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Investigation of Multi-valued, Hysteresis-Type Nonlinearities in Numerical Field Problems

PhD Thesis

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Budapest, 2012
Acknowledgements

I would like to express my thanks towards University of Pécs, Pollack Mihály Faculty of Engineering and Information Technology for the financial support of this work. I am also very thankful for the help of the Department of Information Technology, the Head of Department Géza Várady and the former Head of Department Lajos Szakonyi, and for my colleagues Ildikó Jancskár, Adam Schiffer, and Attila Sipeky, who helped a lot with their insight, creativity and useful advices. I am very thankful to my supervisor Prof. Amalia Ivanyi as well, who helped me a lot, not only in understanding methods of scientific research and rigorous approach of problems, but to improve myself and overcome many difficulties along my first journeys into science. And last, but not the very least I would like to thank for the support and patience of my family, in particular my wife Ibolya, and my three sons Kornél, Dávid and Bálint.
Nyilatkozat

Alulírott Sári Zoltán kijelentem, hogy doktori értekezésemet magam készítettem és abban csak a magadott forrásokat használtam fel. Minden olyan részt, amelyet szó szerint, vagy azonos tartalommal, de átfogalmazva más forrásból átvettem, egyértelműen, a forrás megadásával megjelölttem.

Budapest, 2012.09....

Sári Zoltán

Tájékoztató

A jelen értekezésről készült hivatalos bírálatok, valamint a doktori munka védéséről készült jegyzőkönyv, a védést követően a Budapesti Műszaki és Gazdaságtudományi Egyetem Villamosmérnöki és Informatikai Karának Dékáni Hivatalában érhetők el.
(Budapest, XI. Magyar Tudósok körútja 2. Q ép. B szárny mfsz.)
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Chapter 1

Introduction

The analysis of nonlinear problems containing hysteresis operator, and the numerical solution of these kind of problems are deep and beautiful topics of applied engineering sciences. Recently there are many different hysteresis models with well known properties and behavior, which can be applied in numerous different fields of scientific research, or engineering, and there are well-established numerical methods as well, equipped with rigorous convergence results, which can also be applied for the solution of various classes of problems. Nevertheless approaching a numerical field calculation problem containing a hysteresis operator is still a challenging task, because the intersecting part of the above mentioned areas of hysteresis modeling and numerical solution methods is quite small, and the results concerning the applicability and stability of hysteresis models arising in these kind of problems are either restricted to very special cases, or too ‘strict’ in the sense, that mathematical results formulated for hysteresis operators mostly assume the operator being ‘smooth’ and ‘well-behaved’.

In my thesis I want to investigate a certain class of hysteresis operators in numerical field calculation problems. I want to carry out analytical and numerical examinations concerning the stability properties of the hysteresis operator itself and the numerical solution containing the operator, furthermore I want to develop and implement some improvements of the known models and methods concerning their stability, usability and application.

First of all, I want to develop a hysteresis model which is flawlessly applicable to field calculation problems. At first I describe the model philosophy in the frame of scalar hysteresis, which has all intrinsic features of hysteresis phenomena, and which is really helpful in numerical field calculation due to its compact representation and analytic nature, and later I extend it to a vector model. The model structure enables easy identification (measurement-based, or analytical) of the model parameters, while the measurement-based validation proves the accuracy of the model, making it a good candidate for application in various field calculation problems, where a hysteresis operator is needed. Though the
development of the model is based on generalized concepts from ferromagnetism, the proposed model is general, phenomenological one, thus the application possibilities are not restricted to the field of modeling the nonlinear behavior of ferromagnetic media.

In my investigations and numerical experiments I want to point out the possible stability problems arising during the application of hysteresis models in numerical field calculation, and intend to give suggestions to their solution. There are several physical problems described by nonlinear (partial) differential equations; indeed almost all of the real physical problems are described by nonlinear equations. Some of these nonlinear problems are nonlinear by nature, like fluid dynamics, some of them are nonlinear because nonlinear (inter-)dependencies of physical parameters. Regardless of the type of nonlinearity, the solution of nonlinear equations is not an easy task. Aside from a few very special cases, the analytical solution of nonlinear differential equations are not known and cannot even be derived by any known methods, thus the numerical solutions play a very important role in this field. In order to obtain the numerical solution of a nonlinear equation some form of discretization is needed, like discretization of the continuous differential equations (Finite Difference Method), or discretization of the continuous solution (Finite Element Method), and the discretized nonlinear equations has to be typically solved by some form of iteration. The iteration procedure can be stable, thus leading to a solution, or unstable, resulting in divergence, unwanted periodicity, or even chaotic behavior. Since stability (or instability) is a very important property of any iteration method, it is also important to develop methods of investigation of the behavior of these kind of iteration methods. In the case if the nonlinearity appears in the form of a multi-valued hysteresis operator the situation is even more difficult. The main aim of my research here is to investigate, analyze and improve the numerical solution method of a few special but important problems containing hysteresis operator, and to obtain some useful results concerning their stability properties.

Finally I have developed some application approaches of hysteresis models in numeric simulation in a few important areas such as modeling of the behavior of conducting magnetic material concerning penetration depth, and a two-phase flow problem with dynamic phase transition. Besides the case studies proving the applicability of the proposed hysteresis operator I have extended the formulation of the apparent heat capacity method for convective problems with an appropriate source term by the application of a hysteresis operator, and furthermore I will outline a specific engineering problem concerning the investigation of a two-phase flow network as an actual practical application of the hysteresis model developed.
Chapter 2

Statistical Approach of Scalar and Vector Hysteresis Modeling

2.1 Introduction to hysteresis and hysteresis modeling

In the past few decades hysteresis modeling has become an increasingly important and heavily researched field of applied engineering, mathematical and physical sciences. The main reason behind this is probably the extremely widespread demands of consideration of hysteresis in very diverse fields of science ranging from magnetism, magnetic materials and porous media flow through biology and other life sciences to even economics. This large diversity makes it very important to have a better understanding of hysteresis phenomenon itself and encourages of developing better modeling tools in order to apply hysteresis models in certain simulation tasks connected to hysteretic problems.

The term hysteresis means to lag behind and it originates from the ancient Greeks (800 BC). First pioneers of hysteresis modeling and studying of nonlinear material behavior were seemingly the early researchers of ferromagnetism, Weber [100], Wiedemann [101], and Maxwell [60] in the late 19th century, but the term was first introduced apparently by Ewing [27, 28]. The term hysteresis as it is used nowadays usually describes some kind of ‘memory’ dependence or lagging effect, which can be formally represented by a multi-valued nonlinear operator in mathematics. A typical hysteresis loop with simple branching can be seen in Fig. 2.1, representing the behavior of the hysteresis operator $v = \mathcal{H}\{u\}$.

Mathematical definition of a hysteresis operator in its most basic form describes hysteresis as a rate independent memory [5], having the following properties,

1. (memory) at any time instant $t$, the output $v(t)$ of the operator depends on
the time evolution of the input $u(t)$ and the initial state $v^0$ of the system

$$v(t) = [\mathcal{H}(u, v^0)](t) \quad \forall t \in [0, T],$$

where $\mathcal{H}(., v^0)$ is an operator acting between spaces of time-dependent functions, with the assumption of causality, thus for any $t \in [0, T]$, $v(t)$ is independent of $u|_{[t,T]}$.

2. (rate independece) the time evolution is invariant with respect to any increasing diffeomorphism $\varphi : [0, T] \to [0, T]$

$$\mathcal{H}(u \circ \varphi, v^0) = \mathcal{H}(u, v^0) \circ \varphi \quad \text{in } [0, T],$$

where the symbol $\circ$ denotes the composition of functions.

Though hysteresis effects and engineering applications with hysteresis was already well known in the late 19th century, - and active research is ongoing in the field since then - the rigorous mathematical description and analysis started in the 1970s, apparently with the works of Krasnoselskii, Pokrovskii and their co-workers. In these early papers and monographs [48] they have laid the base of today’s operator formalism and nonlinear functional analysis approach for the definition and examination of hysteresis.

### 2.1.1 Hysteresis models and applications

Taking a look at the palette of different scientific areas looking for hysteretic phenomena, there are numerous fields of applications where the occurrence of
hysteresis is quite obvious, just like ferromagnetism [42], elastic materials [50], or fatigue of mechanical structures [66], but on the other hand there are also many areas, where hysteretic phenomena are not so trivial at first glance, for example vapor-liquid phase transitions [3], porous media flow (soil hysteresis) [29], biology [59], neuroscience (hysteresis effects in muscles and tendons), economics [57] and the traffic analysis of computer networks [44]. Besides, that hysteresis can be interpreted as a general mathematical description of a large group of phenomena exhibiting this kind of nonlinear ‘branching’ behavior with memory, hysteresis phenomena is particularly important in the field of magnetism and magnetic materials. The exact and adequate modeling of ferromagnetic materials is required in very important applications. It is enough to think of transformer cores or steel laminations where the estimation of losses due to hysteresis can only be accomplished with the aid of some kind of hysteresis model representing the nonlinear lossy behavior of the core material [23], or studying of rotating machines where hysteresis losses also have a significant impact on the behavior of machines.

