatrix}
\begin{bmatrix}
F_1 \\
F_2 \\
V_1
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]  
(3-4-1)

For obtaining the natural frequency the moment (external load) has to be set to zero, in other word the free vibration is needed to be taking into account. The next step is related to find the critical buckling moment. Here the process is as same as finding the frequencies but with this difference that the frequency has to be zero. In general both problems shall lead to an Eigen-value problem. As it has been known for this type of problems the determinant must be zero for having a non trivial solution. In the current thesis as far as symmetric cross sections were taken into account for illustrating the graphs, The geometrical eccentricity equals to zero (\( e=0 \)).

#### 3.4.1 Natural frequencies

Non-trivial solution for (3-4-1) Eigen-value problem, while the geometrical eccentricity equals to zero (Fig 3-2) for symmetric cross sections, exists if and only if (3-4-1) determinant set to be zero (3-4-2). By assuming that there is no moment load on the structure, \( M=0 \) (free vibration), and setting the determinant to zero, the first and second natural bending frequencies and first natural torsional frequency can be obtained.

\[
\begin{vmatrix}
EI \frac{\pi^4}{L^4} - \omega^2 \rho A & 0 & 0 \\
0 & 16EI \frac{\pi^4}{L^4} - \omega^2 \rho A & 0 \\
0 & 0 & \frac{\pi^2}{L^2} \left(\frac{EI_w}{L^2} + GJ\right) - \omega^2 \rho I_{ps}
\end{vmatrix}
= 0
\]  
(3-4-2)
Here, $\omega_{b1}$, $\omega_{b2}$ and $\omega_{t1}$ are the first and second natural lateral bending frequencies and first natural torsional frequency respectively. Ratio of first natural lateral bending frequency over first natural torsional frequency in comparison with one, determines the behavior of the structure i.e. dominantly bending or torsional. If the ratio being greater than one, the structure frequency is dominantly torsional otherwise, dominantly bending will occur.

As far as assumed that the cross section area and material properties are not changing among the length, the only factor which put effect on first natural lateral bending frequency over first natural torsional frequency ratio is the length of structure. In other word if the length of beam being lower than a specified length, $L_c$ (3-4-5), dominantly torsion frequency should be dealt with. While having longer beams, leads dominantly bending one.

\[
\frac{\omega^2_{b1}}{\omega^2_{t1}} = \left(\frac{\pi}{L}\right)^4 \frac{EI_z}{\rho A} \left[1 + \left(\frac{\pi}{L}\right)^2 \frac{EI_{\omega}}{GJ}\right] = 1
\]  

(3-4-4)

\[
L_c = \pi \sqrt{\frac{GJ}{E} \left(\frac{I_{ps} I_z}{A} - I_{\omega}\right)}
\]  

(3-4-5)

Fig (3-7) shows that for every length, CS1 cross section has dominantly bending mode. The first natural bending frequency over first natural torsional frequency ratio for it is lower than one for any length. In Fig (3-8) the ratio of first natural lateral bending frequency over first natural torsional one for CS4 cross sections is given.

![Graph showing the ratio of first natural lateral bending frequency over first torsional one for CS1 cross section.](image)
As it is illustrated in (Fig 3-8) two different regions, dominantly torsion and bending can be indicated. Confluence points of \( \frac{\omega_{bl1}}{\omega_{t1}} \) curve and one, can be calculated by using (3-4-5), which will be called \( L_c \).

Using (3-4-3) helps, to express the equation of motion (3-4-1) as follow, (3-4-6). In the present research the geometrical eccentricity substituted with zero \( (e=0) \) because of selecting symmetric cross sections.

\[
\begin{bmatrix}
\rho A \left( \omega_{bl1}^2 - \omega^2 \right) & 0 & -\frac{M (\psi - 1) \left( \frac{\pi}{L} \right)}{2} \\
\rho A \left( 16 \omega_{bl1}^2 - \omega^2 \right) & \frac{64 M}{9 L^2} (\psi + 1) & \rho I_{ps} \left( \omega_{bl1}^2 - \omega^2 \right)
\end{bmatrix}
\begin{bmatrix}
F_1 \\
F_2 \\
V_1
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(3-4-6)

3.4.2 Critical buckling moment

The next step is to find the critical buckling moment of the structure. In this dissertation as shown in (fig 3-2), a beam which is loaded by a gradient moment is considered. For this propose the circular frequency must be omitted, \( \omega = 0 \).

\[
\begin{bmatrix}
\rho A \omega_{bl1}^2 & 0 & -\frac{M (\psi - 1) \left( \frac{\pi}{L} \right)}{2} \\
16 \rho A \omega_{bl1}^2 & \frac{64 M}{9 L^2} (\psi + 1) & \rho I_{ps} \omega_{bl1}^2
\end{bmatrix}
\begin{bmatrix}
F_1 \\
F_2 \\
V_1
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(3-4-7)
The critical static buckling moments are the Eigen-values of the (3-4-1). It depends on the \( \psi \) (gradient moment factor) generally. With this assumption that, the currently defined beam structure is subjected by the uniform moment load \( (\psi = -1) \), the critical moment load is expressed,

\[
M_{cr0}^2 = \rho^2 AI_{ps} \left( \frac{L}{\pi} \right)^4 \alpha_{n1}^2 \omega_{n1}^2
\]  
(3-4-8)

Replacing first bending and torsional frequency from (3-4-3) in to (3-4-8) represents that density, area and polar moment respect to shear center have no direct effect on the critical moment load.

\[
M_{cr0}^2 = EI_\tau \left( \frac{\pi}{L} \right)^2 GJ \left[ 1 + \left( \frac{\pi}{L} \right)^2 \frac{EI_\omega}{GJ} \right]
\]  
(3-4-9)

For finding the end moment critical value (3-4-10), the (3-4-7) characteristic equation’s determinant, set to zero and the equation solution for moment was represented by \( M_{cr} \). Now, the critical buckling moment can be expressed easily by the help of (3-4-8). While the (3-4-1) determinant supposed to be zero in the absence of circular frequency, \( \omega \) (there will be no vibration). The general formula for the critical moment shall be,

\[
M_{cr}^2 = \frac{\rho^2 AI_{ps} \alpha_{n1}^2 \omega_{n1}^2}{\left( \frac{1 - \psi}{2} \right)^2 \left( \frac{\pi}{L} \right)^4 \left( \frac{16(1 + \psi)}{9L^2} \right)^2 + \frac{16(1 + \psi)}{9\pi^2}} = \frac{\rho^2 AI_{ps} \left( \frac{L}{\pi} \right)^4 \alpha_{n1}^2 \omega_{n1}^2}{\left( \frac{1 - \psi}{2} \right)^2 + \left( \frac{16(1 + \psi)}{9\pi^2} \right)^2}
\]  
(3-4-10)

\[
M_{cr}^2 = C^2(\psi) M_{cr0}^2
\]  
(3-4-11)

Here the \( M_{cr} \) represents the end moment critical value (Fig 3-1). \( M_{cr0} \) and \( C(\psi) \) are critical buckling value of the uniform bending moment \( (\psi = -1) \) and the moment gradient coefficient respectively. While \( n \) and \( m \) in (3-3-4) set to be 2 and 1 respectively [40] the moment gradient coefficient is represented by:

\[
C^2(\psi) = \frac{1}{\left( \frac{1 - \psi}{2} \right)^2 + \left( \frac{16(1 + \psi)}{9\pi^2} \right)^2}
\]  
(3-4-12)

As the only effective term in moment gradient coefficient \( C(\psi) \) (3-4-12), is the gradient factor, \( \psi \), there will no difference between dominantly bending mode and dominantly torsional one. for finding \( (M_{cr}/M_{cr0}) \) ratio, different sets of Ritz terms numbers were selected\(^1\). Substituting them in (3-4-1) and assume that angular velocity equals to zero \( (\omega = 0) \) it will lead to \((n+m)\) by \((n+m)\) Eigen-value problem. By setting the determinant to zero a moment gradient factor, \( \psi \), dependent

\(^1\) These tables are given in the appendix.
characteristic equation for critical moment load will have been obtained. Using the latterly found equation helps to fill out the tables with the approximated value of \((M_{cr}/M_{cr0})\). It has to be noted that \(M_{cr0}\) here is calculating by replacing moment gradient factor, \(\psi\), with minus one as a uniform moment load characteristic (\(\psi = -1\)). These tables were proven that increasing Ritz terms number has no significant effect on the moment gradient coefficient. In 2001, Pi and Bradford published a paper where they suggested three different approximations for the moment gradient coefficient [40]. The approximated value for different amount of gradient moment factor, \(\psi\), basis on their job are given in table (3-1).

<table>
<thead>
<tr>
<th>(\psi)</th>
<th>-1.0</th>
<th>-0.9</th>
<th>-0.7</th>
<th>-0.6</th>
<th>-0.5</th>
<th>-0.3</th>
<th>-0.2</th>
<th>-0.1</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equ. 40</td>
<td>1.05</td>
<td>1.17</td>
<td>1.24</td>
<td>1.32</td>
<td>1.51</td>
<td>1.62</td>
<td>1.78</td>
<td>1.99</td>
<td>2.20</td>
<td>2.37</td>
<td>2.72</td>
<td>2.85</td>
<td>2.93</td>
<td>2.89</td>
<td>2.78</td>
<td>1.05</td>
</tr>
<tr>
<td>Equ. 43</td>
<td>1.05</td>
<td>1.18</td>
<td>1.25</td>
<td>1.33</td>
<td>1.53</td>
<td>1.65</td>
<td>1.85</td>
<td>2.11</td>
<td>2.41</td>
<td>2.71</td>
<td>3.52</td>
<td>4.06</td>
<td>4.67</td>
<td>5.61</td>
<td>5.55</td>
<td>1.05</td>
</tr>
<tr>
<td>Equ. 44</td>
<td>1.05</td>
<td>1.16</td>
<td>1.23</td>
<td>1.30</td>
<td>1.46</td>
<td>1.55</td>
<td>1.68</td>
<td>1.82</td>
<td>1.97</td>
<td>2.09</td>
<td>2.35</td>
<td>2.49</td>
<td>2.63</td>
<td>2.94</td>
<td>3.10</td>
<td>1.05</td>
</tr>
</tbody>
</table>

Table 3-1 Moment gradient coefficient [40] suggested equations

A survey of different expressions derived for \(C\) is presented in [40], including the formulas were given in Eurocode 3, equations 40, 43 and 44. From finite element comparisons Mohri at al. in [42] concludes that (3-4-10) is the best approximation. Increasing the \(n\) and \(m\) numbers of trial functions in (3-3-5) more accurate solution is given, but the alternatives of deriving a simple closed form like (3-4-10) were vanished. In (Fig 3-9) a comparison is given of different grade of approximations has been illustrated. With respect to (Fig 3-10), enlarging the Ritz terms numbers in (3-3-4) effects on moment gradient coefficient \(C(\psi)\) (3-4-12) is not too much and it can be approximated with the given formula in Bioshiop et.al [43] . Selecting the minimal numbers in(3-3-4), \(n=2\) and \(m=1\), is the correct and optimal choices.

![Fig 3-9 Variation of moment gradient coefficient C(\(\psi\)) vs. moment gradient factor](image-url)
In the appendix the Ritz terms number effect are given for several cross sections which elaborates that moment gradient coefficient $C(\psi)$ changes versus moment gradient factor, $\psi$, is independent from cross section type and how many terms are taken into account in (3-3-4).

### 3.5 Forced Vibration

The assumed beam forced vibration subjected to initial gradient moment can be investigated. For this reason, a steady state moment load factor, $\mu$, has to be introduced,

$$M = \mu M_{cr} C(\psi)$$

(3-4-13)

Substituting (3-4-13) into (3-4-6) and setting the determinant to zero for obtaining a non-trivial solution will have been leaded to next third order characteristic equation,

$$\begin{bmatrix}
\rho A (\omega_{b1}^2 - \omega^2) & 0 & -\frac{\mu M_{cr} C(\psi - 1)}{L} \left(\frac{\pi}{L}\right)^2 \\
0 & \rho A (16\omega_{b1}^2 - \omega^2) & \frac{64\mu M_{cr} C(\psi + 1)}{9L^2} \\
\text{symmetric} & \rho l_{ps} (\omega_1^2 - \omega^2) & 0
\end{bmatrix}
\begin{bmatrix}
F_1 \\
F_2 \\
V_1
\end{bmatrix}
= 0$$

(3-4-14)

$$\begin{align*}
\left(\omega_{b1}^2 - \omega^2\right) & \left(16\omega_{b1}^2 - \omega^2\right) \left(\omega_1^2 - \omega^2\right) \\
-\mu^2 C^2 & \omega_{b1}^2 \omega_1^2 \left(\omega_{b1}^2 - \omega^2\right) \left(\frac{64(1 + \psi)}{9\pi^2}\right)^2 + \left(16\omega_{b1}^2 - \omega^2\right) \left(\frac{\psi - 1}{2}\right)^2 = 0
\end{align*}$$

(3-4-15)
All three roots of (3-4-15) are the function of steady state moment load factor, $\mu$, and moment gradient parameter, $\psi$. The vibration frequency of the structure is two dimensions function $\omega(\mu, \psi)$. The character of the smallest root of (3-4-15), initial bending moment and gradient relation depends on the first free mode of the beam, i.e. different if it is flexural or torsional (Figs (3-7) and (3-8)). In following steps as the first limited case dominantly bending mode is going to be discussed. Two moment loading boundary conditions will be considered separately (uniform and asymmetric initial moment). The investigation process will have been followed by taking dominantly torsional mode to account. The same procedure shall be applied in this case as well.

3.5.1 Dominantly Bending mode $\omega_{b1} < \omega_{t1}$

Before going thru the details of finding the roots of (3-4-15) which will be discuss later. By setting $(\psi = -1)$ for uniform bending moment load in (3-4-15),

$$
(\omega_{b1}^2 - \omega^2)(16\omega_{b1}^2 - \omega^2)(\omega_{t1}^2 - \omega^2) - (16\omega_{b1}^2 - \omega^2)\mu^2 C^2 \bigg|_{\psi \rightarrow -1} \omega_{b1}^2 \omega_{t1}^2 = 0
$$

(3-4-16)

The equation can be written as:

$$
(16\omega_{b1}^2 - \omega^2)\left[(\omega_{b1}^2 - \omega^2)(\omega_{t1}^2 - \omega^2) - \mu^2 C^2 \bigg|_{\psi \rightarrow -1} \omega_{b1}^2 \omega_{t1}^2\right] = 0
$$

(3-4-17)

By dividing (3-4-17) over $\omega_{b1}^6$, the non dimension equation shall be as follows$^1$. From now on the ratio of natural frequency square over the first bending one square will be designated by $Z_1$.

$$
(16 - Z_1) \left[1 - Z_1\right] \left(\frac{\omega_{t1}^2}{\omega_{b1}^2} - Z_1\right) - \mu^2 C^2 \bigg|_{\psi \rightarrow -1} \frac{\omega_{t1}^2}{\omega_{b1}^2} = 0
$$

(3-4-18)

As it can be seen in (3-4-18), one of the roots is constant number and two other ones are the second order equation roots that can be expressed as follows:

$$
\begin{align*}
Z_1 & = 16 \\
Z_1 & = \frac{1}{2} \left(1 + \frac{\omega_{t1}^2}{\omega_{b1}^2}\right) \pm \sqrt{\left(1 + \frac{\omega_{t1}^2}{\omega_{b1}^2}\right)^2 - 4\frac{\omega_{t1}^2}{\omega_{b1}^2} \left(1 - \mu^2 C^2 \bigg|_{\psi \rightarrow -1}\right)} \\
\end{align*}
$$

(3-4-19)

In next two illustrations the different roots of (3-4-19), were designated for different cross sections. In Figs (3-10 and 11) the natural frequency changing versus steady state factor are given for CS1 and CS4 cross sections. In these graphs the structure is loaded uniformly $(\psi = -1)$. The curves characters represent the coupling between bending and torsional frequency. When $\mu$ (steady state factor) equals to zero, it means the beam is subjected by no external moment (free

$^1$ This case happening for the structures, which their first free mode is bending one ($\omega_{b1} < \omega_{t1}$)
vibration). The structure frequency is the first bending one. By increasing the steady state factor up to one the coupled frequency goes to be zero. At this point the buckling phenomenon will be started (it is called the sound of structure).