As it comes to analytical or numerical simulation of behavior of systems with hysteresis, the various hysteresis models certainly have their own pros and cons. One of the most important inherent properties of hysteresis models, which makes a huge difference during their implementation is the so called memory structure of the model. The concept of memory appears naturally, since hysteresis by definition is a kind of multi-valued nonlinearity, which is not only dependent on its current input and state, but it also depends on its prehistory as well (2.1). This feature greatly increases the difficulties of both the implementation of hysteresis models into numerical schemes and the adequate model constructions of hysteretic nonlinearities as well.

In any particular problem several hysteresis model can be choosen to accomplish a certain task, and the performance, flexibility or precision of the solution will strongly depend on the selected hysteresis model. As nowadays Finite Element Method (FEM) is the main numerical tool for solving field calculation problems, it became a more important feature of a hysteresis model to be ‘easily’ implementable into a FEM numerical frame. Considering memory handling and implementation efforts, some type of hysteresis models are extremely superior with respect to other models in FEM computations because of their structure and mathematical representation.

According to many directions of development of hysteresis models, several application areas and model requirements, several types of hysteresis models exists, and the classification of these models into categories can be accomplished in a large variety of ways according to the particular point of view on which the categorization is based. The following brief overview is mostly based on two fundamental monographs concerning hysteresis: Ivanyi [33] and Visintin [97].
Analytical models

The simplest analytical models are basically some kind of function approximations of the major and minor hysteresis curves, like step function approximation [16], application of idealized magnetic characteristics [104], power-series and Fourier series approximations [70], or approximation by transcendent functions [89]. Among these probably the oldest ‘real’ analytical model is the Rayleigh model [58] in which the magnetic permeability $\mu(H)$ is represented by the first two terms of a power series approximation

$$\mu(H) = \mu_{in} + \eta H,$$  \hspace{1cm} (2.3)

where $\mu_{in}$ is the initial permeability and $\eta$ is the Rayleigh-constant. This approach results in the following simple parabolic relation between the field strength $H$ and the flux density $B$ as follows

$$B = \mu_{in}H + \frac{1}{2}\eta H^2.$$  \hspace{1cm} (2.4)

It is very interesting to note that, the famous, widely used Preisach-model [61] defines a parabolic relation as well between its output and input in the case of applying constant-valued (‘flat’) weighting distribution function.

Dynamical models and the Langevin-theory

One of the most notable models in the group of dynamical models of hysteresis dates back to the end of the 19th century, and it has been constructed by the French physicist Pierre Duhem [25, 26]. The model has a simple analytical representation as the output $v(t)$ of the model can be obtained as the solution of the following Cauchy-problem

$$\frac{dv}{dt} = g_1(u, v) \left( \frac{du}{dt} \right)^+ - g_2(u, v) \left( \frac{du}{dt} \right)^- \hspace{1cm} (2.5)$$

where $g_1$ and $g_2$ are given positive functions corresponding to the rate of change of the increasing and decreasing branches (denoted by $^+$ and $^-$ respectively) of the hysteresis curve. The direction field defined by (2.5) gives the geometric interpretation of the possible paths of evolution of the hysteresis curves in the $(u, v)$ plane.

Besides the phenomenological approach represented by the Duhem-model, there are many other possible ways of examination of hysteresis (branching) phenomenon. A particularly interesting one which is based on the statistical physics description of magnetization [16, 24, 30, 88], resulting in the Langevin-theory.
Given a freely rotating dipole moment with magnetization $m_0$ situated in a magnetic field with given flux density $B$. The interaction energy of the dipole is $E(\theta, \phi) = -m \cdot B = -m_0 B \cos(\theta)$. In the state of thermal equilibrium the dipole has a probability of selecting one of the possible directions based on the Boltzmann statistics with probability density $P(\theta, \phi) = \exp(-\beta E(\theta, \phi))$, where $\beta = 1/T$ is the inverse temperature. The partition function of the system is defined as

$$Z = \int_0^\pi \sin(\theta) \int_0^{2\pi} P(\theta, \phi) \, d\phi \, d\theta = \frac{4\pi \sinh(\beta m_0 B)}{\beta m_0 B},$$

and the expected value $\bar{m}$ of magnetization can be obtained by

$$\bar{m} = \frac{1}{Z \beta} \frac{dZ}{dB} = m_0 \left( \coth(\beta m_0 B) - \frac{1}{\beta m_0 B} \right) = m_0 L(\beta m_0 B),$$

with the introduction of the Langevin function $L(x) = \coth(x) - 1/x$. From (2.7) assuming the linear dependence $M = \chi H = \chi B/\mu_0$, and applying the first term of Taylor-expansion of the Langevin function $L(x) = x/3 + O(x^3)$ with the substitution $M = \rho \bar{m} M / \rho = \frac{\beta m_0^2 \mu_0 M}{3 \chi} \rightarrow \chi = \frac{\beta m_0^2 \mu_0 \rho}{3} \propto \frac{1}{T},$

it follows the inverse temperature dependence of susceptibility $\chi$ known as Curie-law, which was investigated by Curie [19] for various paramagnetic substances.

Based on the Langevin-theory outlined above, Jiles and Atherton constructed a model [43] for the simulation of hysteresis characteristic of ferromagnetic materials taking into account the interactions between the domains of ferromagnetic substances (Weiss-correction). They have extended the improved Langevin theory with energy terms originating from domain wall movements, and the magnetization characteristic in their model is derived by decomposition of magnetization process into a reversible and an irreversible component and applying the concept of differential susceptibility for the dynamic representation [42].

### Preisach-type models

The most well known, and surely the most widely applied models are the variants of the classical Preisach-model of hysteresis. The basic form of this model had been lay down by F. Preisach in his landmark paper [68] in 1935. This model

---

1Sum of all of the Boltzmann factors $\exp(-\beta E_s)$ corresponding to possible microstate energies $E_s$ of an ensemble (system)

2Since generally, if the energy of microstates $E_s$ depends on a quantity of interest, say $A_s$ for all microstates like $E_s = E_s^0 + \lambda A_s$, then expected value of $A$ can be found by $A = -\frac{1}{Z} \frac{dZ}{dA}$. 

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gives a very large degree of freedom and a quite simple reversal mechanism during the construction of branching hysteresis curves. The behavior of the model is basically defined by the following expression

\[ M(t) = \int \int_{h_a \geq h_b} P(h_a, h_b) \gamma_{h_a, h_b} |H(t)| \, dh_a \, dh_b, \quad (2.9) \]

where \( \gamma_{h_a, h_b} \) is the elementary hysteresis operator (hysteron\(^3\)), and \( P(h_a, h_b) \) is the weighting function or Preisach-function\(^4\).

The minor curves are defined by the aid of a special phase space, the so-called Preisach-plane or Preisach-triangle on which the magnetization process is followed. The Preisach-model has infinite memory structure, which is represented by a segmented broken line in the Preisach-plane, this line separates the region of the magnetized and the demagnetized elementary relay operators.

There exist many other variants of this model which were derived from the classical Preisach-model. The detailed description of these Preisach-variants can be found in [61] among several other sources. One of the main drawbacks of the application of Preisach-type models in numerical computations is the difficult handling and large memory consumption of the memory of the model. This feature can be a very serious restriction of solving nonlinear problems with this hysteresis model, especially in the case of larger two- or three-dimensional simulations, or in the case of convection-dominated problems, like fluid flow.

### 2.2 Aim of the thesis

The main aim of my research is the development of a phenomenological model of scalar hysteresis which possesses the following important properties:

- relatively simple mathematical description,
- easy identification procedure,
- easy implementation into numerical field calculation problems.

I propose a scalar hysteresis model based on statistical considerations, and I give a possible vector extension of the proposed model as well. I want to demonstrate the behavior and applicability of the model, the identification procedure, and I want to prove how the model can be a practical tool of modeling hysteretic phenomena.

\(^3\)A hysteron can be interpreted as a non-ideal relay with two separate switching points \( h_a \) and \( h_b \) for OFF=0 and ON=1 switching respectively.

\(^4\)It can be interpreted as a probability density function corresponding to the distribution of the various kinds of hysterons with different switching points
2.3 The statistical approach

Capturing the fundamental concepts of hysteretic behavior, and taking into account the most important properties of hysteresis operators, I develop a hysteresis modeling approach based on statistical considerations, which finally evolves into an improved variant of the general Duhem model of hysteresis \cite{26, 97} maintaining the advantageous property of ODE representation and extending the scope of the model by improving the characterization of the shape of the minor loops based on measurement data, while preserving the simplicity of the model structure as much as possible. The statistical model is developed here by the purpose of obtaining a model which is easy to implement into various numerical schemes and approximates real material behavior very well in the range of acceptable tolerances of engineering applications. The model construction is a quite simple approach of hysteresis phenomena, but finally it results in a versatile and sophisticated hysteresis model, particularly taking into account that the identification of model parameters is very easy, the measurement-based validation of the model shows very good agreement with actual measured behavior, and the vector extension of the model can be constructed naturally.

The basic principles of the developed hysteresis model are defined using terms of ferromagnetic hysteresis, but the very same concepts can be applied in the case of any other system consisting of elementary bistable units interacting with each other. This approach can be extremely useful for the modeling and implementation of hysteresis phenomena in a very widespread area. As an unusual example from economy, in the case of a firm, which is continuously losing and gaining customers, each customer can be regarded as an elementary unit, a ‘hysteron’ with two switching values corresponding to gaining or losing a kind of loyalty towards the firm, and the collective behavior of customers can be handled as a statistical ensemble. An other example can be a fluid system, where a sufficiently small volume of the fluid (a so called ‘cluster’) \cite{3, 39} can be handled as an elementary unit considering the non-equilibrium vapor-liquid first order phase transition with some degree of supersaturation.

2.4 The main concept of the model

As simplified magnetic domain/cluster of a ferromagnetic medium can be represented with a delayed relay operator\footnote{Elementary bistable unit.} \cite{97} with two states corresponding to the magnetized and the demagnetized state as it can be seen in Fig. 2.2. Since the domains differ a bit from each other, the actual value of the switching field \( H^+, H^- \) (up/down) is different for each of them, but it can be assumed, that it is statistically distributed among the domains of the material and the expected value of the switching field is the same for all of them. Under the above
assumptions the switching field can be represented by a random variable from an arbitrary distribution with expected value \( \tilde{\mu} \) and standard deviation \( \tilde{\sigma} \). Since the process of magnetization affects many domains with the same expected value of the switching field the net effect can be represented by a Gaussian random variable, because central limit theorem states that the distribution of the sum or the average of the Independent, Identically Distributed (IID) random variables is the normal or Gaussian distribution.

Let \( X_i \sim D(\tilde{\mu}, \tilde{\sigma}) \) be a random variable from an arbitrary distribution \( D \), corresponding to the switching field \( H^+ \). Then \( X_i \in [-H_s, H_s] \), where \( H_s \) is the saturation field strength\(^6\) and the expected value \( \mathbb{E}[X_i] = \tilde{\mu} \) with standard deviation \( \tilde{\sigma} \). Assuming that \( X_i \)'s are IID, the average of them is a Gaussian random variable \( \bar{X} \sim N(\mu, \sigma) \). It means that the net effect of the magnetization process can be described by the Cumulative Distribution Function (CDF) of the normal distribution. Applying the concept with various number of individual domains with random switching fields can be seen in Fig. 2.3 with 200 switching elements (domains) and in Fig. 2.4 using 2000 relay elements. The parameters of the model at this level are \( \mu \) corresponding to the coercive field \( H_c \), and \( \sigma \) which depends on the maximal susceptibility \( \chi_{\text{max}} \).

Looking at Fig. 2.3, it can be found, that the ‘structure’ of the curve is very similar to a magnified portion of a hysteresis curve showing the Barkhausen-jumps (or Barkhausen-noise) suggesting, that this kind of representation has a relation to the physical origins of hysteresis phenomena as well. This statistical description can be useful in examining of other avalanche-like phenomena and investigation

\(^6\)Theoretically the random variable lies in the interval \([-\infty, \infty]\), but in the case of practical applications the model input is always finite.
of hysteresis could be extended to various areas of scientific research just like the possible connection of earthquakes, Barkhausen-noise and hysteresis [20].

Fig. 2.3. A CDF corresponding to a major hysteresis curve built from 200 individual relays with random switching fields

Fig. 2.4. A CDF corresponding to a major hysteresis curve built from 2000 individual relays with random switching fields
2.4.1 The reversal mechanism

The above pair of parameters \((\mu, \sigma)\) is enough to determine the major hysteresis loop with a strong saturation effect. The reversal curves can be determined in the following way.

Magnetization from negative to positive saturation means nothing more than switching the states of all the domains. According to the concepts above, this process can be described in a purely statistical way, and from this point of view the starting point of the magnetization determines the whole magnetization process to the saturation. It means that a certain hysteresis curve is uniquely characterized by the last reversal point \((H_r, M_r)\), where the direction of the applied field \(H\) changes, thus the model has local memory.

For example if the magnetization process starts from negative saturation \((-H_s, -M_s)\) and the direction of the applied field \(\delta \in \{-1, 1\}\) is positive, accessing a certain level of \(M(t)\) by applying a given, monotonous \(H(t)\) in a given time instant means the switching of the state of a certain percent of domains. Thus the actual \(M(t)\) is proportional to the value of cumulative distribution function of \(\overline{X}\) for the given \(H(t)\). If the direction of the applied field changes at this certain level of magnetization, the process is very similar except that demagnetization corresponds only to those domains which have been magnetized before, because the ones that have not been magnetized are already in the desired state. It follows that a given \((H_r, M_r)\) reversal point and the direction \(\delta\) of the actual applied field determines the cumulative distribution function, which describes the hysteresis curve from \((H_r, M_r)\) to \((\delta H_s, \delta M_s)\) saturation point.

2.4.2 Formulating the model

Let the cumulative distribution function of normal distribution with parameters \((\mu, \sigma)\) be the following:

\[
F(x) = \frac{1}{\sqrt{2\pi}\sigma^2} \int_{-\infty}^{x} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) dt, \quad (2.10)
\]

where \(\sigma\) can be calculated from the maximal susceptibility \(\chi_{\text{max}}\), and \(\mu = H_c\). The dependence of \(\sigma\) on \(\chi_{\text{max}}\) can be formulated according to the fact that the maximum of the derivative of \(F(x)\) - denoted by \(F'(x)\) - is \(\chi_{\text{max}}\), and from this it follows that

\[
\sigma = \frac{1}{\chi_{\text{max}} \sqrt{2\pi}}. \quad (2.11)
\]

Substituting the actual applied field into \(F(x)\), (2.10) gives the percent of the domains that changed their states under the applied field \(H\), thus \(F(H) \in [0, 1]\) is the ‘magnetization ratio’ which is denoted by \(M^*\). From this function the
actual value of $M$ can be calculated in view of the $M_s$ saturation magnetization\footnote{In the case of simulated hysteresis loops ie. $M_s \approx 1.6 \cdot 10^6 [\text{A/m}].$} by a simple linear transformation as $M = 2M_s M^* - M_s.$

\begin{align*}
M &= 2M_s M^* - M_s \tag{2.12}
\end{align*}

Because of $H$ field reversal the expected value of $X$ changes to $-\mu$ since the switchings now will take place around the $H$ switching point, and (2.12) takes

\begin{align*}
M_r^*(H_r) &= \frac{1}{\sqrt{2\pi}\sigma^2} \int_{-\infty}^{H_r} \exp \left( \frac{-(t - \mu)^2}{2\sigma^2} \right) dt.
\end{align*}

In the case of field direction reversal, the area under the probability density function of $X$ on the interval $[-\infty, H_r]$ gives the ratio of the magnetized domains

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{field_reversal.png}
\caption{The field reversal mechanism}
\end{figure}
the form
\[ M'_r(H_r) = \frac{\xi}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{H_r} \exp\left(-\frac{(t + \mu)^2}{2\sigma^2}\right) dt, \]  
(2.13)

where \( \xi \) is needed to maintain the level of magnetization at the reversal point. After calculating \( \xi \) from the fact, that the right hand sides of (2.12) and (2.13) are equal
\[ \xi = \frac{\int_{-\infty}^{H_r} \exp\left(-\frac{(t - \mu)^2}{2\sigma^2}\right) dt}{\int_{-\infty}^{H_r} \exp\left(-\frac{(t + \mu)^2}{2\sigma^2}\right) dt}, \]  
(2.14)

the hysteresis curve can be described by
\[ M''_{Hr,Mr,\delta}(H) = \frac{\xi_{Hr,Mr}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{H} \exp\left(-\frac{(t - \delta\mu)^2}{2\sigma^2}\right) dt. \]  
(2.15)

Fig. 2.5 shows the hysteresis curves (scaled to \( M_s \)) and the corresponding Probability Density Functions (PDFs) during the reversal of the direction of the applied field. The area under the PDFs in the lower figure is proportional to the value of magnetization in the upper figure. It can be seen in the figure, that as the reversal point is reached, the flipped, scaled PDF is used to calculate the reversal magnetization curve according to (2.13) and (2.14).