![Graph](image1)

Fig 3-11 Natural frequency changing vs. steady state moment load factor, \( \psi = -1 \) (uniform) (bending mode) (CS1)

The other limit case is asymmetric moment loading. The natural frequency finding process is as same as what has been mentioned. Here the moment gradient factor needs to be set to one \( (\psi = 1) \). In this way (3-4-15) is changing to,

\[
(\omega_{k1}^2 - \omega^2) (16 \omega_{k1}^2 - \omega^2) (\omega_{k1}^2 - \omega^2) - \mu^2 C^2 (\omega_{k1}^2, \omega_{k1}^2 (\omega_{k1}^2 - \omega^2) \left( \frac{128}{9 \pi^2} \right)^2 = 0
\]

After simplification, as same as what has been done on(3-4-16), the non-dimensional equation for asymmetric loading is,

\[
(1 - Z_i) \left( 16 - Z_i \right) \left( \frac{\omega_{k1}^2}{\omega_{k1}^2} - Z_i \right) - \mu^2 C^2 (\omega_{k1}^2, \omega_{k1}^2 (\omega_{k1}^2 - \omega^2) \left( \frac{128}{9 \pi^2} \right)^2 = 0
\]

Three different, steady state moment load factor dependent roots of (3-4-21) are:
Here (3-4-22) equations can be plotted and compared with one which is one of the (3-4-21) roots to find the minimal frequency as the natural one while the structure is loaded by the asymmetric moment. Figs (3-13) and (3-14) give this concept for two cross section groups respectively.

In figures (3-13) and (3-14) solid line represents the first natural bending frequency. By enlarging the steady state moment load factor from zero (no moment) to one (fully loaded), the characteristic of the natural frequency is changing. Figs (3-13) and (3-14) show that the minimal frequency equals to the first lateral bending frequency up to a specified moment load factor, \( \mu \), depend on cross section type and after that it goes down on a parabolic path until the frequency being zero at beginning of the buckling behavior of the structure (\( \mu = 1 \)). Specified moment load factor for a beam element where the frequency coupling will happen after that while asymmetric moment is acting at the end of the beams can be determined by the help of (3-4-22).
\[ \mu_c = \sqrt{\frac{15}{16} \left( 1 - \frac{\omega_{b1}^2}{\omega_{t1}^2} \right)} \]  
\[ (3-4-23) \]

![Graph 1](Fig 3- 15 Critical steady state moment load factor, \( \psi = 1 \) (asymmetric) (bending mode) (CS1))

![Graph 2](Fig 3- 16 Critical steady state moment load factor, \( \psi = 1 \) (asymmetric) (bending mode) (CS4))

In general, if assumed that the natural lateral bending frequency is smaller than the natural torsional frequency \( \omega_{b1} < \omega_{t1} \), (3-4-15) in non-dimensional format shall be given as:

\[
\left( 1 - \left( \frac{\omega_{\min}}{\omega_{b1}} \right)^2 \right)^2 \left( 16 - \left( \frac{\omega_{t1}}{\omega_{b1}} \right)^2 \right) \left( \frac{\omega_{b1}^2}{\omega_{t1}^2} \right) - \left( \frac{\omega_{\min}}{\omega_{b1}} \right)^2 
\]

\[ -\mu^2 C^2 \left( \frac{\omega_{b1}}{\omega_{t1}} \right)^2 \left[ 1 - \left( \frac{\omega_{\min}}{\omega_{b1}} \right)^2 \left( \frac{64 (1 + \psi)}{9 \pi^2} \right)^2 + \left[ 16 - \left( \frac{\omega_{\min}}{\omega_{b1}} \right)^2 \left( \frac{\psi - 1}{2} \right)^2 \right] \right] = 0 \]  
\[ (3-4-24) \]

Modifying (3-4-24) by using the later introduced non dimensional parameter, \( Z_1 \), leads to (3-4-25).
\[(1 - Z_1)(16 - Z_1) \left( \frac{\omega_1}{\omega_{b1}} \right)^2 - Z_1 \right)
- \mu^2 C^2 \left( \frac{\omega_1}{\omega_{b1}} \right)^2 \left[ (1 - Z_1) \left( \frac{64(1 + \psi)}{9 \pi^2} \right)^2 + (16 - Z_1) \left( \frac{\psi - 1}{2} \right)^2 \right] = 0 \quad (3-4-25)\]

As it is clear in (3-4-25), the non dimensional variable, \(Z_1\), depends on the gradient moment factor and steady state moment load factor. After the natural frequency was found (3-4-25), the modal amplitudes can be calculated in the same way. For demonstrating the coupling of flexural-torsional mode shapes, it is convenient to introduce the modal mixing factor \(\eta_1\), which is the ratio of first bending and torsional amplitudes. By using the first row of (3-4-14) the modal mixing factor shall be (3-4-26). Here \(i_{ps}\) equals to \(\sqrt{I_{ps}/A}\). Using the same notation was used in (3-4-25), \(Z_1\), (3-4-26) will be modified to non-dimension new format (3-4-27).

\[
\eta_1 = \frac{F_1}{i_{ps} V_1} = \frac{\mu C (\psi - 1)}{2} \frac{\omega_1}{\omega_{b1}^2} \frac{\omega_1}{\omega_{b1}^2 - \omega_{min}^2}
\]

\[
\eta_1 = \frac{\mu C (\psi - 1)}{2} \frac{\omega_1}{1 - \omega_{b1}^2/\omega_{b1}^2} = \frac{\mu C (\psi - 1)}{2(1 - Z_1)} \frac{\omega_1}{\omega_{b1}^2}
\]

\[\eta_1 = \frac{\mu C (\psi - 1)}{2} \frac{\omega_1}{1 - \omega_{b1}^2/\omega_{b1}^2} = \frac{\mu C (\psi - 1)}{2(1 - Z_1)} \frac{\omega_1}{\omega_{b1}^2}
\]

3.5.2 Dominantly torsional mode \(\omega_{b1} < \omega_{b1}\)

As another case, when the natural torsional frequency being smaller than natural lateral bending frequency \(\omega_{b1} < \omega_{b1}\), the characteristic equation (3-4-15) will be needed to be given in new format as like what has been done in (3-4-24). Before expressing the general equation, two boundary limitations in gradient moment load must be considered. Supposing the uniform loading (3-4-15) will be (3-4-17) and by dividing it over \(\omega_{b1}^2\), the non dimension equation shall be as follows. From now on the ratio of natural frequency square over the first torsional one square will be designated by \(Z_2\).

\[\left( 16 \frac{\omega_{b1}^2}{\omega_{b1}^2} - Z_2 \right) \left[ \left( \frac{\omega_{b1}^2}{\omega_{b1}^2} - Z_2 \right) \left( 1 - Z_2 \right) - \mu^2 C^2 \bigg|_{\psi=-1} \frac{\omega_{b1}^2}{\omega_{b1}^2} \right] = 0 \quad (3-4-28)\]
The later equation, \((3-4-28)\) roots are;

\[
\begin{align*}
Z_2 &= 16 \frac{\omega_{b1}^2}{\omega_{t1}^3} \\
Z_2 &= \frac{1}{2} \left( 1 + \frac{\omega_{b1}^2}{\omega_{t1}^3} \right) \pm \sqrt{\left( 1 + \frac{\omega_{b1}^2}{\omega_{t1}^3} \right)^2 - 4 \frac{\omega_{b1}^2}{\omega_{t1}^3} \left( 1 - \mu^2 \right)_{\psi=1}}
\end{align*}
\]

\[(3-4-29)\]

The minimum root which is nominated as the non dimensional frequency is illustrated in Fig (3-17). While the asymmetric moment is considered, \((3-4-15)\) reformulated and after simplification, the non-dimensional equation for asymmetric loading is,

\[
\left( \frac{\omega_{b1}^2}{\omega_{t1}^3} - Z_2 \right) \left[ 16 \frac{\omega_{b1}^2}{\omega_{t1}^3} - Z_2 \left( 1 - Z_2 \right) - \mu^2 \left( 128 \frac{\omega_{b1}^2}{\omega_{t1}^3} \right) \right] = 0
\]

\[(3-4-30)\]

There exist three roots for the third order equation \((3-4-30)\) as follows:

\[
\begin{align*}
Z_2 &= \frac{\omega_{b1}^2}{\omega_{t1}^3} \\
Z_2 &= \frac{1}{2} \left( 1 + 16 \frac{\omega_{b1}^2}{\omega_{t1}^3} \right) \pm \sqrt{1 + 16 \frac{\omega_{b1}^2}{\omega_{t1}^3} - 4 \frac{\omega_{b1}^2}{\omega_{t1}^3} \left( 1 - \mu^2 \right)_{\psi=1}}
\end{align*}
\]

\[(3-4-31)\]

Fig 3-17 Natural frequency changing vs. steady state moment load factor, \(\psi=-1\) (uniform) (torsional mode) (CS4)

The natural frequency for asymmetric load, when \(\omega_{t1} < \omega_{b1}\) can be plotted in Fig (3-18).
From later graphs, Figs (3-17 and 18) it can be caught out that, the coupling in frequency happens for both limitation moment loading (uniform and asymmetric one). The general equation for the dominantly torsional case ($\omega_{t1} < \omega_{b1}$) is (3-4-32):

$$\left(\frac{\omega_{b1}}{\omega_{1}}\right)^2 - \left(\frac{\omega_{\text{min}}}{\omega_{1}}\right)^2 \left(16 \left(\frac{\omega_{b1}}{\omega_{1}}\right)^2 - \left(\frac{\omega_{\text{min}}}{\omega_{1}}\right)^2 \right) \left(1 - \left(\frac{\omega_{\text{min}}}{\omega_{1}}\right)^2 \right)$$

$$-\mu^2 C^2 \left(\frac{\omega_{b1}}{\omega_{1}}\right)^2 \left(\frac{\omega_{b1}}{\omega_{1}}\right)^2 \left(\frac{64 \left(1 + \psi\right)}{9\pi^2}\right)^2$$

$$-\mu^2 C^2 \left(\frac{\omega_{b1}}{\omega_{1}}\right)^2 \left(\frac{\omega_{b1}}{\omega_{1}}\right)^2 \left(\frac{\psi - 1}{2}\right)^2 = 0$$

By using a dimension-less parameter as like what has been introduced in periviously, $Z_2 = \left(\frac{\omega_{\text{min}}}{\omega_{1}}\right)^2$, (3-4-32) will be modified as follows,

$$\left(\frac{\omega_{b1}}{\omega_{1}}\right)^2 - Z_2 \left(16 \left(\frac{\omega_{b1}}{\omega_{1}}\right)^2 - Z_2 \left(1 - Z_2 \right)\right)$$

$$-\mu^2 C^2 \left(\frac{\omega_{b1}}{\omega_{1}}\right)^2 \left(\frac{\omega_{b1}}{\omega_{1}}\right)^2 \left(\frac{64 \left(1 + \psi\right)}{9\pi^2}\right)^2$$

$$-\mu^2 C^2 \left(\frac{\omega_{b1}}{\omega_{1}}\right)^2 \left(\frac{\omega_{b1}}{\omega_{1}}\right)^2 \left(\frac{\psi - 1}{2}\right)^2 = 0$$

Calculating the modal mixing factor for this situation where ($\omega_{t1} < \omega_{b1}$) is quite similar to the earlier mentioned case ($\omega_{b1} < \omega_{t1}$), but with a small modification that should be done on (3-4-24). $Z_1$ has to be replaced with $Z_2$, because now the minimum natural frequency being selected for every $\psi$ and $\mu$ from (3-4-33). The non dimensional mixed mode factor for dominantly torsional
mode can be represented as (3-4-34). The process of finding the mixed mode factor is as same as what has done for (3-4-27). The first row of (3-4-14) was considered. The subsequent illustration represents the changing frequency versus to moment gradient factor (ψ) and steady state moment load factor (μ). Related mixed mode shape for each frequency is also plotted.

\[ \eta_2 = \frac{\mu C (\psi - 1)}{2} \left( \frac{\omega_{b1}}{\omega_{b1}} - \frac{\omega_{min}}{\omega_{b1}} \right) = \frac{\mu C (\psi - 1)}{2} \left( \frac{\omega_{b1}}{\omega_{b1}} \right)^2 - Z_2 \]

(3-4-34)

On figures (3-19-22) dominantly bending mode effect on natural frequency changes versus moment load parameters, ψ and μ, are given for the assumed beam in this dissertation. Figures (3-13 and 24) are shown the frequency and mixed mode distribution for a case when dominantly torsional mode is taken into account only. In the appendix, frequency and mixed mode distribution are given for some other cross sections.

**Fig 3- 19 Change of frequency vs. moment load parameters, (bending mode) (CS1)**

**Fig 3- 20 Change of mixed mode ratio vs. moment load parameters, (bending mode) (CS1)**
Fig 3-21 Change of frequency vs. moment load parameters, (bending mode) (CS4)

Fig 3-22 Change of mixed mode ratio vs. moment load parameters, (bending mode) (CS4)

Fig 3-23 Change of frequency vs. moment load parameters, (torsional mode) (CS4)
3.6 Summarize

In the present part, natural frequencies and critical buckling moment of an assumed beam structure were determined, firstly. By the help of these two amounts the forced vibration of the beam was investigated. According to the equations that were derived and plotted graphs, the main points of this chapter can be reviewed as follow:

- With returns to fig (3-7) and (3-8), it could be caught out, by selecting the length of a beam structure which is loaded by an initial gradient moment, while the cross section properties cannot be changed, dominant mode of beam can be, torsional or bending one. As shown in fig (3-8) the dominantly torsional zone occurs for beams where their the length is shorter than a specified number, \( L_c \) (3-4-5). Meanwhile there are some cross sections which have only one mode as like what was given in fig (3-7).
- With returns to fig (3-10), it can be deducted that increasing the Ritz terms number (\( n \) and \( m \)) in (3-3-4) has no magnificent effect on the moment gradient coefficient \( C(\psi) \). The equation which was derived for \( C(\psi) \) (3-4-12), is very close to equation (40) in [40].
- By considering the dominant bending mode of the structure (figures (3-13) to (3-16)), when uniform moment (\( \psi = -1 \)) applying on beam the frequency is coupled meanwhile in asymmetric moment loading up to certain value of steady state moment factor (\( \mu \)) first bending frequency is the main frequency of forced vibration analysis of the beam. By enlarging steady state moment factor the frequency goes down on parabolic curve (Figs (3-15) and (3-16)) which represent the coupling of frequencies. The curve equation is given by(3-4-22).
- Bearing dominantly torsional mode in mind (figures (3-17) and (3-18)), causes the frequency of the beam being coupled for any values of steady state moment load factor, \( \mu \), and moment gradient parameter, \( \psi \).
Chapter 4

4.1 Introduction

The coupled vibration and critical buckling moment of a beam structure have been found and discussed up to now. This chapter as it was mentioned earlier in chapter two goes to investigate the damping effect on the coupled vibration and critical buckling moment as well. The beam with symmetric cross section subjected to linearly varying steady state initial moment loads.

What was done in later chapter, for obtaining the frequency of a beam which was subjected to initial gradient moment based on an idealistic assumption, no damping effect on the structure. In real engineering problems the damping effect exists and cannot be neglected almost always. A model is usually suggested for taking the damping into account. Rayleigh damping model for solution process will have been selected in current research. This model has a benefit, which did not change the solution procedure and the previous equations were derived in last chapter for un-damped forced vibration of the beam will be kept. The damping effect is needed to be added to them. Two cross sections were considered in last part, will be taken into account here once more for making the plots. In these illustrations increasing damping coefficient effect on natural frequency, critical buckling moment and forced vibration frequency and relevant shape modes will be presented.