2.4.3 Behavior of the model

In this section some test results are presented to demonstrate the behavior of the hysteresis model developed and described above. The results presented here first, are derived from the simple, two-parameter \((\mu, \sigma)\) model. Fig. 2.6 shows simulated first order reversal loops. From the figure it can be seen that the model is capable to capture the most important properties of hysteresis: saturation and branching.

The accommodation property of the model can be seen in Fig. 2.7. Accommodation is a real material property, and there are experimental observations of accommodation [22] in ferrites and soft magnetic materials. Accommodation comes into effect when the amplitude of the excitation field is smaller than the width of the major loop. The larger the amplitude of the excitation the faster the hysteresis loop becomes stable, as it can be seen in Fig. 2.7 the minor loop is shifted upwards and becomes stable after a few cycles. Since the hysteresis loop is derived from the CDF of the normal distribution the model exhibits strong saturation and accommodation properties as the figures suggest.

The minor loops simulated by the model developed are non-congruent loops, which is the kind of behavior that can be shown experimentally [96]. The

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8An in-depth examination concerning stability of minor loops is discussed in Chapter 3.
developed model thus holds all important characteristic features of scalar hysteresis with non-congruent minor loops, any kind of reversal loops and furthermore the model exhibits accommodation property for excitations smaller than $H_s$ ‘saturation field’, the excitation required for saturated state. Fig. 2.8 shows first order reversal curves and concentric minor loops generated by the model.

The accommodation property of the model can be seen in Fig. 2.9 for different past histories of the material. The first curve is generated from a demagnetized starting state, the second one is constructed after negative saturation state
branching the minor loop from the ascending major curve.

The simulation results obtained by the above model seem to be appropriate, however the model is not versatile enough since all the magnetic domains/clusters are supposed to be uniform (which is certainly a coarse assumption).

### 2.5 Extending the scope of the model

The developed model is capable of describe hysteretic behavior, but with the limitation, that the major hysteresis curve is represented by the CDF of the normal distribution, thus the shape of the hysteresis loop is quite limited. One step forward to make the model much more customizable is, that a certain level of deviation from the above described ‘common’ expected value of switching field is enabled. Assuming that the magnetic moments of the units are not the same for all domains, the distribution of moments can also be described by some kind of statistical distribution. The most straightforward approach is to apply the normal distribution for this purpose as well. Let $Y \sim N(\mu_2, \sigma_2)$ be a random
variable corresponding to the distribution of magnetic moments of the domains. The statistical approach can be further extended by considering the consequence of the distribution of moments, thus the applied magnetic field will force the ‘weakest’ domains first to change their states and as the magnetic field strength increases further, the domains will change their states according to their moments.

Since the switching fields and magnetic moments are described by distribution functions the magnetization process can be modeled by embedding these distributions into each other. The construction can be formulated as

\[
F(x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \int_{-\infty}^{x} \exp\left(-\frac{(t - \mu_1)^2}{2\sigma_1^2}\right) dt, \tag{2.16}
\]

\[
\hat{F}(x) = -\cot(\pi F(x)), \tag{2.17}
\]

\[
G(x) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \int_{-\infty}^{\hat{F}(x)} \exp\left(-\frac{(t - \mu_2)^2}{2\sigma_2^2}\right) dt, \tag{2.18}
\]

where the intermediary equation (2.17) is required to map \( F(x) \) from the interval \([0, 1]\) to \( \hat{F}(x) \in [-\infty, \infty] \). The parameters \( \mu_1, \sigma_1 \) and \( \mu_2, \sigma_2 \) are correspond to the distributions of the switching fields and the moments of domains/clusters respectively.

This extension is a possible way of enabling the representation of asymmetric hysteresis loops by the model, since the indirection provided by the composition of the distributions yields the possibility of having major loop shapes deviating from the shape of normal CDF originating from the basic form of the model.

The magnetization ratio in the case of uniform magnetic moments can be calculated by (2.16), and this value can be fed into the second distribution function \( G(.) \) corresponding to random variable \( Y \) representing the distribution of the magnetic moments. Substituting the actual applied field \( H \) into the \( G(x) \) function gives the magnetization ratio \( M^*(H) = G(H) \in [0, 1] \) based on the assumption that the distribution of the magnetic moments is the normal distribution. At this point the actual value of magnetization can be calculated from the magnetization ratio \( G(H) \) in view of the \( M_s \) saturation magnetization by a scaling, as it has been done before in the case of the simple two-parameter model. Since the resulting distribution can be asymmetric, in the case of field reversal the probability density function has to be flipped from left to right instead of the changing the expected value only.

### 2.5.1 Parameter identification

Identification of the model parameters is untroubled for the model structure above. In the case of the simple model the determination of the values of parameters \( (\mu, \sigma) \) goes in a straightforward manner because \( \mu = H_c \) and \( \sigma \) can

\(^9\)‘Weakest’ means that the smallest amount of energy needed to change its state.
be calculated from the maximum susceptibility $\chi_{\text{max}}$ as it was described earlier (2.11). The extended model has four parameters and the values of parameters are not so trivially related to the shape of the hysteresis curve as in the case of the former model. Let $\eta = [\mu_1, \sigma_1, \mu_2, \sigma_2]$ be a vector containing the four parameters of the model. The identification procedure can be carried out in view of coercive field $H_c$, remanent magnetization $M_{\text{rem}}$, maximum susceptibility $\chi_{\text{max}}$, and susceptibility at point $(H = 0, M = M_{\text{rem}})$ denoted by $\chi_{\text{rem}}$. The following non-linear system of equations can be constructed for these values and vector $\eta$

\begin{align*}
G(x = H_c, \eta) &= \frac{1}{2}, \\
G'(x = H_c, \eta) &= \frac{\chi_{\text{max}}}{2M_s}, \\
G(x = 0, \eta) &= \frac{M_{\text{rem}}}{2M_s}, \\
G'(x = 0, \eta) &= \frac{\chi_{\text{rem}}}{2M_s}.
\end{align*}

(2.19)

Since the function $G(H)$ describes the shape of the major hysteresis curve, the above system is defined based on the characteristic properties of this curve. The first equation covers the case, when the field excitation is equal to the coercive field $H_c$ and the level of magnetization is exactly half way between the positive and negative saturation values corresponding to the numerical value of $1/2$, because the $G(H)$ curve is normalized to $[0, 1]$. Each of the remaining equations also correspond to one characteristic feature of the major curve in a similar fashion.

The non-linear system (2.19) can be handled as a non-linear minimization problem and can be solved for example by trust region methods [63]. The solution of the system is the $\eta$ parameter vector which contains the parameters of the hysteresis model.

Fig. 2.10 shows the behavior of the model for the following set of properties of the major hysteresis loop

\begin{align*}
H_c &= 1500 \text{ A/m}, \quad M_{\text{rem}} = 1.325 \cdot 10^6 \text{ A/m}, \\
\chi_{\text{max}} &= 1500, \quad \chi_{\text{rem}} = 500,
\end{align*}

(2.20)

and resulting parameter vector

\begin{align*}
\eta &= \begin{bmatrix} -1411.7, \ 3660.5, \ 1.2629, \ 0.5511 \end{bmatrix}.
\end{align*}

(2.21)

From the figure it can be seen that applying the modified distribution function enables that the shape of the hysteresis loop can be quite different from the ones created by the CDF of the normal distribution. In this way the model becomes more versatile and wider variety of hysteresis loops can be modeled.
2.6 Further generalization of the model

Since the model applies a distribution function to form the major hysteresis loop, the model can be generalized further by substituting the cumulative distribution function by a properly scaled measured major hysteresis loop. This generalization will allow to simulate practically any kind of hysteresis loops, and makes the measurement-based identification extremely easy.

This natural further generalization step leads to the building of the model from not only a nested distribution (2.18), but a linear combination of several distributions. This step is basically an other way of introducing, that the domains are not uniform, but they belong to a certain number of different ‘types’ or ‘categories’. This extension gives even more versatility for the model as it is demonstrated in the followings.

The hysteresis loop in this case is built from superposed normal cumulative distribution functions (CDFs), which can be formalized as follows

\[
F_i(x) = \frac{1}{\sqrt{2\pi}\sigma_i^2} \int_{-\infty}^{x} \exp \left( \frac{-(t - \mu_i)^2}{2\sigma_i^2} \right) dt, \tag{2.22}
\]

\[
G(x) = \frac{\sum_{i=1}^{n} w_i F_i(x)}{\sum_{i=1}^{n} w_i}, \tag{2.23}
\]

where the functions \(F_i(x)\) are the individual CDFs with their own set of parameters \(\mu_i\) and \(\sigma_i\), and \(G(x)\) is the weighted sum of them, since the characterizing function of the model is a set of cumulative distribution functions from normal distribution according to the philosophy of the model. Just like in the previous cases, \(M^*(H) = G(H)\) from which the actual value of magnetization
can be obtained by scaling to \([-M_s, M_s]\). In most of the practical situations it is required to apply not only one CDF, since the shape of it is rather limited due to the fact that it has only two parameters \(\mu\) and \(\sigma\). As it can be seen in Fig. 2.11 the shape of the measured hysteresis curve cannot be fit properly with only one CDF.

Fig. 2.11. The process of identification

According to the approach of constructing the hysteresis curve from weighted sum of normal CDFs, as it has been formalized in (2.23), a much better fit can be achieved. In the case of this particular identification, combining three distribution functions is enough. The actual number of CDFs describing the hysteresis curve depends on the shape of the hysteresis curve\(^{10}\), and the desired accuracy of the fit to be achieved. The balance of these two factors affects the size of the final set of CDFs required to construct the hysteresis model. In order to identify the model parameters, the following optimization has to be carried out for function \(G(x)\) defined in equation (2.23)

\[
\eta_{opt} = \min_{\eta} ||G(H_m, \eta) - M^*_m||, \tag{2.24}
\]

where \(\eta = [\mu_1, \ldots, \mu_n, \sigma_1, \ldots, \sigma_n, w_1, \ldots, w_n]\) is the vector of the parameters, \(H_m\) is the measured magnetic field strength, \(M^*_m\) is the measured normalized magnetization curve, a major curve ranging from the negative to the positive saturation.

Normalization is required because the model - and also the identification procedure - handles the major hysteresis curve as a weighted sum of CDFs thus its values must be in the interval \([0, 1]\).

2.7 Differential description of the model

In this section the mechanisms and some supporting functions of the model are described in details on an ODE-based approach, where the model is described

\(^{10}\)The important factor is basically the ‘distance’ of major curve from the CDF of normal distribution.
by an differential equation. This kind of representation can be extremely useful during the implementation of the hysteresis model into actual field calculation problems.

### 2.7.1 The magnetization process

The process of magnetization can be described by the model in the form of a differential equation formalized for the magnetization probability $M^* \in [0, 1]$ based on the statistical distributions corresponding to the model. The main assumption is that the value of magnetization during a full magnetization cycle can be determined by the CDF of the model, as it was already described earlier. Under this assumption the actual value of magnetization can be calculated from the CDF, which can be handled as the normalized magnetization

$$
M^*(t) = \frac{M(t) + M_s}{2M_s} \in [0, 1],
$$

where $M(t)$ is the actual value of magnetization in [A/m] and $M_s$ is the saturation magnetization also in [A/m].

Introducing the field strength in the direction of increasing

$$
\tilde{H}(t) = H(t)\delta(t),
$$

where $\delta(t) : [t_0, t_E) \rightarrow \{-1, 1\}$ is the direction of the increasing of the applied field defined as

$$
\delta(t) = \text{sgn}\left(\frac{dH(t)}{dt}\right),
$$

where $H(t)$ is the applied actual magnetic field, the CDF of the hysteresis model can be interpreted as

$$
G(\tilde{H}(t)) = \begin{cases} 
P(m_i = m^+ | H(t)), & \text{if } \delta(t) = 1, \\
P(m_i = m^- | H(t)), & \text{if } \delta(t) = -1, \\
\forall i,
\end{cases}
$$

where $P(m_i = m^+ | H(t))$ is the probability of that the $i$-th magnetic moment is in magnetized state under the applied field $H(t)$, in other words $G(\tilde{H}(t))$ represents that whether the state of the moment is magnetized ($m^+$), or demagnetized ($m^-$).

Since the above defined $M^*(t)$ ‘probability’ of magnetization has its values in the interval $[0, 1]$ for any $t$, it can be substituted as one of the probabilities in (2.28) corresponding to one particular direction as follows

$$
G(\tilde{H}(t)) = \begin{cases} 
M^*(t), & \text{if } \delta(t) = 1, \\
1 - M^*(t), & \text{if } \delta(t) = -1,
\end{cases}
$$

(2.29)
according to the fact that 
\[ P(m_i = m^+|H(t)) = 1 - P(m_i = m^-|H(t)). \]
Calculating the time derivative of (2.29) yields
\[
\frac{dG(\tilde{H}(t))}{dt} = \begin{cases} 
\frac{dM^*(t)}{dt}, & \text{if } \delta(t) = 1, \\
-dM^*(t)/dt, & \text{if } \delta(t) = -1,
\end{cases} \tag{2.30}
\]
which can be written as
\[
\frac{dG(\tilde{H}(t))}{dt} = \frac{dM^*(t)}{dt} \delta(t). \tag{2.31}
\]
Applying the chain rule and expanding results in
\[
g(\tilde{H}(t)) \frac{d\tilde{H}(t)}{dt} = \frac{dM^*(t)}{dt} \delta(t), \tag{2.32}
\]
where \( g(\tilde{H}) = dG(\tilde{H})/d\tilde{H} \) is the PDF corresponding to the model.

From this, the following form of the differential equation of the magnetization probability can be derived
\[
\frac{dM^*(t)}{dt} = g(\tilde{H}(t)) \frac{d\tilde{H}(t)}{dt} \delta(t), \tag{2.33}
\]
which applying the denoted substitution yields the following equation
\[
\frac{dM^*(t)}{dt} = g(\tilde{H}(t)) \frac{dH(t)}{dt} \tag{2.34}
\]
The solution of the above differential equation describes the time evolution of magnetization in full magnetization cycles. Considering minor hysteresis loops during the magnetization process requires extending of (2.34) with an additional term \( \gamma(t) \), which maintains the proper level of magnetization in minor loops based on the philosophy of the model in the case of reversals as it has been introduced above. This extended form of the differential equation can be written as
\[
\frac{dM^*(t)}{dt} = \gamma(t) g(\tilde{H}(t)) \frac{dH(t)}{dt}, \tag{2.35}
\]
where
\[
\gamma(t) = \frac{\tilde{M}(t)}{1 - G(H(t))}, \tag{2.36}
\]
containing \( \tilde{M}(t) \) the magnetization probability in the direction of the increasing field defined as
\[
\tilde{M}(t) = -\delta(t)M^*(t) + \frac{1 + \delta(t)}{2}. \tag{2.37}
\]
The function $\gamma(t) \equiv 1$ on the major loop, so adding an exponent to it enables to construct different sets of minor loops to the same major curve, thus it results in more freedom, and leads to a more general form of the model as follows

$$\frac{dM^*(t)}{dt} = \gamma^q(t) g(\tilde{H}(t)) \frac{dH(t)}{dt},$$  \hspace{1cm} (2.38)

where the exponent $q$ can be a constant or even a function of $H$ or $M$ giving a quite large versatility for the model. The measurement based identification considering this additional parameter is a simple two step process. The first step is to identify the model parameters corresponding to the major hysteresis curve, the second step is to find the proper value of $q$ by the minimization of error between a set of measured and simulated minor hysteresis curves (first order reversal curves or concentric loops). This model structure and identification procedure is a unique feature of this model, providing very high level of versatility and utility in actual engineering applications regarding both the identification and implementation as well.

The above described model of hysteresis is capable of modeling non-congruent minor loops, accommodation property as it was demonstrated before, and it is very easy to implement into field calculation problems as it is shown in the following chapters. In order to obtain the solution of (2.38) the only quantities need to be known are the initial value of magnetization $M^*(t_0)$ and the field excitation $H(t)$.

### 2.8 Validating the model with measurement data

The model developed in the preceding sections describes hysteresis loops very well. In this section it will be proven that the model can be fit to measured data, and it is capable of modeling the behavior of the actual magnetic specimen, which the identification is based on.

In order to identify the model parameters by the minimization (2.24), first of all the measured increasing major curve has to be normalized to $[0, 1]$ since the model handles this curve as the superposition of normal CDFs with varying set of parameters ($\mu_i, \sigma_i$ and $w_i$ grouped into the vector $\eta$). The normalized major curve is a single valued function which is basically approximated by the linear combination of CDFs of the model. In this way the major curve can be approximated very accurately, and based on the major curve, by the aid of the reversal mechanism of the model, simulations can be carried out. Recalling the model differential equation

$$\frac{dM^*(t)}{dt} = \gamma^q(t) g(\tilde{H}(t)) \frac{dH(t)}{dt},$$  \hspace{1cm} (2.39)
it can be seen that there is an ‘shaping’ parameter $q$. Substituting different values in $q$ results in different minor loop shapes. As it has been mentioned before, the parameter $q$ does not affect the shape of the major curve since the function $\gamma(t)$ is identically one on the major curve, thus the parameter $q$ can be utilized to set the shape of the minor curves only.