Fig 4-1 Chapter 4 process
4.2 Natural damped frequencies and Critical Buckling Moment

The motions equation for a damped system, in this dissertation, a beam with simply support boundary conditions at the ends which is subjected to an initial gradient moment (Fig 3-2) is moderately similar to the un-damped system. In this way the procedure was used in chapter three for governing (3-3-15) is also valid here.

Without losing the generality of the solution, mentioned in previous subsection, Rayleigh damping in the form of can be used.

\[ D_n = 2 \xi M \]  

(4-2-1)

Here \( D_n \) is the damping matrix and \( \xi \) represents the damping coefficient. The \( \xi \) dimension is over second. Here it should be mentioned that, damping coefficient can be selected in an interval. By defining \( \zeta \) as the ratio of damping\(^1\) [41],

\[ \xi = \frac{\text{damping}}{\text{critical damping}} = \frac{2\xi M}{2\omega_n M} = \frac{\xi}{\omega_n} \]  

(4-2-2)

In(4-2-2), \( \omega_n \) is the natural frequency, in this survey it will be either first lateral bending or first torsional one depending on mode case (figures 3-8 and 9). The damping ratio is changing from zero to one. The oscillatory response of a system can be determined as:

- **Over damped** (\( \zeta > 1 \)): The system returns (exponentially decays) to equilibrium without oscillating. Larger values of the damping ratio \( \zeta \) return to equilibrium more slowly.
- **Critically damped** (\( \zeta = 1 \)): The system returns to equilibrium as quickly as possible without oscillating.
- **Under damped** (\( 0 < \zeta < 1 \)): The system oscillates (at reduced frequency compared to the un-damped case) with the amplitude gradually decreasing to zero.
- **Un-damped** (\( \zeta = 0 \)): The system oscillates at its natural frequency (\( \omega_n \)).

What has been discussed in present research is under damped case (\( 0 < \zeta < 1 \)). By considering(4-2-2), the damping coefficient must be changed between zero and natural frequency (\( \omega_n \)).

\[ \xi = \zeta \omega_n \quad \text{and} \quad 0 < \zeta < 1 \quad \Rightarrow \quad 0 < \xi < \omega_n \]  

(4-2-3)

\(^1\) The damping ratio is a dimensionless parameter that characterizes the frequency response of a second order ordinary differential equation. It provides a mathematical means of expressing the level of damping in a system relative to critical damping.
In this regard (3-3-15) needs to be modified by adding the damping term to it,

\[
M \ddot{X} + D \dot{X} + \left( K_x + M S \right) X = 0
\]  \hspace{1cm} (4-2-4)

(4-2-4) is a second order differential matrix equation and one of the possible solutions in this case is to assume an exponential function as a trial solution. For damped system, as like this research, a function such as (4-2-5) is needed to be assumed.

\[
X(t) = e^{-\xi t} q(t)
\]  \hspace{1cm} (4-2-5)

Here \(q(t)\) is the un-damped solution of (4-2-4). The first and second derivatives of (4-2-5) respect to time are,

\[
\ddot{X}(t) = -\xi e^{-\xi t} q(t) + e^{-\xi t} \ddot{q}(t)
\]

\[
\dddot{X}(t) = \xi^2 e^{-\xi t} q(t) - 2\xi e^{-\xi t} \dot{q}(t) + e^{-\xi t} \ddot{q}(t)
\]  \hspace{1cm} (4-2-6)

Substituting (4-2-5) and (4-2-6) in (4-2-4) gives the equation of motion with considering the damping.

\[
M \ddot{q}(t) + \left( -\xi^2 M + K_x + M S \right) q(t) = 0
\]  \hspace{1cm} (4-2-7)

The later second order differential equation (4-2-7) can be solved by using the modal Eigen-value problem analysis of any literatures in vibration theory [e.g. [41]]. (4-2-7) is modified to:
\[
\begin{align*}
\left(-\omega^2 M - \xi^2 M + K_e + M S\right)q(t) &= 0 \\
\left(-\left(\omega^2 + \xi^2\right)M + K_e + M S\right)q(t) &= 0
\end{align*}
\] (4-2-8)

In (4-2-8) the non-trivial solution exists if and only if the matrix determinant set to zero. This process has been used in the current dissertation for finding the critical buckling and natural frequency since last chapter. Because of this from now on the equations will be expressed briefly to avoid misunderstanding of the readers. The motion equation expanded matrix form is:

\[
\begin{bmatrix}
\frac{E I}{L} \frac{\pi^4}{L^4} & 0 & -\rho A e \left(\omega^2 + \xi^2\right) \\
-\rho A \left(\omega^2 + \xi^2\right) & \frac{M \left(\psi - 1\right) \left(\frac{\pi}{L}\right)^2}{2} \\
16\frac{E I}{L} \frac{\pi^4}{L^4} & 64M \left(\psi + 1\right) \frac{\pi}{L} & \frac{\pi^2}{L^2} \left(\frac{E I}{L} \frac{\pi^2}{L^2} + GJ\right) \\
-\left(\omega^2 + \xi^2\right) \rho A & 0 & -\left(\omega^2 + \xi^2\right) \rho I_p s
\end{bmatrix}
\begin{bmatrix}
F_1 \\
F_2 \\
V_1
\end{bmatrix}
= \begin{bmatrix} 0 \\
0 \\
0
\end{bmatrix}
\] (4-2-9)

4.2.1 Natural frequency

As symmetric cross sections are selected for plotting the graphs, the geometrical eccentricity, \(e\), is zero. The natural frequencies of the beam will be quiet similar to the case that there was no damping.

\[
\begin{align*}
\left(\omega_{b1}^d\right)^2 &= \left(\frac{\pi}{L}\right)^4 \frac{E I}{\rho A} - \xi^2 = \omega_{b1}^2 - \xi^2 \\
\left(\omega_{b2}^d\right)^2 &= 16 \left(\frac{\pi}{L}\right)^4 \frac{E I}{\rho A} - \xi^2 = 16\omega_{b1}^2 - \xi^2 \\
\left(\omega_{t1}^d\right)^2 &= \left(\frac{\pi}{L}\right)^2 \frac{GJ}{\rho I_p s} \left[1 + \left(\frac{\pi}{L}\right)^2 \frac{E I_{mh}}{GJ}\right] - \xi^2 = \omega_{t1}^2 - \xi^2
\end{align*}
\] (4-2-10)

With considering (4-2-3) and assuming bending mode, (4-2-10) can be divided by first natural frequency as follows:

\[
\begin{align*}
\left(\frac{\omega_{b1}^d}{\omega_{b1}}\right)^2 &= 1 - \xi^2 \\
\left(\frac{\omega_{b2}^d}{\omega_{b1}}\right)^2 &= 16 - \xi^2 \\
\left(\frac{\omega_{t1}^d}{\omega_{b1}}\right)^2 &= \left(\frac{\omega_{t1}}{\omega_{b1}}\right)^2 - \xi^2
\end{align*}
\] (4-2-11)

But for torsional mode, first torsional frequency should be taken into account in(4-2-3), and if (4-2-10) is divided by first torsional frequency, one can have:
\[
\left( \frac{\omega_{ni}}{\omega_{n1}} \right)^2 = \left( \frac{\omega_{b1}}{\omega_{n1}} \right)^2 - \xi^2 \quad \left( \frac{\omega_{n2}}{\omega_{n1}} \right)^2 = 16 \left( \frac{\omega_{b1}}{\omega_{n1}} \right)^2 - \xi^2 \quad \left( \frac{\omega_{t1}}{\omega_{n1}} \right)^2 = 1 - \xi^2
\]  

(4-2-12)

It was given in chapter 3, fig (3-7), that the CS1\(^1\) cross section frequencies are in dominantly bending mode. In this sense the frequencies are changing with respect to increasing the damping coefficient as like figures (4-3).

Fig 4-3 Changing the natural frequency vs. damping ratio (CS1) (bending mode)

As the second cross section group that was introduced in chapter 3, and with respect to fig (3-8), there will be two different cases in changing the frequency which must be discussed. Fig (4-4) illustrates the mainly bending and torsional frequency changes with respect to the changes of damping ratio for CS4. Fig (4-5) represents the frequency changes versus damping ratio for torsional dominant mode.

Fig 4-4 Changing the natural frequency vs. damping ratio (CS4) (bending mode)

\(^1\) Sectional properties are given in the appendix.
Finding the natural frequencies equation help to rearrange the equation of motion in a convenient way,

\[
\begin{bmatrix}
\rho A \left( \left( \omega_{x_1} \right)^2 - \omega^2 \right) & 0 & -\frac{M}{2} \left( \psi - 1 \left( 2 \pi \right)^2 \right) \\
\rho A \left( \left( \omega_{x_2} \right)^2 - \omega^2 \right) & \frac{64M}{9L^2} (\psi + 1) & 0 \\
symmetric & \rho I_{ps} \left( \left( \omega_{x_1} \right)^2 - \omega^2 \right) & 0 \\
\end{bmatrix}
\begin{bmatrix}
F_1 \\
F_2 \\
V_1 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]  
\[(4-2-13)\]

(4-2-13) can be expressed in another way. The difference between un-damped(3-4-14) and damped(4-2-13) motion equation for a symmetric cross section can be determined clearly.

\[
\begin{bmatrix}
\omega_{x_1}^2 - \omega^2 \\
\left( 16 \omega_{x_1}^2 - \omega^2 \right) \\
symmetric & \frac{I_{ps}}{A} \left( \omega_{x_1}^2 - \omega^2 \right)
\end{bmatrix}
\begin{bmatrix}
\frac{M}{2} \left( \psi - 1 \left( 2 \pi \right)^2 \right) \\
\frac{64M}{9L^2} (\psi + 1) \\
\rho I_{ps} \left( \left( \omega_{x_1} \right)^2 - \omega^2 \right)
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \frac{I_{ps}}{A}
\end{bmatrix}
\begin{bmatrix}
F_1 \\
F_2 \\
V_1 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]  
\[(4-2-14)\]

### 4.2.2 Critical buckling moment

The next step is finding the critical buckling moment while the damping effect is considered also. For this reason the circular frequency, \( \omega \), supposed to be zero. Replacing circular frequency with zero in (4-2-13) leads to(4-2-15). The critical damped static buckling moments are being obtained from the Eigen-value problem(4-2-15).
The critical buckling moment depends on the $\psi$ (gradient moment factor) and damping coefficient $\xi$. With the assumption that, the currently defined beam structure is subjected by the uniform moment load ($\psi = -1$) the critical damped moment load due to what have been mentioned is,

$$
\left( M_{cr0}^d \right)^2 = \rho A l s \left( \frac{L}{\pi} \right)^4 \left( \frac{\omega_{b1}}{\psi} \right)^2 \left( \frac{\omega_{t1}}{\psi} \right)^2 = \rho^2 A l s \left( \frac{L}{\pi} \right)^4 \left( \omega_{b1}^2 - \xi^2 \right) \left( \omega_{t1}^2 - \xi^2 \right) \quad \text{(4-2-16)}
$$

Here, if (4-2-16) is divided by (3-4-8) and taking into account (4-2-3), after some simplification the subsequent non-dimension ratio is obtained for dominantly bending and torsional respectively:

$$
\left( \frac{M_{cr0}^d}{M_{cr0}} \right)^2 = \left( 1 - \xi^2 \right) \left( 1 - \left( \frac{\omega_{b1}}{\omega_{t1}} \right)^2 \xi^2 \right) \quad \text{(4-2-17)}
$$

$$
\left( \frac{M_{cr0}^d}{M_{cr0}} \right)^2 = \left( 1 - \xi^2 \right) \left( 1 - \left( \frac{\omega_{b1}}{\omega_{t1}} \right)^2 \xi^2 \right) \quad \text{(4-2-18)}
$$

(4-2-17) and (4-2-18) are plotted in fig 4-6 for two different cross sections while bending mode and torsional is applying. 

![Fig 4-6 M_cr0/M_c0 vs. damping ratio](image-url)
The critical damped buckling moment can be expressed easily by the help of (4-2-16). While the (4-2-13) determinant supposed to be zero in the absence of circular frequency, $\omega$ (there is no vibration). The general formula for the critical moment expresses, 

$$
\left( \frac{M}{d_{cr}} \right)^2 = \left( \frac{M_{d,0}}{d_{cr}} \right)^2 \left( \frac{64(1+\psi)}{9\pi^2} \right)^2 \left( \frac{\omega_b}{\omega_{bs}} \right)^2 + \left( \frac{1-\psi}{2} \right)^2 
$$

(4-2-19)

$$
\left( \frac{M}{d_{cr}} \right)^2 = C^2(\psi, \xi) \left( \frac{M_{d,0}}{d_{cr}} \right)^2 
$$

(4-2-20)

where $M_{d,0}$ represents the end damped moment critical value. $M_{d,0}$ and $C(\psi, \xi)$ are damped critical buckling value of the uniform bending moment ($\psi = -1$) and the damped moment gradient coefficient respectively. By taking (4-2-3) into consideration, damped moment gradient coefficient can be given as follows:

$$
C^2(\psi, \xi) = C_d^2 = \frac{1}{\left( \frac{64(1+\psi)}{9\pi^2} \right)^2 \left( \frac{1-\xi^2}{16-\xi^2} \right) + \left( \frac{1-\psi}{2} \right)^2} 
$$

(4-2-21)

Here it has worth to mention that, damped moment gradient coefficient $C(\psi, \xi)$ can be different if the dominant mode of the beam is different, i.e. bending or torsional. It occurs because of presenting first bending frequency in the equation. In dominantly bending one the first bending frequency is the smallest frequency of the structure meanwhile in torsional mode the later mentioned frequency is the second frequency of the structure. In figures (4-7 and 8) damped moment gradient coefficient is plotted for dominantly bending and torsional mode respectively.

Fig 4- 7 Moment gradient coefficient $C(\psi, \xi)$ vs $\psi$ (bending mode)
4.3 Forced Vibration

As like what has been done in chapter three, the basic point for finding the natural frequency of the structure while the damping effect is taken into consideration is using (3-4-13). Of course here this equation is needed to be modified to cover the damping. By having (4-2-20) and keeping in mind that, \( \mu \), is steady state moment load factor the moment load shall be,

\[
M = \mu M_{cr0} C_d
\]  

(4-3-1)

Replacing (4-3-1) into (4-2-13) and setting the determinant to zero for obtaining a non-trivial solution, next third order characteristic equation will have been derived(4-3-4),

\[
\begin{vmatrix}
\rho A \left( (\omega_{h1}^d)^2 - \omega^2 \right) & 0 & -\frac{\mu^4 M_{cr0} C_d (\psi - 1)}{2} \left( \frac{\pi}{L} \right)^2 \\
\rho A \left( (\omega_{h2}^d)^2 - \omega^2 \right) & 64 \mu^4 M_{cr0} C_d (\psi + 1) & 9L^2 \\
\rho I_p s \left( (\omega_{h1}^d)^2 - \omega^2 \right) & \rho I_p s \left( (\omega_{h2}^d)^2 - \omega^2 \right) & \rho I_p s \left( (\omega_{h1}^d)^2 - \omega^2 \right)
\end{vmatrix}_{\text{symmetric}} = 0
\]  

(4-3-2)

\[
\rho^3 A^2 I_{p s} \left( (\omega_{h1}^d)^2 - \omega^2 \right) \left( (\omega_{h2}^d)^2 - \omega^2 \right) \left( (\omega_{h1}^d)^2 - \omega^2 \right)
\]

\[
-\rho A \left( (\omega_{h1}^d)^2 - \omega^2 \right) \left( \frac{64 \mu (1 + \psi) C_d}{9L^2} \right) \left( M_{cr0}^d \right)^2
\]  

(4-3-3)

\[
-\rho A \left( (\omega_{h2}^d)^2 - \omega^2 \right) \left( \frac{\mu (\psi - 1) C_d}{2} \right) \left( \frac{\pi}{L} \right)^4 \left( M_{cr0}^d \right)^2 = 0
\]
Substituting critical uniform moment load \((4-2-16)\) in\((4-3-3)\) and simplifying the equation returns,

\[
\left(\omega_b^2 - \omega_1^2\right)\left(\omega_b^2 - \omega_t^2\right)\left(\omega_b^2 - \omega_1^2\right) - \left(\omega_b^2 - \omega_1^2\right)\left(\frac{64\mu(1+\psi)C_x}{9\pi^2}\right)^2\left(\omega_b^2 - \omega_1^2\right) = 0
\]

\[(4-3-4)\]

There exist three roots for the third order equation\((4-3-4)\), and they are the function of steady state moment load factor, \(\mu\), moment gradient parameter \(\psi\), and damping coefficient \(\xi\). The character of the smallest root of \((4-3-4)\) depends on the first free mode of the beam, i.e. different if it is flexural or torsional\(^1\).