As Fig. 2.12 shows, the identification procedure according to (2.24) results in a set of weighted PDFs, which will represent the shape of the major hysteresis curve on the whole interval based on the measured curve. It is clear from the figure, that the PDFs with highest weights are centered around the coercive field value (see Fig. 2.13 for reference), but it is also apparent, that there are other density functions as well, shifted from the coercive field having significant weight to contribute to the linear combination of CDFs describing the major hysteresis curve. Fig. 2.12 provides an intuitive insight to the model philosophy showing how much, and which way each of the components affects the final hysteresis curve. In addition to the identification of the major hysteresis curve, the shape of the minor loops can be fit by the aid of parameter $q$ of the hysteresis model with another minimization. The value of $q$ in this case (see Fig. 2.13 and Fig. 2.14) was found to be $q = 2.54$.

A set of measured and simulated first order reversal curves can be seen in Fig. 2.13 from the middle region of the major hysteresis loop. The upper figure shows the normalized magnetization versus time curves, both the measured and simulated ones, and it is clear from the figure, that the simulation follows the measurement very well. The lower figure contains the hysteresis curves in the usual $M-H$ plane.

It can be seen from the figure, that the simulation approximates the
Fig. 2.13. Comparison of a set of measured and simulated reversal curves in the middle region of the hysteresis loop

measurement very accurately, and it can also be noticed that the largest deviation of the simulated curves from measured ones is in the reversal region where eddy current phenomena affects the measurement. The reason behind the larger deviation in this region is that the hysteresis model is rate independent, and does not take dynamic eddy current effects into account.

The behavior of the model is also very good in the case of smaller reversal curves. Fig. 2.14 shows a set of small first order reversal curves, where the size of the minor loops are less than 10% of the size of the major hysteresis curve. It can be seen from the figure that the simulated minor loops almost coincide with the measured ones, and it is also interesting to notice that in this region the eddy current effects are negligible.
current effects does not affect the loops that much, since the magnitude of eddy currents is proportional to $dB/dt$ which is not particularly large in this region.

Fig. 2.15 shows the relative error - in percentage of the magnitude of the major loop - between the measured and simulated magnetization curves versus time. It can be clearly seen from the figure, that the accuracy of the simulation is fairly high, it remains below 5% even in the region of higher eddy currents. The periodic nature of the error plot reflects the periodic excitation, where the highest error values coincide with highest values of $dB/dt$. For smaller first order reversal curves (smaller than one tenth of the largest curve) the accuracy is more superior, showing, that the model is capable of handling not only the major curve, but small minor curves as well due to the model construction and
identification procedure. It has to be emphasized here, that the identification is based on the *major hysteresis curve only*, and the shaping of the minor loops is handled by one scalar parameter $q$, so the identification is not a curve fitting procedure and the accurate approximation suggests, that the developed model describes hysteretic behavior at its fundamental level.

![Relative error plot](image.png)

Fig. 2.15. Relative error plot showing the difference between the measured and the simulated first order reversal curves for the middle and lower regions of the hysteresis loop respectively.
2.9 Remark on model characterization

Though the general concepts behind the presented model are similar to those of the classical Preisach model of hysteresis, the recent model belongs to the class of Duhem-type operators concerning its local memory representation, wiping properties and accommodation features.

Duhem-type operators are in general not order preserving (in a sense of Visintin’s definition [97]), they do not have any inner variables, and they are characterized by non-congruent minor loops. On the other hand, Preisach-type operators have the congruency and wiping-out properties, they do have inner variables for the representation of memory, and they are generally order preserving if the Preisach measure is nonnegative. According to the characterization theorem [10, 61], ‘under minor restrictions any rate-independent causal operator fulfilling the congruency and wiping-out properties is a Preisach operator.’ [98]

Nevertheless it is important to note, that there are many improved/modified variants of the classical Preisach model of hysteresis, which have similar properties to those of the Duhem-type models (like accommodation), and these model categories can overlap each other, making the rigorous distinction between them generally an extremely complicated task, which can be only achieved in the case of strictly defined subtypes of the above models having exact, and thoroughly analyzed functional analytic description.

2.10 Vector generalization of the model

In this section I want to introduce the construction of the vector generalization of the above developed model, demonstrating, that the vector extension arises in a very natural way and leads to an easy to use vectorial model of hysteresis.

As it is known, the description of the magnetic states of the domains of the material relies on statistical considerations. Considering a magnetic specimen as a collection of freely rotating magnetic dipoles (dipole-moments), the magnetic moment $m$ can be assumed to be the same for all of them for the sake of simplicity. With these assumptions, the only factor that affects the magnetic state of the material represented by the whole ensemble of rotating dipoles is the orientation of these magnetic dipole moments.

At first let the description of ideas be restricted to two space dimensions, later the model can be extended to three dimensions. In two dimensions the possible direction of the dipoles is the angular interval $[0, 2\pi)$. At this point I suggest, that the best way of describing the state of the material (which is certainly equivalent with the sum of the states of the individual dipole moments) is to introduce a probability density function $f : [0, 2\pi) \to \mathbb{R}^+$, which operates on the directions and results in probability densities, thus the integral of the above function over
a certain angle span $\Delta\varphi$ gives the probability of that the direction of a magnetic dipole (denoted by $\angle m_i$) falls into this angle interval

$$P(\angle m_i \in \Delta\varphi) = \int_{\Delta\varphi} f(\alpha) d\alpha. \quad (2.40)$$

The above function $f$, which can be interpreted as a directional density will be referred as the state-function throughout the section. With this representation the calculation of the magnetization is extremely easy as follows

$$M_x = \int_0^{2\pi} f(\alpha) \cos \alpha \, d\alpha,$$

$$M_y = \int_0^{2\pi} f(\alpha) \sin \alpha \, d\alpha. \quad (2.41)$$

It is really interesting and important to point out at this point, that this formulation can be the base of a connection of this description to the family of Fourier series, and there can be found decent models in the literature based on space-wise Fourier decomposition technique [54]. As it can be clearly seen from (2.41), the components of the magnetization vector can be obtained from the base harmonics of the state-function corresponding to the memory of the material.

The mechanism of changing the state-function according to the changes in the excitation field is relatively easy. Consider a starting state of uniformly distributed magnetic dipole moments, which results in zero magnetization and constant state-function on the whole interval $[0, 2\pi)$. Let the magnetic field excitation $H(t)$ be a sinusoidal function of time starting from zero and with amplitude high enough for saturating the material. This gradually changing excitation field causes the change of the state-function in the way described below.

The memory of the system consists of two components, the state-function $f_0(\alpha)$, and the excitation field $H_0$. This two component forms the set $\mathcal{M} = \{f_0(\alpha), H_0\}$. In the case of increasing the amplitude of the excitation field in a certain direction $\varphi$, the probability corresponding to this particular direction will increase (the state function reflects this change), thus the probability corresponding to any other direction will slightly be decreased, since they form a complete probability field, thus the integral of $f(\alpha)$ over the interval $[0, 2\pi)$ has to remain one.

### 2.10.1 The state-function

The memory structure of the material is treated as a probability density function (or discrete probability distribution) on the interval $[0, 2\pi)$, since the magnetization vector $M$ can be represented as the vectorial sum of the statistically distributed domains in the space [16, 95]. This directional density or state function accounts for the probabilities of the alignment of the domains to certain space directions. Theoretically this function can be continuous, but
in the case of its application i.e. in a numerical field calculation problem, it is computationally more convenient to treat it as a discrete distribution on $[0, 2\pi)$. In the latter case an appropriately dense segmentation is required over the interval. Fig. 2.16 shows the distributions corresponding to the demagnetized material with uniformly distributed domains (dashed line) and the near saturation state for a given direction $\varphi = \pi$ (solid line). The function corresponding to the saturated case is the PDF of a normal distribution with expected value $\varphi$. In the latter case if field strength is increased more the distribution theoretically becomes a $\delta(\alpha - \varphi)$-function (Dirac-$\delta$) with zero variance corresponding to perfectly aligned domains.

![Graph showing density functions](image)

Fig. 2.16. Density functions corresponding to uniformly distributed state (dashed line), and close to saturation state (solid line)

This memory handling method enables to describe well the possible states of the material, and shows that the model has so called infinite memory which according to the literature can be quite useful in vectorial description of hysteresis phenomena [33, 95].

### 2.10.2 The magnetization process

After defining the memory structure, the magnetization process can be defined in order to obtain basic simulation results. In this representation the process of magnetization can be handled as modifying the probability density function (state-function) or discrete distribution corresponding to the magnetization vector $\mathbf{M}$. Considering an external applied field $\mathbf{H}$ with a direction $\varphi$ and a certain magnitude in this direction, the $\mathbf{H}$ vector can be treated in a way that it is always increasing toward a certain direction or remains constant, but never decreases. Decreasing the amplitude along $\varphi$ can be regarded as increasing the
amplitude along $\varphi + \pi$. Based on this concept, the direction of the applied field is

$$\varphi = \hat{\varphi} + \pi \left[ \text{sgn} \left( \frac{\partial|H|}{\partial t} \right) - 1 \right] / 2,$$  \hspace{1cm} (2.42)

where $\hat{\varphi}$ denotes the actual direction of the field vector $H$, so this is the angle of the ‘real’ vector, and $\varphi$ is the growing direction of field vector $H$. It is important to handle $\varphi$ as it is introduced above, because applying an increasing magnetic field in an arbitrary direction $\varphi$ is equivalent to growing the weights in the distribution function around this $\varphi$ in the model.