### 4.3.1 Dominantly bending mode \(\omega_{b1}<\omega_{t1}\)

As the first possibility for searching the minimum root of \((4-3-4)\), dominantly bending mode can be assumed. By this way if \((4-3-4)\) is divided over \(\omega_{b1}\), it will be changed to a non-dimension equation. \((4-3-5)\) can be represented in other format \((4-3-6)\) by using \((4-2-3)\).

\[
\begin{align*}
&\left(1 - \left(\frac{\xi}{\omega_b}\right)^2 - Z_i\right)\left(16 - \left(\frac{\xi}{\omega_b}\right)^2 - Z_i\right)\left(\frac{\omega_1}{\omega_b}\right)^2 - \left(\frac{\xi}{\omega_b}\right)^2 - Z_i \\
&-\mu^2C_x^2\left(1 - \left(\frac{\xi}{\omega_b}\right)^2\right)\left(\frac{\omega_1}{\omega_b}\right)^2 - \left(\frac{\xi}{\omega_b}\right)^2\left(1 - \left(\frac{\xi}{\omega_b}\right)^2 - Z_i\right)\left(\frac{64(1+\psi)}{9\pi^2}\right)^2 \\
&-\mu^2C_x^2\left(1 - \left(\frac{\xi}{\omega_b}\right)^2\right)\left(\frac{\omega_1}{\omega_b}\right)^2 - \left(\frac{\xi}{\omega_b}\right)^2\left(16 - \left(\frac{\xi}{\omega_b}\right)^2 - Z_i\right)\left(\frac{\psi - 1}{2}\right)^2 = 0
\end{align*}
\]

\[(4-3-5)\]

\[
\begin{align*}
&\left(1 - \xi^2 - Z_i\right)\left(16 - \xi^2 - Z_i\right)\left(\frac{\omega_1}{\omega_b}\right)^2 - \xi^2 - Z_i \\
&-\mu^2C_x^2\left(1 - \xi^2\right)\left(\frac{\omega_1}{\omega_b}\right)^2 - \xi^2\left(1 - \xi^2 - Z_i\right)\left(\frac{64(1+\psi)}{9\pi^2}\right)^2 \\
&-\mu^2C_x^2\left(1 - \xi^2\right)\left(\frac{\omega_1}{\omega_b}\right)^2 - \xi^2\left(16 - \xi^2 - Z_i\right)\left(\frac{\psi - 1}{2}\right)^2 = 0
\end{align*}
\]

\[(4-3-6)\]

\(^1\) Figures (3-7) and (3-8)
The damped frequencies in (4-3-4) were replaced with the help of (4-2-10). \( Z \) in (4-3-5) represents the ratio of minimum frequency over first un-damped bending one \((\omega / \omega_{b1})^2\). Before having the natural frequency distribution due to moment load parameters \((\mu, \psi)\), the boundary limitations are needed to be considered as same as what mentioned in chapter three\(^1\). Due to this background the (4-3-4) can be separated to two different parts. The first section will investigate the natural frequency graph while the structure is loaded by an asymmetric moment \((\psi = 1)\).

\[
\left(\left(\omega_{b1}^d\right)^2 - \omega^2\right)\left(\left(\omega_{b21}^d\right)^2 - \omega^2\right)\left(\left(\omega_{b1}^d\right)^2 - \omega^2\right) - \mu^2 C_d^2 \left[\frac{128}{9 \pi^2}\right]^2 = 0 \quad (4-3-7)
\]

By dividing (4-3-7) over \( \omega_{b1}^d \), then using (4-2-10) and (4-2-3), the subsequent third order frequency distribution equation is derived.

\[
(1 - \zeta^2 - Z_i)\left(16 - \zeta^2 - Z_i\right) \left[\frac{\omega_{b1}^d}{\omega_{b1}} - \zeta^2 - Z_i\right] - (1 - \zeta^2 - Z_i) \mu^2 C_d^2 \left[\frac{128}{9 \pi^2}\right]^2 = 0 \quad (4-3-8)
\]

Three different, steady state moment load factor dependent, roots of (4-3-8) are:

\[
Z_i = 1 - \zeta^2 \quad Z_i = \frac{-b \pm \sqrt{b^2 - 4ac}}{2} \quad (4-3-9)
\]

\[
a = 1 \quad b = -16 - \frac{\omega_{b1}^2}{\omega_{b1}^2} + 2\zeta^2
\]

\[
c = \left(\frac{\omega_{b1}^2}{\omega_{b1}^2} - \zeta^2\right) \left[\left(16 - \zeta^2\right) - \mu^2 C_d^2 \left[\frac{128}{9 \pi^2}\right]^2\right] \quad (4-3-10)
\]

The roots which were given by (4-3-9), are plotted in figures (4-9) to (4-10) for CS1 and CS4 cross sections\(^2\). Specified moment load factor for a beam element where the frequency coupling will happening after that while asymmetric moment is acting at the end of the beams can be determined by the help of (4-3-8).

\[
\left(\mu^2\right)^2 = \frac{15 \left(\frac{\omega_{b1}^2}{\omega_{b1}^2} - 1\right)}{\left(16 - \zeta^2\right) \left(\frac{\omega_{b1}^2}{\omega_{b1}^2} - \zeta^2\right)} \quad (4-3-11)
\]

---

\(^1\) Equations (3-4-19), (3-4-22), (3-4-29) and (3-4-31)

\(^2\) These cross section properties are given in the appendix
By taking uniform moment ($\psi = -1$) into consideration, (4-3-4), must be rearranged by replacing the gradient moment factor with minus one:

\[
\left( (\omega_{b_2}^d)^2 - \omega^2 \right) \left[ \left( (\omega_{b_1}^d)^2 - \omega^2 \right) \left( (\omega_{b_3}^d)^2 - \omega^2 \right) - \mu^2 C_i^2 \right]_{\psi = -1} \left( \omega_{b_1}^d \right)^2 \left( \omega_{b_1}^d \right)^2 = 0
\]  

(4-3-12)

Once more by dividing the later equation over $\omega_{b_1}^d$, and taking (4-2-10) and (4-2-3) into account once more,

\[
\left( 16 - \zeta^2 - Z_1 \right) \left( 1 - \zeta^2 - Z_1 \right) \left( \frac{\omega_{b_1}^2}{\omega_{b_1}^2} - \zeta^2 - Z_1 \right) \\
- \left( 16 - \zeta^2 - Z_1 \right) \mu^2 C_i^2 \left( 1 - \zeta^2 \right) \left( \frac{\omega_{b_1}^2}{\omega_{b_1}^2} - \zeta^2 \right) = 0
\]

(4-3-13)
The third order characteristic equation for uniformly initial moment loaded beam structure has three different roots. These roots are given in (4-3-14):

\[
Z_i = 16 - \zeta^2, \quad Z_i = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}
\]  

(4-3-14)

The parameters used in (4-3-14) are defined as follows,

\[
a = 1, \quad b = -1 - \frac{\omega_1^2}{\omega_{b1}^2} + 2\zeta^2, \quad c = \left(1 - \zeta^2\right)\left(1 - \zeta^2\right)\left(1 - \mu^2 C_i^2\right)
\]

(4-3-15)

Illustrations (4-11) and (4-12) are representing the changes of minimum frequency versus changing the steady state moment factor \(\mu\), is varying between zero\(^1\) (no moment) and one\(^2\) (full moment).

---

\(^1\) Free vibration

\(^2\) Buckling
With returns to the first row of (4-3-2), the damped modal mixing factor for the dominantly bending case shall be,

\[
\frac{d \eta_l}{i \sqrt{V_i}} = \frac{\mu C_d (\psi - 1)}{2} \sqrt{\left(\omega_{b1}^2 - \zeta^2 \right) \left(\omega_{l1}^2 - \zeta^2 \right)} \left(\omega_{b1}^2 - \zeta^2 - \omega_{min}^2 \right)
\]

(4-3-16)

For having a better view, and make it easier to compare the results, (4-3-16) can be expressed in a new form as like what has been done previously by using (4-2-3).

\[
\frac{d \eta_l}{i \sqrt{V_i}} = \frac{\mu C_d (\psi - 1)}{2} \sqrt{\left(1 - \zeta^2 \right) \left(\omega_{b1}^2 - \zeta^2 \right)} \left(1 - \zeta^2 - Z_i \right)
\]

(4-3-17)

Figures 4-13 to 4-18 are elaborating the damped frequency changes (4-3-5) and damped mixed mode factor (4-3-17) of a structure subjected to an initial gradient moment load (Fig 3-2) for three different damping ratios. In these graphs the assumed cross section is CS1.

---

1 Cross section details is given in appendix
Fig 4-15 Natural Frequency vs. moment load parameters (bending mode), $\zeta = 0.6$ (CS1)

Fig 4-16 Mixed mode factor vs. moment load parameters (bending mode), $\zeta = 0.6$ (CS1)

Fig 4-17 Natural Frequency vs. moment load parameters (bending mode), $\zeta = 0.8$ (CS1)
Graphs which are shown later (figures 4-19 to 4-24) are plotted to represent the damping ratio changing effect on frequency and mixed mode ratio of CS4.
Fig 4-21 Natural Frequency vs. moment load parameters (bending mode), $\zeta = 0.6$ (CS4)

Fig 4-22 Mixed mode factor vs. moment load parameters (bending mode), $\zeta = 0.6$ (CS4)

Fig 4-23 Natural Frequency vs. moment load parameters (bending mode), $\zeta = 0.8$ (CS4)
4.3.2 Dominantly torsional mode $\omega_{t1} < \omega_{b1}$

For the second limited case in vibration analysis, occurs when the natural torsional frequency being smaller than natural lateral bending frequency $\omega_{t1} < \omega_{b1}$. By starting from (4-3-4) and having the later assumption $(\omega_{t1} < \omega_{b1})$ the characteristic equation is,

$$
\left( \frac{\omega_{b1}}{\omega_{t1}} \right)^2 - \left( \frac{\xi}{\omega_{t1}} \right)^2 - Z_2 \right) \left( 16 \left( \frac{\omega_{b1}}{\omega_{t1}} \right)^2 - \left( \frac{\xi}{\omega_{t1}} \right)^2 - Z_2 \right) \left( 1 - \left( \frac{\xi}{\omega_{t1}} \right)^2 - Z_2 \right)
$$

$$
-\mu^2 C_i \left( \frac{\omega_{b1}}{\omega_{t1}} \right)^2 - \left( \frac{\xi}{\omega_{t1}} \right)^2 \left( 1 - \left( \frac{\xi}{\omega_{t1}} \right)^2 \right) \left( \frac{\omega_{b1}}{\omega_{t1}} \right)^2 - \left( \frac{\xi}{\omega_{t1}} \right)^2 - Z_2 \right) \left( 64 \left( 1 + \frac{\psi}{9\pi^2} \right) \right)^2

$$

$$
-\mu^2 C_i \left( \frac{\omega_{b1}}{\omega_{t1}} \right)^2 - \left( \frac{\xi}{\omega_{t1}} \right)^2 \left( 1 - \left( \frac{\xi}{\omega_{t1}} \right)^2 \right) \left( \frac{\omega_{b1}}{\omega_{t1}} \right)^2 - \left( \frac{\xi}{\omega_{t1}} \right)^2 - Z_2 \right) \left( \frac{\psi - 1}{2} \right)^2 = 0
$$

(4-3-18)

Procedures which were used for governing (4-3-9) and (4-3-14), are also valid for dominantly torsional mode $(\omega_{t1} < \omega_{b1})$ as well. While the structure is subjected to the asymmetric bending $(\psi = 1)$, (4-3-9) is needed to be reformed. Having (4-2-10) and dividing (4-3-4) over $\omega_{t1}^6$, leads:

$$
\left( \frac{\omega_{b1}}{\omega_{t1}} \right)^2 - \left( \frac{\xi}{\omega_{t1}} \right)^2 - Z_2 \right) \left( 16 \left( \frac{\omega_{b1}}{\omega_{t1}} \right)^2 - \left( \frac{\xi}{\omega_{t1}} \right)^2 - Z_2 \right) \left( 1 - \left( \frac{\xi}{\omega_{t1}} \right)^2 - Z_2 \right)

$$

$$
-\mu^2 C_i \left( \frac{\omega_{b1}}{\omega_{t1}} \right)^2 - \left( \frac{\xi}{\omega_{t1}} \right)^2 \left( 1 - \left( \frac{\xi}{\omega_{t1}} \right)^2 \right) \left( \frac{\omega_{b1}}{\omega_{t1}} \right)^2 - \left( \frac{\xi}{\omega_{t1}} \right)^2 - Z_2 \right) \left( 128 \left( 1 + \frac{\psi}{9\pi^2} \right) \right)^2 = 0

$$

(4-3-19)

Now by considering (4-2-3) and stress this point that here $\omega_n$ equals to the first torsional frequency (4-3-19) can be rewritten:
Three different, steady state moment load factor dependent roots of (4-3-20) are:

\[ Z_2 = \frac{\omega_{b2}^2}{\omega_{b1}^2} - \zeta^2, \quad Z_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2} \]  

(4-3-21)

Here in (4-3-21) parameters are defined as,

\[ a = 1, \quad b = -16 \frac{\omega_{b1}^2}{\omega_{b2}^2} + 2\zeta^2 - 1 \]

(4-3-22)

\[ c = \left(1 - \zeta^2\right) \left[16 \frac{\omega_{b1}^2}{\omega_{b2}^2} - \zeta^2\right] - \mu^2 C_d^{\psi} \left[1 - \zeta^2\right] ^2 \]  

(4-3-23)

Dividing (4-3-23) over \( \omega_{b1}^2 \), and taking (4-2-10) and (4-2-3) into account, the subsequent third order characteristic equation is derived:

\[ \left( \frac{\omega_{b2}^2}{\omega_{b1}^2} - \omega^2 \right)^2 \left[ \left( \frac{\omega_{b1}^2}{\omega_{b1}^2} - \omega^2 \right)^2 - \mu^2 C_d^{\psi} \left[1 - \zeta^2\right] ^2 \right] = 0 \]  

(4-3-24)

Dividing (4-3-23) over \( \omega_{b1}^2 \), and taking (4-2-10) and (4-2-3) into account, the subsequent third order characteristic equation is derived:

\[ \left( \frac{\omega_{b2}^2}{\omega_{b1}^2} - \omega^2 \right)^2 \left[ \left( \frac{\omega_{b1}^2}{\omega_{b1}^2} - \omega^2 \right)^2 - \mu^2 C_d^{\psi} \left[1 - \zeta^2\right] ^2 \right] = 0 \]  

(4-3-24)
The third order characteristic equation for uniformly initial moment loaded beam structure has three different roots. These roots are given in (4-3-25):

\[ Z_2 = 16 \frac{\omega_{b1}^2}{\omega_{1}^2} - \zeta^2, \quad Z_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2} \]  

(4-3-25)

Parameters that were used in (4-3-25), are introduced as:

\[ a = 1, \quad b = -1 - \frac{\omega_{b1}^2}{\omega_{1}^2} + 2\zeta^2, \quad c = \left( \frac{\omega_{b1}^2}{\omega_{1}^2} - \zeta^2 \right) \left( 1 - \zeta^2 \right) \left( 1 - \mu^2 C_d \right) \]  

(4-3-26)

![Figure 4-26 Natural frequency changing vs. steady state moment load factor, \( \psi = -1 \) (CS4) (torsional mode)](image)

The modal amplitudes (mixed mode ratio) for different damping factor can be expressed by the next equation,

\[ d \eta_2 = \frac{F_1}{i \mu V_1} = \frac{\mu C_d (\psi - 1)}{2} \left( \frac{\omega_{b1}^2}{\omega_{1}^2} - \zeta^2 \right) \left( 1 - \zeta^2 \right) \left( \frac{\omega_{b1}^2}{\omega_{1}^2} - \zeta^2 - Z_2 \right) \]  

(4-3-27)

Mixed mode ratio in dominantly torsional mode (4-3-27) can be expressed by replacing the ratio of damping coefficient over first torsional frequency by damping ratio (4-2-3),

\[ d \eta_2 = \frac{F_1}{i \mu V_1} = \frac{\mu C_d (\psi - 1)}{2} \left( \frac{\omega_{b1}^2}{\omega_{1}^2} - \zeta^2 \right) \left( 1 - \zeta^2 \right) \left( \frac{\omega_{b1}^2}{\omega_{1}^2} - \zeta^2 - Z_2 \right) \]  

(4-3-28)

Figures (4-27) to (4-29) illustrates the CS4 cross section frequency changes and the relevant mixed mode ratio versus to changing the moment gradient factor \( \psi \), goes between uniform (minus one)
and asymmetric (one) moment loading (fig 3-3). Meanwhile the steady state moment factor $\mu$ is changing from zero (free vibration) to one (full load).

Fig 4-27 Natural Frequency vs. moment load parameters (torsional mode), $\zeta = 0.4$ (CS4)

Fig 4-28 Mixed mode factor vs. moment load parameters (torsional mode), $\zeta = 0.8$ (CS4)

Fig 4-29 Natural Frequency vs. moment load parameters (torsional mode), $\zeta = 0.6$ (CS4)
Fig 4-30 Mixed mode factor vs. moment load parameters (torsional mode), $\zeta = 0.6$ (CS4)

Fig 4-31 Natural Frequency vs. moment load parameters (torsional mode), $\zeta = 0.8$ (CS4)

Fig 4-32 Mixed mode factor vs. moment load parameters (torsional mode), $\zeta = 0.8$ (CS4)
4.4 Summarize

In present chapter the damping effect on frequency coupling of the structure has been discussed. The matrix form of motion’s equation was taken into account here also. Without losing the generality, the Rayleigh damping (damping matrix was proportional to the mass matrix) added to matrix form of motion’s equation. For governing the dimensionless equations damping ratio, $\zeta$, was applied. Increasing the damping ratio effect on Natural frequencies, critical buckling moment, moment gradient coefficient factor, structure frequency and relevant mixed mode factors were discussed by the help of different cross sections to evaluate the equations. With returns to numerical examples graphs, following points can be figured out.

- Damped moment gradient coefficient $C(\psi, \zeta)$ equation (4-2-21), was given as a promoted form of (3-4-12).
- Enlarging the damping ratio causes, the damped moment gradient coefficient $C(\psi, \zeta)$ being increased as well in comparison with free case (there is no damping applied). (Figures (4-7) and (4-8)).
- In figure (4-9) to (4-10) while the asymmetric moment was applied on the beam and bending mode was taken into account. The beam frequency changes versus gradient moment factor $\psi$ and steady state moment parameter $\mu$, has same behavior as like what has shown in chapter three (Figures (3-15) and (3-16)). The minimal frequency of the structure is equals to its first damped bending one and after a specified point $\mu^d$, the frequency will goes to zero on a parabolic structure. Increasing the damping coefficient has a reverse influence on the frequency graph.
- Illustrations (4-11) to (4-12), which were plotted for uniform bending moment load in bending mode represent the coupling of the frequencies. Selecting higher values for damping ratio, $\zeta$, leads to coupled frequency curves being lower in comparison with undamped case.
- When the dominantly torsion mode was chosen\(^1\) the frequency distribution over gradient moment factor $\psi$ and steady state moment parameter $\mu$ plane did not change. The frequency coupling happened for any gradient moment factor, and steady state moment parameter. Here the damping effect is decreasing the value of the frequency and it does not change the behavior of the curve.
- In bending mode, having higher amount for damping ratio means that the mixed mode factor distribution surface versus gradient moment factor $\psi$ and steady state moment parameter $\mu$ plane will be shifted upward. For dominantly torsional mode it has been moved down in presence of the damping with comparison with undamped situation.

\(^1\) In current dissertation CS4 cross sections have this mode only.
Chapter 5

5.1 Introduction

The present section is going to investigate one of the most important researchers interest among the years. The stability of structures under the periodic load has been essential in designing the structures. As far as finding analytical solution for the differential equation are not possible always, by the help of numerical assumption functions the possibility domains for solution(s) can be drawn and further discussion shall be done on these areas.

What is going to be done in this chapter is to find the regions where the assumed beam in chapter three subjected to in initial gradient moment will be instable if the acting moment being dependent to time. For this purpose, (3-3-15) has been taken into account once more. After that, time dependent moment acting on the beam shall be given by new format. The moment equation contains static and dynamic terms. Solution procedure of the motion equation leads to an equation which is known as a Mathieu-Hill. A sinusoidal function will be applied as a possible trail function. Two independent matrix equations will be derived. The last but one step is to plot the stability borders of the beam and represent the instability regions. The final point is to discuss about the parameters, that instable regions can be affected by them. These factors effect will be represented by relevant graphs as well.

![Fig 5- 1 Chapter 5 process](image-url)
5.2 Periodic excitation

As, there exist no analytical solution for second order differential equation with periodic coefficient of periodically excited system like current research. Different solutions methods have been explored for this type. In this regard two main subjects were mostly interested for scientists, existence of periodic solutions and their stability.

A beam which is analyzed in this paper subjected to a concentrated periodical moment at its ends,

\[ M(t) = M_s + M_t \cos(\Omega t) \] (5-2-1)

Here \( M_s \) and \( M_t \) are static and dynamic amplitude of time dependent moment respectively. The excitation frequency is designated by \( \Omega \). With returns to (3-2-2) the initial moment distribution is:

\[ M = M_{cr} \left( \lambda + \kappa \cos(\Omega t) \right) \] (5-2-2)

\( \lambda \) is static buckling moment percentage and \( \kappa \) is for dynamic one. \( \lambda \) and \( \kappa \) are changing between zero and one. Critical moment designated by \( M_{cr} \). Equations (3-3-15) and (5-2-2) return:

\[ \ddot{X} + \left( K_s + M_{cr} \left( \lambda + \kappa \cos(\Omega t) \right) \right) X = 0 \] (5-2-3)

The later equation is known as a Mathieu-Hill. There are several possible approximated solution techniques in this field such as Bolotin\(^1\) based on Floquet’s theory, Galerkin method, the Lyapunov second method, asymptotic techniques by krylov, and perturbation and iteration are taken into consideration as well. One of the most used processes is to be assumed a periodic function. Among lots of periodic functions in time the sinusoidal series has been applied by lots of researchers.

5.3 Periodic solution

In this dissertation, this time dependent periodic function is taken into account based on Brown [19] with \( 2T \) period (first region of stability). Approximated periodic solution, which is advised for(5-2-3), is [44]:

\[ X = \sum_{k=1,3,5} a_k \sin \left( \frac{k\Omega t}{2} \right) + b_k \cos \left( \frac{k\Omega t}{2} \right) \] (5-2-4)

\( ^1 \) Vladimir Bolotin (1926 Tambov -2008) - One of the outstanding experts in the sphere of structural mechanics, Doctor of Technical Sciences, professor, Corresponding Member of the Academy of Sciences of the USSR, chief of the laboratory at Blagonravov Engineering Sciences Institute.
The second time derivative has a close relation with (5-2-4),

\[ \ddot{x} = -\left( \frac{k\Omega}{2} \right)^2 x \]

(5-2-5)

Replacing (5-2-4) and (5-2-5) into (5-2-3) leads,

\[
\sum_{k=1,3,5}^{\infty} \left[ -M \left( \frac{k\Omega}{2} \right)^2 + K_c + \lambda M_{cr} S \right] \left[ a_k \sin \left( \frac{k\Omega t}{2} \right) + b_k \cos \left( \frac{k\Omega t}{2} \right) \right] \\
+ M_{cr} \kappa S \left[ a_k \sin \left( \frac{k\Omega t}{2} \right) \cos \left( \Omega t \right) + b_k \cos \left( \frac{k\Omega t}{2} \right) \cos \left( \Omega t \right) \right] = 0
\]

(5-2-6)

With returns to mathematic rules in trigonometric function,

\[
\sin \left( \frac{k\Omega t}{2} \right) \cos \left( \Omega t \right) = \frac{1}{2} \left[ \sin \left( (k+2) \frac{\Omega t}{2} \right) + \sin \left( (k-2) \frac{\Omega t}{2} \right) \right]
\]

\[
\cos \left( \frac{k\Omega t}{2} \right) \cos \left( \Omega t \right) = \frac{1}{2} \left[ \cos \left( (k+2) \frac{\Omega t}{2} \right) + \cos \left( (k-2) \frac{\Omega t}{2} \right) \right]
\]

(5-2-7)

By the help of (5-2-7), (5-2-6) will be,

\[
\sum_{k=1,3,5}^{\infty} \left[ -M \left( \frac{k\Omega}{2} \right)^2 + K_c + \lambda M_{cr} S \right] \left[ a_k \sin \left( \frac{k\Omega t}{2} \right) + b_k \cos \left( \frac{k\Omega t}{2} \right) \right] \\
+ \frac{1}{2} M_{cr} \kappa S \left[ a_k \sin \left( (k+2) \frac{\Omega t}{2} \right) + a_k \sin \left( (k-2) \frac{\Omega t}{2} \right) \right] \\
+ \frac{1}{2} M_{cr} \kappa S \left[ b_k \cos \left( (k+2) \frac{\Omega t}{2} \right) + b_k \cos \left( (k-2) \frac{\Omega t}{2} \right) \right] = 0
\]

(5-2-8)

Since trigonometric functions, are linearly independent they can be separated easily:

\[
\sin \left( \frac{k\Omega t}{2} \right) \left[ -\left( \frac{k\Omega}{2} \right)^2 M + K_c + \lambda M_{cr} S \right] a_k \\
+ \frac{1}{2} M_{cr} \kappa S a_k \left[ \sin \left( (k+2) \frac{\Omega t}{2} \right) + \sin \left( (k-2) \frac{\Omega t}{2} \right) \right] \\
+ \cos \left( \frac{k\Omega t}{2} \right) \left[ -\left( \frac{k\Omega}{2} \right)^2 M + K_c + \lambda M_{cr} S \right] b_k \\
+ \frac{1}{2} M_{cr} \kappa S b_k \left[ \cos \left( (k+2) \frac{\Omega t}{2} \right) + \cos \left( (k-2) \frac{\Omega t}{2} \right) \right] = 0
\]

(5-2-9)
While (5-2-9) has infinite terms, and independent terms that should be set to zero, it can be written in two different matrices. One contains the a’s coefficient and the other one is for b’s. Selecting the first two terms of approximated assumption series (5-2-4):

\[
\sin\left(\frac{\Omega t}{2}\right)\left[-\left(\frac{\Omega}{2}\right)^2Ma_1 + \frac{\lambda b}{2}a_1, -\frac{1}{2}M_{cr}Sb_1 + \frac{1}{2}M_{cr}\kappa Sb_1\right] \\
+ \sin\left(\frac{3\Omega t}{2}\right)\left[-\left(\frac{3\Omega}{2}\right)^2Ma_1 + \frac{\lambda b}{2}a_1, -\frac{1}{2}M_{cr}Sb_1 + \frac{1}{2}M_{cr}\kappa Sb_1\right] \\
+ \cos\left(\frac{\Omega t}{2}\right)\left[-\left(\frac{\Omega}{2}\right)^2Mb_1 + \frac{\lambda b}{2}b_1, -\frac{1}{2}M_{cr}Sb_1 + \frac{1}{2}M_{cr}\kappa Sb_1\right] \\
+ \cos\left(\frac{3\Omega t}{2}\right)\left[-\left(\frac{3\Omega}{2}\right)^2Mb_1 + \frac{\lambda b}{2}b_1, -\frac{1}{2}M_{cr}Sb_1 + \frac{1}{2}M_{cr}\kappa Sb_1\right] = 0
\]  

(5-2-10) can be represented in matrix form by two different terms whose separating a and b coefficients.

\[
\begin{bmatrix}
-\left(\frac{\Omega}{2}\right)^2M + \frac{\lambda b}{2}a_1, -\frac{1}{2}M_{cr}\kappa Sb_1, \left(\lambda - \frac{\kappa}{2}\right)S \\
\frac{1}{2}M_{cr}\kappa Sb_1, -\left(\frac{3\Omega}{2}\right)^2M + \frac{\lambda b}{2}b_1, \left(\lambda + \frac{\kappa}{2}\right)S \\
\end{bmatrix}
\begin{bmatrix}
a_1 \\
b_1
\end{bmatrix}
\]

\[
\begin{bmatrix}
-\left(\frac{\Omega}{2}\right)^2M + \frac{\lambda b}{2}a_1, -\frac{1}{2}M_{cr}\kappa Sb_1, \left(\lambda - \frac{\kappa}{2}\right)S \\
\frac{1}{2}M_{cr}\kappa Sb_1, -\left(\frac{3\Omega}{2}\right)^2M + \frac{\lambda b}{2}b_1, \left(\lambda + \frac{\kappa}{2}\right)S \\
\end{bmatrix}
\begin{bmatrix}
a_1 \\
b_1
\end{bmatrix} = 0
\]  

(5-2-11) is an Eigen-value problem whose for having non trivial solution the coefficient determinant must be set to zero. As far as a and b are independent, there must be two different characteristic equations. One for a’s and the other one for b’s.

\[
\begin{bmatrix}
-\left(\frac{\Omega}{2}\right)^2M + \frac{\lambda b}{2}a_1, -\frac{1}{2}M_{cr}\kappa Sb_1, \left(\lambda - \frac{\kappa}{2}\right)S \\
\frac{1}{2}M_{cr}\kappa Sb_1, -\left(\frac{3\Omega}{2}\right)^2M + \frac{\lambda b}{2}b_1, \left(\lambda + \frac{\kappa}{2}\right)S \\
\end{bmatrix}
\begin{bmatrix}
a_1 \\
b_1
\end{bmatrix} = 0
\]

\[
\begin{bmatrix}
-\left(\frac{\Omega}{2}\right)^2M + \frac{\lambda b}{2}a_1, -\frac{1}{2}M_{cr}\kappa Sb_1, \left(\lambda - \frac{\kappa}{2}\right)S \\
\frac{1}{2}M_{cr}\kappa Sb_1, -\left(\frac{3\Omega}{2}\right)^2M + \frac{\lambda b}{2}b_1, \left(\lambda + \frac{\kappa}{2}\right)S \\
\end{bmatrix}
\begin{bmatrix}
a_1 \\
b_1
\end{bmatrix} = 0
\]  

77
5.4 Instability graphs

It has been mentioned already, that (5-2-12) dimensions are infinite. By using the first element of the each matrix, the first approximation of first stability region can be obtained. In this way the first step that must be clarified is to find the optimum number of terms for having the stability domains. With returns to the chapter 3 and ref [40] the minimal number should be set to two terms for lateral deflection \( (n=2) \) and one term for the twisting \( (m=1) \). For making simplification in plots, the non-dimensional ratio of excitation frequency over fundamental natural frequency has used. This factor which is called frequency ratio is given by:

\[
\phi = \frac{\Omega}{\omega}
\]  

(5-2-13)

It has worth to be mentioned, \( \omega \) in (5-2-13) is first lateral bending moment in dominantly bending mode and first torsional one if dominantly torsional mode is considered.