The change in the state-function due to change of the excitation can be handled by the aid of the above defined direction and a corresponding directional excitation $H_\varphi$, which can be formalized as

$$H_\varphi = \text{sgn} \left( \frac{d|H(t)|}{dt} \right) |H|,$$  \hspace{1cm} (2.43)

which is analogous with (2.26) of the scalar model. This special variable enables to describe the state-change corresponding to the change of the field excitation over time. The effect responsible of the state change is the change of the directional probability $\xi_\varphi$, which is basically the alignment probability to direction $\varphi$

$$\xi_\varphi = \int_0^{2\pi} f(\alpha)\delta(\alpha - \varphi) d\alpha,$$  \hspace{1cm} (2.44)

where $\delta(\alpha - \varphi)$ is a shifted $\delta$-distribution, which holds the property that

$$\int_0^{2\pi} \delta(\alpha) d\alpha = 1,$$  \hspace{1cm} (2.45)

and has a role of separating the densities in direction $\varphi$ from the other directions. This definition of the alignment probability assumes that the actual most probable directional configuration of the magnetic dipoles is the one which is defined by the memory of the system, and if the state-change occurs, this probability is used as a starting point for the change.

During the time evolution of the system the differential probability is introduced as

$$\Delta \xi_\varphi = \int_{\Delta H_\varphi} g(H_\varphi) dH_\varphi,$$  \hspace{1cm} (2.46)

where $g$ is the probability distribution function corresponding to magnetization process (its role is the very same as it has been described above in the case of the scalar models (2.34)), and $dH_\varphi$ is the differential of the $H_\varphi$ excitation. This
differential probability corresponds to the differential of the field excitation in the direction $\varphi$.

With the aid of the above introduced quantities, the change of the state-function can be described as

$$f(\alpha) = f_0(\alpha) \left(1 - \frac{\Delta \xi_\varphi}{1 - \xi_\varphi}\right) + \delta(\alpha - \varphi) \frac{\Delta \xi_\varphi}{1 - \xi_\varphi},$$

(2.47)

where $f_0(\alpha)$ is the previous state-function, $\xi_\varphi$ is the above defined alignment probability, and $\delta(\alpha - \varphi)$ is a shifted $\delta$-distribution.

### 2.10.3 The case of rotational excitation

Under rotational excitation the state function $f(\alpha)$ is a very natural and easy way of handling the effect of field rotation and representing the memory of the system.

The rotation of the applied magnetic field can be regarded as weighting and shifting operations carried out on the probability distribution function, since a mod-$2\pi$ shifting is equivalent to rotation in this representation. Considering a nearly $\delta$-like distribution corresponding to perfectly aligned domains (saturated material), the shift results in the rotation of the magnetic field vector which is in agreement with the results published in literature [9, 45]. On the other hand if the applied field is not large enough to align the domains around any particular direction, the shifting of the distribution function will not affect the state of the material so much [9, 45].

Basically the case of rotational excitation is handled the same way as it had been described in the previous section, the only difference is that the direction $\varphi$ also changes. During the process of rotating magnetization the actual weight $w_\varphi$ that the rotating $H$ vector drags with itself around is

$$w_\varphi = f_{\text{an}}(\varphi) \mathcal{H}_{[0,1]}\{|H|, \mathcal{M}\},$$

(2.48)

where $f_{\text{an}}$ is the anisotropy function defined on $[0, 2\pi]$ and the inherent anisotropy function assuming uniaxial anisotropy can be defined as

$$f_{\text{an}}(\alpha) = \gamma |\cos(\alpha)| + (1 - \gamma),$$

(2.49)

where $\gamma \in [0, 1]$ is the strength of anisotropy.

### 2.11 Simulation results

In this section I present some simulation results obtained by applying unidirectional field and rotational field, respectively. The results show that the model has the potency of 2D hysteresis modeling and it can be a good
candidate for using in field calculation problems, where rotational fields, or other multidimensional phenomena can arise. It can be seen from the following simulations and the model structure described above, that the developed model obeys to the fundamental physical principles as the *saturation property*; meaning, that the magnetization calculated by the model in any direction cannot be larger than the saturation magnetization, and the *loss property*; ensuring, that hysteresis losses for rotating fields tend to decrease with increasing magnitude of excitation fields [22].

### 2.11.1 Unidirectional field

The behavior of the hysteresis model investigated along the easy axis direction, the magnetization is started from demagnetized state, which can be represented by either uniformly distributed domains or polarized domains. In both cases, it results in zero net magnetization $M = 0$. Fig. 2.17 shows a major hysteresis loop along the easy axis direction $\varphi = 0$.

![Fig. 2.17. Major hysteresis loop](image)

It can be seen that the hysteresis curve is perfectly matches the one that the identification was based on (scalar curve in Fig. 2.11), as it should be, since for the shaping of the loop the same $g(.)$ function is applied as in the case of the scalar model. In addition to the straightforward similarity of the major hysteresis loop, the vector model also shows accommodation due to its reversal mechanism described above, Fig. 2.18 shows a minor loop obtained by a triangle shaped excitation in the $\varphi = 0$ direction.
2.11.2 Rotational field

The investigation of the model for rotational field conditions offers much more possible excitations and conditions taking into consideration. First of all in the case of a large rotating excitation the hysteresis loss (the area enclosed by the hysteresis loop) tends to strongly decrease, which coincides with theoretical and experimental results found in the literature [45]. The resulting hysteresis loops - constructed by the projection of the $M - H$ trajectory to the easy axis - can be seen in Fig. 2.19 and Fig. 2.20. As it can be seen in Fig. 2.20, the hysteresis loops corresponding to rotational excitations with magnitudes $1.8H_c$ and $1.2H_c$ does not result in closed loops due to the lagging of the magnetization vector $M$ behind the excitation vector $H$.

This kind of lagging of the magnetization vector $M$ behind the vector of rotational exciting field $H$, and the impact of the coercive field of the material can be tracked by the aid of the trajectories depicted in Fig. 2.21 and Fig. 2.22.

As Fig. 2.21 shows, the trajectory of magnetization vector $M$ moves along with the vector of applied field in case of large field (dotted curve). In the case of a smaller rotational excitation the vector of magnetization will lag behind the applied field vector, and furthermore its magnitude decreases during the rotation (see dashed, and solid curves corresponding to $1.8H_c$ and $1.2H_c$ respectively). The magnitude of the applied rotational field is given as a multiple of the coercive field $H_c$ of the material in order to emphasize the impact of $H_c$ on the rotational behavior. The simulation results presented here meet with published measurements on rotational magnetization process [34, 45]. In the case, if the field excitation is smaller than the coercive field of the material the trajectory no longer encircles the origin as it was proved by Kahler and Della Tore [45],
Fig. 2.19. Major loop formations in easy direction for large rotational $H$

Fig. 2.20. Rotating excitation vector $H$ and $M$ fields in easy direction

does this essentially means, that the magnetic field has not enough energy to rotate the domains, resulting in small deviations of magnetization vector. Further decreasing of the magnitude of the rotating excitation field results in very tiny movements of the magnetization vector around its original position as it can be seen in Fig. 2.22. Keeping in mind the representation of the state function corresponding to the memory of the model, the smaller the magnitude of the excitation, the smaller amount of change occurs in $f(\alpha)$, which implies small change of the overall magnetization vector.
Fig. 2.21. The effect of rotation of large magnetic field $H$

Fig. 2.22. Rotation of smaller $H$ fields
2.12 Thesis 1.

I have developed a general phenomenological scalar hysteresis model by the aid of a statistical approach based on simple statistical considerations of the behavior of simple bistable units (domains, clusters) of the media, which resulted in a versatile model of hysteresis having useful properties, furthermore I have also developed a possible vector extension of the model, and I have established an efficient measurement-based identification method validated by actual measurements, which proved to be very accurate [78, 73, 79, 77, 76].

1.1. I have constructed an ODE-based scalar hysteresis model by the aid of statistical approach of hysteresis phenomena, which resulted in an improved variant of general Duhem-like models of hysteresis [78]. In the developed model I have introduced a mathematical description of the hysteresis curves, which enables of shaping the minor hysteresis loops independently from the major curve offering large versatility to the model, while preserving the advantageous properties of the closed form analytical formulation.

1.2. I have developed a two-step, measurement-based identification procedure of the model, applying linear combination of probability density functions for precisely shaping the major hysteresis curve by optimization, and a one-parameter minimization for approximating the shape of the minor loops [73, 79]. The identification of minor loops can be accomplished by either first order reversal curves or concentric minor loops.