Figs (5-2) show the effect of the selecting different number of Ritz terms \((3-3-4)\) on instability region of the beam structure which is under periodic load, Fig (3-1). This graph is plotted for CS1 as an example, while uniform moment is applying on the beam and static buckling moment percentage is set to 0.5. it has to be stressed that only first instability region is illustrated in this figure.

The graph shows the first stability region of the first approximation analysis which is the most important part that should be taken into account of in estimation of the beam stability analysis(5-2-12). Uniform loading has selected for the numerical example drawings.

Fig 5-2 Comparison of solutions with different Ritz terms, \( \lambda \) sets to 0.5 for \( \psi = -1 \) (CS1)
It could be seen in Fig (5-2), that there is no magnificent difference in stability regions\(^1\). In this regard it is better to set the Ritz terms numbers to the lowest ones.

The accuracy of the solution has to be proven before showing the other factors influence on the stability borders and unstable regions of the assumed beam in present survey. As a clear example a point has selected in the ratio of excitation frequency over fundamental natural frequency and buckling moment percentage \((\phi,\kappa)\) plane. Then by plotting a one of displacement time functions which were introduced in as an unknown time function coefficient in Ritz approximation formula (3-3-4) versus time the being stable or instability of the structure shall been elaborated.

For this reason, (5-2-3) has to be considered. By using the space state method the order of the motion equation will be reduced one. The matrix dimensions and number of equations will be doubled consequently. Two points will be chosen from Stable and Unstable domains were mentioned in one of the previously drawn graphs. As an example CS1 cross section (Fig 5-3) is selected. Point’s abscissa and abscissa and ordinate of these points are the ratio of excitation frequency over fundamental natural frequency and buckling moment percentage values of them respectively \((A\ and\ B\ point\ in\ Fig\ 5-3)\) as given in table.

<table>
<thead>
<tr>
<th></th>
<th>(\phi)</th>
<th>(\mu)</th>
<th>(\lambda)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.5</td>
<td>0.6</td>
<td>0.5</td>
</tr>
<tr>
<td>B</td>
<td>1.5</td>
<td>0.6</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 5-1 Selected points details

Pre multiplying (5-2-3) by inverse matrix of mass (3-3-16) returns:

\[
\ddot{X} + M^{-1} \left( K_s + M_{cr} \left( \lambda + \kappa \cos(\Omega t) \right) S \right) X = 0 \quad (5-2-14)
\]

In space state vector method, defining a new vector like (5-2-15), and its first derivative respect to time helps to reformulate (5-2-14).

\(^1\) The instability regions regarding to different Ritz term for some other cross sections are given in he appendix as examples.
\[ Y = \begin{bmatrix} X \\ \dot{X} \end{bmatrix} \]  \hspace{1cm} (5-2-15)

\[ \frac{d}{dt} \begin{bmatrix} X \\ \dot{X} \end{bmatrix} = \begin{bmatrix} \dot{X} \\ \ddot{X} \end{bmatrix} = \begin{bmatrix} 0 \\ M^{-1} \left( K_e + M \kappa \cos(\Omega t) \right) S \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} X \\ \dot{X} \end{bmatrix} \]  \hspace{1cm} (5-2-16)

\[ \ddot{Y} = P \ddot{Y} \]

Here, in the last given equation,

\[ P = \begin{bmatrix} 0 \\ M^{-1} \left( K_e + M \kappa \cos(\Omega t) \right) S \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \]

\[ I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix}_{(n+m) \times (n+m)} \]  \hspace{1cm} (5-2-17)

Solving (5-2-16), numerically, for points A and B (Fig 5-3) after replacing the relevant parameters values which are given in (Table 5-1), subsequent illustrations will represent the time behavior of the first unknown function of the Ritz approximation method, \( F_1(t) \), (3-3-4) as an example for the given points respectively.

Fig 5-4 Time function behavior of a point in Stable domain (A, fig (5-3))

Fig 5-5 Time function behavior of a point in Unstable domain (B, fig (5-3))
It can be seen that, in (fig 5-4) the unknown function’s behavior in time is bounded and a cyclic manners are going to be start after almost 0.15 s. comparing figures (5-4) and (5-5) elaborate, B point time behavior is not bounded and the amplitude is rocketed by enlarging the time. In this way, by the help of figures (5-4) and (5-5), the solution process leads to derive (5-2-12) for seeking the stability borders and instable regions veracity has been proven.

5.5 Loading parameters effect on instability regions

There are some parameters that, their effects are important to be considered on stability domains. Such as static (λ) and dynamic (κ) buckling moment percentage. These parameters are going to be discussed for two different cross section types\(^1\). Figures (5-7) to (5-9) show the instability borders of two moment loading boundary cases for different cross sections separately. In Fig (5-7), CS1 cross section dynamic instability borders will be represented while bending mode is considered. Fig (5-8) and Fig (5-9), illustrate these borders for CS4 in dominant bending and torsional mode respectively. What can be realized from the latterly mentioned figures is, while asymmetric moment (ψ = 1) is applying on the beam with dominantly bending mode, the instability borders are going to be very close to each other. In brief it means, a structure, in this case a beam which is subjected to asymmetric moment is always stable, while it is subjected asymmetrically.

\(^1\) Cross sections properties are given in the appendix.
Fig 5-7 Instability regions for different values of $\psi$, $\lambda = 0.5$ (CS1) (bending mode)

Fig 5-8 Instability regions for different values of $\psi$, $\lambda = 0.5$ (CS4) (bending mode)

Fig 5-9 Instability regions for different values of $\psi$, $\lambda = 0.5$ (CS4) (torsional mode)
Here is another parameter which the stability borders are affected by its changes. This issue is the static buckling moment percentage ($\lambda$). This parameters influence on instability regions are going to be given in following plots. Graphs (5-10) to (5-13) are showing the first region(s) of the first instability approximation\(^1\). Figures (5-7) to (5-8), elaborated that the stability borders can be plotted for uniform loading only, if the first stability approximation(5-2-12) is taken into account and the length of the beam is selecting somehow that dominantly bending mode occurs .With returns to what has been discussed in chapter 3, 3.4.1, the structure fundamental frequency, $\omega$, and the critical buckling moment, $M_{cr}$, are different if dominantly bending or torsional mode applying considering for the structure.

\[\text{Fig 5-10 Instability regions for different values of } \lambda \text{ while } \psi = -1 \text{ (CS1) (bending mode)}\]

\[\text{Fig 5-11 Instability regions for different values of } \lambda \text{ while } \psi = 1 \text{ (CS1) (bending mode)}\]

\(^1\)The first elements of (5-2-12) matrices were selected.
In Figs (5-14 and 5-15) the static buckling moment percentage ($\lambda$) changing effect on instability borders for CS4 cross section, in dominantly torsional mode is given.
Next graphs are representing the variation of stability region borders with respect to changing the moment gradient factor ($\psi$), the non-dimensional ratio of excitation frequency over fundamental natural frequency ($\phi$) and the dynamic buckling moment percentage ($\kappa$) for a certain static buckling moment percentage ($\lambda$). The first stability surfaces and unstable regions of first stability analysis approximation are specified as well. Zones between two surfaces represent the instable regions of the structure, Fig (5-6). Outside of these surfaces, the beam structure under gradient moment load in this dissertation will have a bounded time dependent function changes versus time (the structure is stable). Plots are given for a fixed value of the static buckling moment percentage ($\lambda$). In figures (5-16) to (5-18) this amount is set to 0.5. Moment gradient factor ($\psi$) is changing between minus one (uniform moment) and one (asymmetric moment) (Fig 3-2). The non-dimensional ratio of excitation frequency over fundamental natural frequency ($\phi$) is selected somehow to show the first unstable zone for each cross section. Figures (5-16) to (5-17) are giving first instable regions of the assumed cross sections while their length were selected long enough that bending mode is the main performance of the beam.
5.6 Summarize

In this chapter, the stability regions for a beam structure that was given in chapter 3 (fig 3-2) and was subjected by initial time dependent gradient moment load have been investigated. With respect to two boundary moment loadings and static buckling moment percentage variation as the effective elements on stability of a structure the following points can be elicited:
• Using extra terms, more than two terms for lateral bending and more than one term for torsional deflection, has no significant effect on the stability regions while the first stability analysis approximation is considered, Fig (5-2).

• When bending mode is the dominant mode of a structure and the beam is loading by asymmetric moment, if the first instability region of the first stability analysis approximation (5-2-12) is considered, the beam will be always stable, Figs (5-7), (5-8), (5-16) and (5-17).

• In dominantly torsional mode, instability regions are exist for any value of moment gradient factor, $\psi$ in first instability region of the first stability analysis approximation, figures (5-9) and (5-18).

• By increasing the share of the static buckling moment percentage in initial moment load (selecting higher $\lambda$) the instability regions width and in consequence the instability of the beam will be increased. In other word the stability borders of are going to be further from each other. In this sense the most stable position for the structure is, when no static moment percentage ($\Lambda$) is applying, figures (5-10) to (5-15).
Chapter 6

6.1 Introduction

In previous chapter first dynamic instability regions of the assumed structure which is subjected to gradient initial moment load (Fig 3-2) were found. Up to the investigation which was done among the current research, there was no literature about the effect of damping on the stability borders of a beam that is loaded by non-uniform time periodic initial gradient bending moment. The current section is going to talk about the effect of the damping\(^1\) on the structure stability regions. This chapter procedure’s synopsis is: the equation which was applied in previous chapter (5-2-3) will be modified by considering the Rayleigh damping. For solving the modified Mathieu-Hill equation a same procedure shall be applied as like what has done in chapter five for governing the instability borders equation. The only difference here is except two different equations for the border curves a unique equation will be adopted. In following, effective parameters on instability regions will be discussed. By considering the Rayleigh damping phenomenon, there were three independent factors that their effects must be sought out while the other ones are fixed to a certain number. Two of these parameters were discussed in chapter five in the absence of the damping are, Static buckling moment percentage (\(\lambda\)) and moment gradient factor (\(\psi\)). The extra parameter which its effect must be taken into account is Rayleigh damping’s coefficient.

\[\text{Fig 6-1 Chapter 6 process}\]

\(^1\) Rayleigh damping (Chapter 4)
6.2 Periodic solution

By using the assumption which was taken into account in chapter four (4-2-1), the chapter 5 equations\(^1\) will be modified as follows:

\[
M \ddot{X} + D_a \dot{X} + \left( K_c + M_{cr} \left( \lambda + \kappa \cos(\Omega t) \right) S \right) X = 0 \tag{6-2-1}
\]

(6-2-1) is also a Mathieu-Hill equation and the solution process that was given previously is also valid in this case. A time dependent periodic trail function needs to be selected as like [19], with 2\(T\) period (first region of stability). Approximated periodic solution, which is advised for (6-2-1), is as same as (5-2-3) [44]:

\[
X = \sum_{k=1,3,5} a_k \sin \left( \frac{k\Omega t}{2} \right) + b_k \cos \left( \frac{k\Omega t}{2} \right) \tag{6-2-2}
\]

The first and second time derivation, by mathematical rules being:

\[
\begin{align*}
\dot{X} &= \sum_{k=1,3,5} a_k \left( \frac{k\Omega}{2} \right) \cos \left( \frac{k\Omega t}{2} \right) - b_k \left( \frac{k\Omega}{2} \right) \sin \left( \frac{k\Omega t}{2} \right) \\
\ddot{X} &= -\left( \frac{k\Omega}{2} \right)^2 X
\end{align*}
\tag{6-2-3}
\]

Replacing (6-2-3) and (6-2-2) in (6-2-1) give:

\[
\begin{align*}
\sum_{k=1,3,5} \left[ -M \left( \frac{k\Omega}{2} \right)^2 + K_c + \lambda M_{cr} S \left( a_k \sin \left( \frac{k\Omega t}{2} \right) + b_k \cos \left( \frac{k\Omega t}{2} \right) \right) + M_{cr} \kappa S \left( a_k \sin \left( \frac{k\Omega t}{2} \right) \cos(\Omega t) + b_k \cos \left( \frac{k\Omega t}{2} \right) \cos(\Omega t) \right) \right] \\
+ D_a \left( a_k \left( \frac{k\Omega}{2} \right) \cos \left( \frac{k\Omega t}{2} \right) - b_k \left( \frac{k\Omega}{2} \right) \sin \left( \frac{k\Omega t}{2} \right) \right) &= 0 \tag{6-2-4}
\end{align*}
\]

Using mathematic rules in trigonometric function (5-2-7), (6-2-4) can be expanded.

\(^1\) equations (5-2-3) and (5-2-6) to (5-2-11)
\[ \sum_{k=1,3,5}^{\infty} \left[ -M \left( \frac{k\Omega}{2} \right)^2 + K_s + \lambda M_{cr} S \right] \left[ a_k \sin \left( \frac{k\Omega t}{2} \right) + b_k \cos \left( \frac{k\Omega t}{2} \right) \right] \\
+ \frac{1}{2} M_{cr} \kappa S \left[ a_k \sin \left( \frac{(k+2)\Omega t}{2} \right) + a_k \sin \left( \frac{(k-2)\Omega t}{2} \right) \right] \\
+ \frac{1}{2} M_{cr} \kappa S \left[ b_k \cos \left( \frac{(k+2)\Omega t}{2} \right) + b_k \cos \left( \frac{(k-2)\Omega t}{2} \right) \right] \\
+ D \left[ a_k \left( \frac{k\Omega}{2} \right) \cos \left( \frac{k\Omega t}{2} \right) - b_k \left( \frac{k\Omega}{2} \right) \sin \left( \frac{k\Omega t}{2} \right) \right] = 0 \] (6-2-5)

The linear independence of sinusoidal functions, helps to separate (6-2-5) into two independent parts, Sine and Cosine terms:

\[ \sin \left( \frac{k\Omega t}{2} \right) \left[ \left[ -\left( \frac{k\Omega}{2} \right)^2 M + K_s + \lambda M_{cr} S \right] a_k - D \frac{k\Omega}{2} \right] \\
+ \frac{1}{2} M_{cr} \kappa S a_k \left[ \sin \left( \frac{(k+2)\Omega t}{2} \right) + \sin \left( \frac{(k-2)\Omega t}{2} \right) \right] \\
+ \cos \left( \frac{k\Omega t}{2} \right) \left[ \left[ -\left( \frac{k\Omega}{2} \right)^2 M + K_s + \lambda M_{cr} S \right] b_k + D \frac{k\Omega}{2} \right] \\
\frac{1}{2} M_{cr} \kappa S b_k \left[ \cos \left( \frac{(k+2)\Omega t}{2} \right) + \cos \left( \frac{(k-2)\Omega t}{2} \right) \right] = 0 \] (6-2-6)

Due to (6-2-2) has infinite terms, (6-2-6) is also contains infinite terms, as the first approximation of stability analysis, only first two terms of (6-2-2) were selected.

\[ \sin \left( \frac{\Omega t}{2} \right) \left[ -\left( \frac{\Omega}{2} \right)^2 M a_1 + K_s a_1 + \lambda M_{cr} S a_1 - \frac{1}{2} M_{cr} \kappa S a_1 \right] \\
+ \frac{1}{2} M_{cr} \kappa S a_1 - \Omega \xi M b_1 \right] \\
+ \sin \left( \frac{3\Omega t}{2} \right) \left[ -\left( \frac{3\Omega}{2} \right)^2 M a_3 + K_s a_3 + \lambda M_{cr} S a_3 + \frac{1}{2} M_{cr} \kappa S a_3 \right] \\
+ \frac{1}{2} M_{cr} \kappa S a_3 - 3\Omega \xi M b_3 \right] 

\[ +\cos\left(\frac{\Omega t}{2}\right)\left[ -\left(\frac{\Omega}{2}\right)^2 M b_1 + K_1 b_1 + \lambda M_{cr} S b_1 + \frac{1}{2} M_{cr} \kappa S b_1 \right. \\
+ \left. \frac{1}{2} M_{cr} \kappa S b_1 + 3\Omega \xi M a_1 \right] \]

\[ +\cos\left(\frac{3\Omega t}{2}\right)\left[ -\left(\frac{3\Omega}{2}\right)^2 M b_1 + K_1 b_1 + \lambda M_{cr} S b_1 + \frac{1}{2} M_{cr} \kappa S b_1 \right. \\
+ \left. \frac{1}{2} M_{cr} \kappa S b_1 + 3\Omega \xi M a_1 \right] = 0 \]

(6-2-7)

Terms are in brackets should be zero one by one because of the linear independence of trigonometric functions in (6-2-2), that returns four independent equation in this case, which must be zero.

\[ \begin{cases} 
-\left(\frac{3\Omega}{2}\right)^2 M a_1 + K_2 a_1 + \lambda M_{cr} S a_1 + \frac{1}{2} M_{cr} \kappa S a_1 + \frac{1}{2} M_{cr} \kappa S a_1 - 3\Omega \xi M b_1 = 0 \\
-\left(\frac{\Omega}{2}\right)^2 M a_1 + K_1 a_1 + \lambda M_{cr} S a_1 - \frac{1}{2} M_{cr} \kappa S a_1 + \frac{1}{2} M_{cr} \kappa S a_1 - \Omega \xi M b_1 = 0 \\
-\left(\frac{\Omega}{2}\right)^2 M b_1 + K_1 b_1 + \lambda M_{cr} S b_1 + \frac{1}{2} M_{cr} \kappa S b_1 + \frac{1}{2} M_{cr} \kappa S b_1 + \Omega \xi M a_1 = 0 \\
-\left(\frac{3\Omega}{2}\right)^2 M b_1 + K_2 b_1 + \lambda M_{cr} S b_1 + \frac{1}{2} M_{cr} \kappa S b_1 + \frac{1}{2} M_{cr} \kappa S b_1 + 3\Omega \xi M a_1 = 0 
\end{cases} \]

(6-2-8)

For having a simple form (6-2-8) can be represented as a matrix equation.

\[
\begin{bmatrix}
  a & b & 0 & -c \\
  b & d_1 & 0 & a_1 \\
  0 & e & d_2 & b_1 \\
  c & 0 & b & a
\end{bmatrix}
\begin{bmatrix}
  a_1 \\
  a_1 \\
  b_1 \\
  b_1
\end{bmatrix} = 0
\]

(6-2-9)

where,

\[
a = -\left(\frac{3\Omega}{2}\right)^2 M + K_{cr} + M_{cr} \lambda S \\
b = \frac{M_{cr}}{2} \kappa S \\
d_1 = -\left(\frac{\Omega}{2}\right)^2 M + K_{cr} + M_{cr} \left(\lambda - \frac{\kappa}{2}\right)S \\
c = 3\Omega \xi M \\
d_2 = -\left(\frac{\Omega}{2}\right)^2 M + K_{cr} + M_{cr} \left(\lambda + \frac{\kappa}{2}\right)S \\
e = \Omega \xi M
\]

(6-2-10)
Selecting two by two central elements as the first stability region analyze in damped case (6-2-9) leads to:

\[
\begin{bmatrix}
\left(-\frac{\Omega}{2}\right)^2 M + K_e + M_cr \left(\lambda - \frac{\kappa}{2}\right) S & -\Omega \xi M \\
\Omega \xi M & -\left(-\frac{\Omega}{2}\right)^2 M + K_e + M_cr \left(\lambda + \frac{\kappa}{2}\right) S
\end{bmatrix}
= \begin{bmatrix}
a_1 \\
b_1
\end{bmatrix}
= 0 \quad (6-2-11)
\]

(6-2-11)'s determinant must be zero, for having non-trivial solution. Plotting the characteristic equation gives the stability borders and instability regions for different parameters which has mentioned in brief at the beginning of this chapter in introduction part.

\[
\begin{bmatrix}
\left(-\frac{\Omega}{2}\right)^2 M + K_e + M_cr \left(\lambda - \frac{\kappa}{2}\right) S & -\Omega \xi M \\
\Omega \xi M & -\left(-\frac{\Omega}{2}\right)^2 M + K_e + M_cr \left(\lambda + \frac{\kappa}{2}\right) S
\end{bmatrix}
= 0 \quad (6-2-12)
\]

### 6.3 Damping ratio effect on instability graphs

By the help of (6-2-12), the first approximation stability regions for a selected structure in this dissertation (Fig 3-2) with different cross sections while the damping effect is considered can be plotted in subsequent figures. Here is also needed to use a non-dimensional parameter, frequency ratio (5-2-13) for simplicity as well as what did in chapter 5. The only difference is natural frequency and critical moment have to replaced by the equations were given for damped first natural bending frequency (4-2-10) and the critical moment (4-2-16) in chapter 4. The first group of the sample numerical results (Fig 6-2 to 6-5) is representing the effect of damping ratio $\zeta$ on the first approximation of stability regions when the uniform moment loading ($\psi = -1$) was applied for CS1 and CS4\(^1\), respectively while dominantly bending mode is applying.

![Fig 6-2 Instability regions for different values of damping ratio ($\zeta$), $\psi=-1$, $\lambda=0.7$ (CS1) (dominantly bending)](image)

\(^1\) These abbreviations were used for nominating the cross sections in the appendix.
By selecting the length of the beam Fig (3-2), short enough, Fig (3-7), while CS4 is the selected cross section family, dominantly torsional mode\(^1\) will be occurred. In this case both loading boundary conditions (uniform moment and asymmetric one) influences on the beam instability regions can be seen. In other words the first instable regions of the first stability analyze

\(^1\) First torsional frequency of the beam will be smaller than the first bending one.
approximation can be seen for every moment gradient factors (ψ), Fig (5-9). Figures (6-6) and (6-7) are illustrating the unstable regions and stability borders of CS4 for uniform and symmetric moment separately when dominantly torsional mode of the cross section is selected.

![Graph](image1)

**Fig 6-6 Instability regions for different values of damping ratio (ζ), ψ=1, λ=0.5 (CS4) (dominantly torsional)**

![Graph](image2)

**Fig 6-7 Instability regions for different values of damping ratio (ζ), ψ=1, λ=0.5 (CS4) (dominantly torsional)**

### 6.4 λ effect on instability regions

The other factor which its effect on instability border has to be investigated as it was mentioned earlier in the introduction is the static buckling moment percentage (λ). Following graphs (fig (6-8) to fig (6-11)) will illustrate this influence for two different cross sections (CS1 and CS4) in dominantly bending and torsional mode cases. The uniform moment loadings are discussed in bending mode. As the first regions are the most important ones and with returns to figures (5-7) and (5-8), the uniform moment loading was selected for plotting the examples to represent the static buckling moment percentage effect while the damping is included.
Fig 6-8 Instability regions for different $\lambda$ values, $\psi=-1$, $\zeta=0.1$ (CS1) (dominantly bending)

Fig 6-9 Instability regions for different $\lambda$ values, $\psi=-1$, $\zeta=0.1$ (CS4) (dominantly bending)

In addition, when the torsional mode is considering for the structure, in present survey CS4, both uniform and asymmetric mode can be illustrated.

Fig 6-10 Instability regions for different $\lambda$ values, $\psi=-1$, $\zeta=0.1$ (CS4) (dominantly torsion)
6.5 Summarize

In present chapter, damping effect on stability regions of a beam structure that was given in chapter 3 (fig 3-2) and was subjected by initial time dependent gradient moment was investigated. With respect to two boundary moment loadings (uniform and asymmetric), static buckling moment percentage variation and Rayleigh damping coefficient as the effective elements on stability of a structure (in this dissertation two symmetric cross section families were selected for plotting the examples) the following points can be drawn:

- Applying greater amounts as the damping ratio ($\zeta$) leads to increasing the stability of the structure in general. As it is shown in this chapter figures (figs (6-2) to (6-7)), the stability borders are shifting upward in ($\phi$, $\kappa$) in other word the width of instability borders are decreased.
- Changing static moment percentage ($\lambda$), as like what were given in figures (6-8) to (6-11), has a direct effect on the structure instability. It means that if the static moment percentage being increased, width of instability regions shall be wider in comparison with the cases that static moment percentage value is less. In this sense for having much more stable structure the static moment percentage, needs to be selected as small as possible.
Chapter 7

7.1 Introduction

The current dissertation has been done to fulfill the formal requirement of PhD program in Kandó Kálmán Doctoral School at Budapest University of Technology and Economics (BME). As it has been mentioned and explained in full details among pervious chapters the aim of this research was to investigate the effect of gradient moment loading on the vibration and stability regions of a beam structure. Practically this chapter parts contain the points were found and mentioned in chapter three, four, five and six as outcomes of the survey. Subsequently the survey assumptions and results will be reviewed.

7.2 Assumptions

A thin walled beam model with symmetrical cross section (shear center and cross section are coincided) was taken into account\(^1\). Basic assumptions which this study was founded on their basis are: the beam member is straight and prismatic. Cross section’s area is constant along the length, and the cross-section is rigid in its plane but is subjected to torsional warping, rotations are large, strains are small, and the material is homogeneous, isotropic and linearly elastic\(^2\). The beam was sustained with two simply supported fork like constraints at each ends that allow free warping and prevent torsional rotation.

Initial concentrated moments were acting on end points of the beam (theoretically on surfaces where geometrical boundary conditions were applied) on the strongest principle axis. In this text while these moments acting axes direction are same, the condition is named asymmetrically loading. As another case the uniform loading was used for a situation that end moments acting axes direction are opposite.

7.3 Coupled frequencies and critical buckling moment

By the help of Ritz method and potential energy principle, then applying increment operator a set of equations was derived. The matrix form of these equations that have been obtained is known as the motion equations of the structure. Considering the modal analysis technique for solving an Eigen-value problem the characteristic equation of the beam was returned, beam’s natural

---

\(^1\) Upon this assumption four cross sections in two families (CS1, CS2, CS3 and CS4) were introduced.

\(^2\) Properties for a specified material are given in the appendix.
frequencies were found in the absence of external moment. Regarding to the ratio of first lateral bending frequency over first torsional one, two separated mode could be defined for the beam with previously mentioned conditions. If this ratio being greater than one it was called dominantly torsional mode while this ratio is lower than one the beam mode is known as a dominantly bending one. The critical buckling moment was found as well. A moment gradient coefficient, $C(\psi)$ was defined as the ratio of critical buckling moment over the uniformly loaded critical buckling moment. On the last but one step characteristic equation was modified by the help of natural frequencies, critical buckling moment and steady state moment factor, $\mu$ (represents the share of critical buckling moment on the element) forced vibration analysis was done.

7.3.1 Thesis I

I.a) The optimal selections for Ritz terms are two terms for lateral deflection and one term for torsional one. Higher terms have no significant effect on moment gradient coefficient, $C(\psi)$ [45].

I.b) Considering dominantly bending mode, while the beam is subjected to uniform moment, causes a coupling in the frequency of structure [45].

I.c) When asymmetric moment is acting on the beam, the critical steady state moment factor, $\mu_c$, was defined. The frequency equals to the first lateral bending one up to $\mu_c$ and then it will be coupled by increasing the steady state moment factor value up to one (full load) [45].

I.d) For cross sections which have dominantly torsional mode the coupling occurs for any value of moment gradient parameter $\psi$, and steady state moment load factor $\mu$.

7.4 Damped Coupled frequencies and critical buckling moment

Rayleigh damping\(^1\) effect on structure’s frequencies of the structure and shape modes was discussed to modify latterly derived equation for un-damped case, by adding the damping term to the matrix form equation. Defining the damping ratio, $\zeta$, helped to express equations in dimension less mode, which is more suitable for comparing the results. The generality of the solution was not changed in this sense. Using a trail exponential equation as a mathematical transformation between damped and un-damped case and applying modal analysis for Eigenvalue problem, the damped frequency and critical moment were found. Substituting of last two founded equations in characteristic equation made it possible to investigate the frequency and shape mode of a beam while damping is also taken into account.

\(^1\) Here it was assumed that damping matrix is proportional to the mass matrix and $\zeta$ was the proportionality ratio.
7.4.1 Thesis II

II.a) Moment gradient coefficient, $C(\psi)$ was improved to $C(\psi,\zeta)$ by considering the damping ratio, $\zeta$ [46].

II.b) Selecting higher values for the Rayleigh damping ratio leads, the damped Moment gradient coefficient, $C(\psi, \zeta)$ being shifted upward [46].

II.c) The critical steady state moment factor, $\mu_c$ which had been given, was modified to $\mu_c^d$ to cover up damping ratio effect.

II.d) Frequency and mixed mode ratio distribution’s nature versus moment gradient parameter $\psi$, and steady state moment load factor $\mu$ remained same. But the amplitudes were decreased in both dominantly bending and torsional one [46].

II.e) Mixed mode factors amplitudes were enlarged by increasing the damping coefficient, $\zeta$ while bending mode is taken into account [46]. Meanwhile this ratio is decreased in comparison with undamped case, when the torsional mode is the dominant one.

7.5 Dynamic stability analysis (Un-damped)

For investigating instability regions of the beam structure, the matrix form of the beam’s characteristic equation of motions was considered. As far as the dynamic stability was selected to be dealt with, the initial moment loading should be multiplied by a periodic time dependent function. The latterly mentioned equation form is known as a Mathieu-Hill equation. In the present study the first instability regions of first stability approximation due to its importance were spotlighted. In this way a trigonometric function with full period was suggested to be as a possible solution. By replacing the trail function into the modified motion of equation with periodic time dependent initial load, two independent series were derived. Series could be separated due to sine and cosine terms which were independent functions. As far as these series contain infinite terms, a limitation should be selected for achieving the optimal solution. Moment gradient parameter $\psi$ and static buckling moment percentage $\lambda$, effects were shown on instability regions. For making simplifying graphs, frequency ratio, $\psi$ (the ratio of input frequency over either first lateral bending or torsional one regarding to the beam’s mode i.e. bending or torsional) was introduced.

7.5.1 Thesis III

III.a) Having more than two terms in series of trail sinusoidal function has no significant effect on the first instability regions of first stability approximation [47].
III.b) By selecting the dominantly bending mode, for the asymmetric moment, the beam is stable for any value of frequency ratio, $\phi$, while it is changing around the first instability region [47].

III.c) Instability regions exist for any amount of moment gradient parameter, $\psi$, if dominantly torsional mode is considered.

III.d) Increasing the share of static buckling moment percentage $\lambda$, in initial time periodic moment load causes the beam being more instable.

7.6 Dynamic stability analysis (damped)

The damping effect on stability regions was studied as the last part of this research. The solution process included a series of trigonometric functions with full period. By selecting different values for Rayleigh damping ratio, $\zeta$, and static buckling moment percentage $\lambda$, in initial time periodic moment load, these parameters influence on instability regions were elaborated.

7.6.1 Thesis IV

IV.a) While the Rayleigh damping coefficient, $\zeta$ is increased the stability of the beam with simply supported hinged at ends, subjected to time dependent non-uniform moment is improved [48].

IV.b) Considering the Rayleigh damping into account, has no influence on static buckling moment percentage, $\lambda$, effect on instability of the beam. Higher amounts of $\lambda$, reduces the stability of the beam.

7.7 Results utilization

The present study has been done as a fundamental research on the basis of mechanical modeling, mathematical and engineering calculations. It was followed by computer programming for evaluating equations. Results can be useful for industries that are needed to deal with periodic loadings in their product’s designing process. Particularly, public transportation vehicle structures producers, which are using beam elements. Current outcomes will be also valuable for commercial car chassis producer, airspace and cutting tool machine designers industries to have a better understanding of loading type effect.
In this part of the text necessary cross sections (CS1, CS2, CS3 and CS4) properties which were used for plotting the graphs in chapters 3 to 6 is expressed firstly. Subsequently extra graphs and plots for CS2 and CS3 cross sections will be given separately for each chapter.

### 8.1 Cross section properties

<table>
<thead>
<tr>
<th></th>
<th>CS1</th>
<th>CS2</th>
<th>CS3</th>
<th>CS4</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>h [mm]</strong></td>
<td>200</td>
<td>300</td>
<td>100</td>
<td>200</td>
</tr>
<tr>
<td><strong>b [mm]</strong></td>
<td>100</td>
<td>150</td>
<td>50</td>
<td>100</td>
</tr>
<tr>
<td><strong>t1 [mm]</strong></td>
<td>8.5</td>
<td>10.7</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td><strong>t2 [mm]</strong></td>
<td>5.6</td>
<td>7.1</td>
<td>4.5</td>
<td>7.5</td>
</tr>
<tr>
<td><strong>r [mm]</strong></td>
<td>12</td>
<td>15</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td><strong>Y_{cs} [mm]</strong></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td><strong>Z_{cs} [mm]</strong></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td><strong>A [mm^2]</strong></td>
<td>2848</td>
<td>5381</td>
<td>854.3</td>
<td>2873</td>
</tr>
<tr>
<td><strong>I_y [mm^4]</strong></td>
<td>19431689</td>
<td>83561132</td>
<td>1095874</td>
<td>14946503</td>
</tr>
<tr>
<td><strong>J [mm^4]</strong></td>
<td>68464</td>
<td>197538</td>
<td>895380</td>
<td>12171305</td>
</tr>
<tr>
<td><strong>I_w [mm^6]</strong></td>
<td>12746157814</td>
<td>124256407470</td>
<td>50300540</td>
<td>2702573497</td>
</tr>
<tr>
<td><strong>L_{cs} [mm]</strong></td>
<td>20855372</td>
<td>89598916</td>
<td>1463749</td>
<td>19999175</td>
</tr>
</tbody>
</table>

\[
\rho = 8 \times 10^{-9} \text{[Ns}^2/\text{mm}^4]\quad E = 2 \times 10^5 \text{[N/mm}^2]\quad G = 0.8 \times 10^5 \text{[N/mm}^2]\]
8.2 Chapter three graphs and tables

8.2.1 Critical length

![Graph showing ratio of first lateral bending frequency over first torsional one (CS2-CS3)](image)

Fig app - 1 Ratio of first lateral bending frequency over first torsional one (CS2-CS3)

8.2.2 Frequencies and critical buckling moment

<table>
<thead>
<tr>
<th>Frequency [Hz]</th>
<th>CS1</th>
<th>CS2</th>
<th>CS3</th>
<th>CS3</th>
<th>CS4</th>
<th>CS4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1\textsuperscript{st} bending</td>
<td>43.90</td>
<td>65.77</td>
<td>40.75</td>
<td>16298.02</td>
<td>82.34</td>
<td>8234.22</td>
</tr>
<tr>
<td>2\textsuperscript{nd} bending</td>
<td>175.60</td>
<td>263.09</td>
<td>162.98</td>
<td>65192.08</td>
<td>329.37</td>
<td>32936.89</td>
</tr>
<tr>
<td>1\textsuperscript{st} torsional</td>
<td>66.39</td>
<td>82.00</td>
<td>618.42</td>
<td>13195.58</td>
<td>617.16</td>
<td>6576.23</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Critical buckling Moment (\psi = -1) [kN.m]</th>
<th>CS1</th>
<th>CS2</th>
<th>CS3</th>
<th>CS3</th>
<th>CS4</th>
<th>CS4</th>
</tr>
</thead>
<tbody>
<tr>
<td>90.96</td>
<td>479.5</td>
<td>114.1</td>
<td>2434</td>
<td>1559</td>
<td>1661</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\psi = -1)</th>
<th>dominantly bending</th>
<th>dominantly bending</th>
<th>dominantly bending</th>
<th>dominantly torsion</th>
<th>dominantly bending</th>
<th>dominantly torsion</th>
</tr>
</thead>
</table>

Table app - 1 Frequencies and critical buckling moment for different cross sections
8.2.3  Moment gradient coefficient $C(\psi)$

$$v(x,t) = V(x)\sin\left(\frac{\pi x}{l}\right) + V_t(x)\sin\left(\frac{2\pi x}{l}\right)$$

**Lateral Bending $v$ (n=2)**

$$a(x,t) = F(x)\sin\left(\frac{\pi x}{l}\right)$$

**Torsional warping $\alpha$ (m=1)**

$$v(x,t) = V(x)\sin\left(\frac{3\pi x}{l}\right) + V_t(x)\sin\left(4\frac{\pi x}{l}\right)$$

**Lateral Bending $v$ (n=4)**

$$a(x,t) = F(x)\sin\left(\frac{\pi x}{l}\right)$$

**Torsional warping $\alpha$ (m=1)**

Table app - 2 Moment gradient coefficient ($M_{cr}/M_{cr0}$) for different cross sections while m=1, n=2

<table>
<thead>
<tr>
<th>$\psi$</th>
<th>CS1</th>
<th>CS2</th>
<th>CS3 (dominantly bending)</th>
<th>CS3 (dominantly torsion)</th>
<th>CS4 (dominantly bending)</th>
<th>CS4 (dominantly torsion)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.0</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>-0.9</td>
<td>1.0711</td>
<td>1.0711</td>
<td>1.0711</td>
<td>1.0711</td>
<td>1.0711</td>
<td>1.0711</td>
</tr>
<tr>
<td>-0.7</td>
<td>1.1521</td>
<td>1.1521</td>
<td>1.1521</td>
<td>1.1521</td>
<td>1.1521</td>
<td>1.1521</td>
</tr>
<tr>
<td>-0.6</td>
<td>1.2450</td>
<td>1.2450</td>
<td>1.2450</td>
<td>1.2450</td>
<td>1.2450</td>
<td>1.2450</td>
</tr>
<tr>
<td>-0.5</td>
<td>1.3521</td>
<td>1.3521</td>
<td>1.3521</td>
<td>1.3521</td>
<td>1.3521</td>
<td>1.3521</td>
</tr>
<tr>
<td>-0.3</td>
<td>1.4763</td>
<td>1.4763</td>
<td>1.4763</td>
<td>1.4763</td>
<td>1.4763</td>
<td>1.4763</td>
</tr>
<tr>
<td>-0.2</td>
<td>1.6206</td>
<td>1.6206</td>
<td>1.6206</td>
<td>1.6206</td>
<td>1.6206</td>
<td>1.6206</td>
</tr>
<tr>
<td>-0.1</td>
<td>1.7883</td>
<td>1.7883</td>
<td>1.7883</td>
<td>1.7883</td>
<td>1.7883</td>
<td>1.7883</td>
</tr>
<tr>
<td>0.1</td>
<td>1.9815</td>
<td>1.9815</td>
<td>1.9815</td>
<td>1.9815</td>
<td>1.9815</td>
<td>1.9815</td>
</tr>
<tr>
<td>0.3</td>
<td>2.4340</td>
<td>2.4340</td>
<td>2.4340</td>
<td>2.4340</td>
<td>2.4340</td>
<td>2.4340</td>
</tr>
<tr>
<td>0.5</td>
<td>2.6641</td>
<td>2.6641</td>
<td>2.6641</td>
<td>2.6641</td>
<td>2.6641</td>
<td>2.6641</td>
</tr>
<tr>
<td>0.6</td>
<td>2.8506</td>
<td>2.8506</td>
<td>2.8506</td>
<td>2.8506</td>
<td>2.8506</td>
<td>2.8506</td>
</tr>
<tr>
<td>0.7</td>
<td>2.9552</td>
<td>2.9552</td>
<td>2.9552</td>
<td>2.9552</td>
<td>2.9552</td>
<td>2.9552</td>
</tr>
<tr>
<td>0.9</td>
<td>2.9173</td>
<td>2.9173</td>
<td>2.9173</td>
<td>2.9173</td>
<td>2.9173</td>
<td>2.9173</td>
</tr>
<tr>
<td>1.0</td>
<td>2.7875</td>
<td>2.7875</td>
<td>2.7875</td>
<td>2.7875</td>
<td>2.7875</td>
<td>2.7875</td>
</tr>
</tbody>
</table>

Table app - 3 Moment gradient coefficient ($M_{cr}/M_{cr0}$) for different cross sections while m=1, n=4

<table>
<thead>
<tr>
<th>$\psi$</th>
<th>CS2</th>
<th>CS2</th>
<th>CS3 (dominantly bending)</th>
<th>CS3 (dominantly torsion)</th>
<th>CS4 (dominantly bending)</th>
<th>CS4 (dominantly torsion)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.0</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>-0.9</td>
<td>1.0710</td>
<td>1.0709</td>
<td>1.0706</td>
<td>1.0707</td>
<td>1.0706</td>
<td>1.0709</td>
</tr>
<tr>
<td>-0.7</td>
<td>1.1516</td>
<td>1.1513</td>
<td>1.1498</td>
<td>1.1505</td>
<td>1.1498</td>
<td>1.1511</td>
</tr>
<tr>
<td>-0.6</td>
<td>1.2434</td>
<td>1.2429</td>
<td>1.2385</td>
<td>1.2406</td>
<td>1.2385</td>
<td>1.2422</td>
</tr>
<tr>
<td>-0.5</td>
<td>1.3485</td>
<td>1.3473</td>
<td>1.3377</td>
<td>1.3423</td>
<td>1.3378</td>
<td>1.3460</td>
</tr>
<tr>
<td>-0.3</td>
<td>1.4692</td>
<td>1.4669</td>
<td>1.4484</td>
<td>1.4570</td>
<td>1.4485</td>
<td>1.4642</td>
</tr>
</tbody>
</table>

Lateral Bending $v$ (n=2)
8.2.4 Frequency and mixed mode ratio (CS3) (bending and torsional mode)

\[ \alpha(x,t) = F_x(t) \sin \left( \frac{\pi x}{I} \right) + F_y(t) \sin \left( \frac{2\pi x}{I} \right) \]

\[ F_x(t) \sin \left( \frac{3\pi x}{I} \right) \]

**Torsional warping \( \alpha \) \( m=3 \)**

\[ v(x,t) = V_x(t) \sin \left( \frac{\pi x}{I} \right) + V_y(t) \sin \left( \frac{2\pi x}{I} \right) \]

**Lateral Bending \( v \) \( n=2 \)**

\[ \alpha(x,t) = F_x(t) \sin \left( \frac{\pi x}{I} \right) + F_y(t) \sin \left( \frac{2\pi x}{I} \right) \]

\[ F_x(t) \sin \left( \frac{5\pi x}{I} \right) \]

**Torsional warping \( \alpha \) \( m=5 \)**

<table>
<thead>
<tr>
<th>( \psi )</th>
<th>CS1</th>
<th>CS2</th>
<th>CS3 (dominantly bending)</th>
<th>CS3 (dominantly torsion)</th>
<th>CS4 (dominantly bending)</th>
<th>CS4 (dominantly torsion)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.0</td>
<td>1.7657</td>
<td>1.7586</td>
<td>1.7058</td>
<td>1.7296</td>
<td>1.7061</td>
<td>1.7505</td>
</tr>
<tr>
<td>0.1</td>
<td>1.9445</td>
<td>1.9332</td>
<td>1.8527</td>
<td>1.8882</td>
<td>1.8532</td>
<td>1.9203</td>
</tr>
<tr>
<td>0.2</td>
<td>2.1425</td>
<td>2.1256</td>
<td>2.0111</td>
<td>2.0607</td>
<td>2.0118</td>
<td>2.1068</td>
</tr>
<tr>
<td>0.3</td>
<td>2.3539</td>
<td>2.3305</td>
<td>2.1790</td>
<td>2.2440</td>
<td>2.1800</td>
<td>2.3053</td>
</tr>
<tr>
<td>0.5</td>
<td>2.5651</td>
<td>2.5359</td>
<td>2.3522</td>
<td>2.4318</td>
<td>2.3536</td>
<td>2.5060</td>
</tr>
<tr>
<td>0.6</td>
<td>2.7515</td>
<td>2.7219</td>
<td>2.5198</td>
<td>2.6105</td>
<td>2.5212</td>
<td>2.6900</td>
</tr>
<tr>
<td>0.7</td>
<td>2.8715</td>
<td>2.8480</td>
<td>2.6560</td>
<td>2.7495</td>
<td>2.6574</td>
<td>2.8211</td>
</tr>
<tr>
<td>0.9</td>
<td>2.8778</td>
<td>2.8635</td>
<td>2.7076</td>
<td>2.7935</td>
<td>2.7089</td>
<td>2.8462</td>
</tr>
<tr>
<td>1.0</td>
<td>2.7517</td>
<td>2.7420</td>
<td>2.6116</td>
<td>2.6887</td>
<td>2.6128</td>
<td>2.7298</td>
</tr>
</tbody>
</table>

**Table app - 4 Moment gradient coefficient \( M_{\alpha}/\ell_{\text{inb}} \) for different cross sections while \( m=3, n=2 \)**

**Table app - 5 Moment gradient coefficient \( M_{\alpha}/\ell_{\text{inb}} \) for different cross sections while \( m=5, n=2 \)**

8.2.4 Frequency and mixed mode ratio (CS3) (bending and torsional mode)
8.2.5 Frequency and mixed mode ratio (CS2) (bending mode)

8.3 Chapter four graphs

8.3.1 Frequency and mixed mode ratio (CS3) (bending mode)
8.3.2 Frequency and mixed mode ratio (CS3) (torsional mode)

Fig app - 6 Frequency and mixed mode ratio $\zeta = 0.6$ (bending mode) (CS3)

Fig app - 7 Frequency and mixed mode ratio $\zeta = 0.8$ (bending mode) (CS3)

Fig app - 8 Frequency and mixed mode ratio $\zeta = 0.4$ (torsional mode) (CS3)
8.3.3 Frequency and mixed mode ratio (CS2) (bending mode)
8.4 Chapter five figures

8.4.1 Gradient moment factor effect on Instability regions

Fig app - 14 Instability regions for $\psi$ boundary values, $\lambda = 0.5$ (CS2) (bending mode)
Fig app - 15 Instability regions for different $\psi$, $\lambda = 0.5$ (CS2) (bending mode)

Fig app - 16 Instability regions for $\psi$ boundary values, $\lambda = 0.5$ (CS3) (bending mode)

Fig app - 17 Instability regions for different $\psi$, $\lambda = 0.5$ (CS3) (bending mode)
8.4.2 Static buckling moment percentage effect on Instability regions
Fig app - 21 Instability regions for different $\lambda$, $\psi = 1$ (CS2) (bending mode)

Fig app - 22 Instability regions for different $\lambda$, $\psi = -1$ (CS3) (bending mode)

Fig app - 23 Instability regions for different $\lambda$, $\psi = -1$ (CS3) (torsional mode)
8.5 Chapter six figures

8.5.1 Damping ratio and Gradient moment factor effect on Instability regions

Fig app - 24 Instability regions for different $\lambda$, $\psi = 1$ (CS3) (torsional mode)

Fig app - 25 Instability regions for different $\zeta$, $\psi = -1$ (CS1) (bending mode)

Fig app - 26 Instability regions for different $\zeta$, $\psi = -1$ (CS3) (bending mode)
Fig app - 27 Instability regions for different $\zeta$, $\psi = 1$ (CS3) (bending mode)

Fig app - 28 Instability regions for different $\zeta$, $\psi = -1$ (CS3) (torsional mode)

Fig app - 29 Instability regions for different $\zeta$, $\psi = 1$ (CS3) (torsional mode)
8.5.2 Static buckling moment percentage effect on damped Instability regions

Fig app - 30 Instability regions for different $\lambda$ values, $\psi$=-1, $\zeta$=0.1 (CS2) (dominantly bending)

Fig app - 31 Instability regions for different $\lambda$ values, $\psi$=-1, $\zeta$=0.1 (CS3) (dominantly bending)

Fig app - 32 Instability regions for different $\lambda$ values, $\psi$=-1, $\zeta$=0.1 (CS3) (dominantly torsional)
Fig app - 33 Instability regions for different λ values, ψ=1, ζ=0.02 (CS3) (dominantly torsional)
Bibliography


[56] F. Mohri, A. Brouki and J. C. Roth, "Theoretical and numerical stability analyses of unrestrained, mono-symmetric thin-walled beams," *Journal of Construct and Steel Research,*