1.3. I have proposed a possible vector extension of the established scalar model, which provides a reasonable description of vectorial hysteresis phenomena in two-dimensions [79, 77, 76]. The model captures the most important features of vectorial hysteresis, obeys the general fundamental physical principles of vector hysteresis, and shows good agreement with known experimental results.
Chapter 3

Stability Analysis of Nonlinear, Hysteretic Problems

3.1 Aim of the research

The main aim of my research here is to investigate the numerical solution of a few special but important problems containing hysteresis operator, and to obtain some useful results concerning their stability properties. Here I want to develop two stability investigation methods. At first the accommodation property of the hysteresis model described in the preceding section is investigated in detail, at second a nonlinear diffusion problem with hysteresis is examined. The problem concerning the stability of accommodating minor loops is handled by the aid of a Poincaré-map which provides a closed-form solution [80] for the stable position of a set of accommodating minor loops. With this solution the exact location of the stable minor loop can be obtained instead of estimation [96].

Thus the main goal in the first part of my research is the investigation of the stability properties of a hysteresis model, which is described by the previously developed differential equation and implemented into field calculation applications in FDTD and FEM numerical schemes. On the basis of the structure of the model it belongs to the family of Duhem-type models [25, 33], but the actual construction of the ODE is based on some statistical considerations, and the hysteretic behavior is described by distribution functions corresponding to continuous random variables. In the investigation an important subset of all possible solutions is examined, namely the ones corresponding to periodic excitations of the model. In the case of small amplitude periodic input the model response exhibits transient accommodation and the stability properties of these accommodating minor loops form one of the central topics of my work. I prove by the aid of the phase-space of the model, that the trajectories corresponding to various solutions are smooth, and the transient loops converge to stable periodic solutions (stable non-accommodating minor loops).

In the second part of my work I investigate a diffusion problem in nonlinear
media, focusing on the iterative solution procedure of the problem, and I recommend an appropriate selection of the damping constant of the iteration [81]. In order to investigate the nature and the parameter dependence of the iterative scheme I apply the Lyapunov method for the construction of a Lyapunov exponent-based parameter space. Certain slices of this parameter space clearly show the reasons behind the divergence, periodicity or even the chaotic behavior of the iteration corresponding to some parameter configurations. The most important concepts from the mathematical apparatus applied in the following sections is described in Appendix C.

3.2 The stability analysis of the hysteresis operator

In the followings I examine the behavior of the developed scalar hysteresis operator (Thesis 1.) in the case of periodic excitation, which is probably the most important for engineering applications. As a pre-requisite for the numerical stability of the solution of coupled PDE problems containing an ODE-based hysteresis operator, that the operator itself has to have stable solutions, thus the investigation of the stability of the minor loops obtained from the model is particularly important.

3.2.1 The autonomous system and the phase-space geometry

In order to investigate the behavior of the operator, I have constructed an appropriate three-dimensional representation, where the various solution trajectories in the three-dimensional phase space correspond to hysteresis curves. For periodic excitations the hysteresis model as it has been defined in Section 2.7 can be represented as an autonomous system which has the following form

\[\begin{align*}
x_1 &= x_2, \\
x_2 &= -2\alpha x_2 - (\alpha^2 + \omega^2)x_1, \\
x_3 &= \gamma^q g(\delta x_1)x_2,
\end{align*}\]

(3.1)

where the overdot denotes time derivatives, state variables \(x_1, x_2\) and \(x_3\) correspond to field variables \(H, \dot{H}\) and \(M\) respectively, \(\alpha > 0\) is the decay and \(\omega > 0\) is the frequency of the sinusoidal excitation. The first two equations of (3.1) define a linear oscillator, which produces the input of the actual hysteresis model defined by the 3rd equation (2.38).

Since in the case of exponentially decaying input the output of the hysteresis model is trivially stable (\(\lim_{t \to \infty} \dot{x}_i = 0, \ i = 1, 2, 3\)) the investigations are restricted to the more interesting case of periodic excitation with constant
amplitude. Due to the sinusoidal periodic excitation, the geometry of the phase space corresponding to (3.1) is the surface of a cylinder. One solution can be seen in Fig. 3.1 with initial conditions $H_0 = -700$, $\dot{H}_0 = 0$, $M_0 = 0$.

![3D phase-space diagram](image)

**Fig. 3.1.** A trajectory and the corresponding hysteresis loops in the phase-space of the system represented by (3.1)

Representing the hysteresis operator by an autonomous system of ODEs is advantageous because this way any set of hysteresis loops correspond to a smooth trajectory in the phase-space of the system. By this artificial extension of the hysteresis operator to the three-dimensional phase space, the reversal points, which have the disadvantageous property of being non-differentiable are vanish, and each solution of (3.1) appears as a smooth curve, which can be projected back into the $H - M$ plane as a hysteresis loop (see lower plot in Fig. 3.1).

As it can be seen in Fig. 3.1, the solution, which is represented by a trajectory
in the phase-space describes a series of accommodating hysteresis loops in the $H - M$ plane. The behavior of each of the solutions of the ODE system is similar to the one that can be seen in the figure. Every trajectory ‘wraps’ onto a stable periodic orbit in the phase-space after various amount of accommodation. The accommodation depends on the width of the minor loops and some shape parameters of the major hysteresis curve. In order to examine the exact location of the stable periodic orbits, where accommodation ends up and the minor loop becomes stable, the method of Poincare-sections is introduced \[102\].

### 3.2.2 Examining the accommodation by a Poincare-section

With the aid of a Poincare-section and the corresponding Poincare-map, the stability of periodic orbits corresponding to minor hysteresis loops is examined in the followings. In the case of the three-dimensional state space defined by (3.1) the applied Poincare-section is a plane, specifically the one which satisfies the $x_2 = 0, (\dot{H} = 0)$ equation.

With this selection, the Poincare-map gives the sequence of reversal points starting with the initial point defined by the initial conditions. The sequence is convergent and it has a unique fixed point corresponding to a stable periodic solution of (3.1), thus it exactly defines the location of the stable periodic orbit in the phase-space.

The Poincare-map can be derived from the analytical solution of the system (3.1). In the case of periodic excitation with constant amplitude, the output of the model is a piecewise monotone function increasing ($M'(H) > 0$) in interval $t \in [n2\pi/\omega, (2n + 1)\pi/\omega)$ and decreasing in interval $t \in [(2n + 1)\pi/\omega, (n + 1)2\pi/\omega)$, $n \in \mathbb{N}$. The $x_2 = 0$ Poincare-plane separates the increasing and the decreasing sections of the output curve in the phase space, thus it enables to solve system (3.1) separately for increasing and decreasing branches ($\delta = 1$ and $\delta = -1$ respectively). From these solutions with properly selected initial conditions the Poincare-map can be constructed.

The $z_{n+1} = P(z_n, \mu)$ Poincare-map has the form (see Appendix B)

$$P(z_n, \mu) = \frac{(-1 + a) ((1 - 2 z_n + a z_n) b + z_n (-1 + a)^2) (-1 + a + b)}{(-a^2 + a - 1 + z_n) b^2 + (-2 a^3 + 5 a^2 - 4 a + 1) b - (-1 + a)^4}, \tag{3.2}$$

where $\mu = [a, b]$ is the vector of parameters defined as $a = G(A)$ and $b = G(A) - G(-A)$ corresponding to a sinusoidal excitation defined as $H(t) = -A \cos(\omega t)$, and $z_n = x_3(t)_{t=2n\pi/\omega}$ stands for the output of the model in the $n$-th period starting width $z_0 = x_3(0)$ from the initial condition. The Poincare-map (3.2) correctly describes the sequence of intersection points of the solution curve and the Poincare-plane. Comparing the analytical results to the numerical solution, the error remains below the tolerance of the numerical solver.
In Fig. 3.2 the applied Poincare-section can be seen with a trajectory corresponding to a transient solution. The sequence of intersection points of the trajectory and the Poincare-plane clearly approaches the fixed-point corresponding to the periodic solution representing the stable minor loop.

In the construction of the model, there is a pair of functions $g : \mathbb{R} \to \mathbb{R}^+$ and $G : \mathbb{R} \to (0, 1)$, which has the role of defining the shape of the major hysteresis loop. $G$ is a linear combination of cumulative distribution functions (2.23), which is strictly monotonous on the whole $\mathbb{R}$, thus its derivative $g(x) > 0, \forall x$.

One of the most interesting properties, originating from the model structure is that the location of the stable periodic solutions is independent of the starting point of the loop and the exact shape of the functions $G$ and $g$, it is enough to know the width of the minor loop and a few points of the $G$ curve, as it is clear from (3.2).

In Fig. 3.3 a minor loop can be seen, corresponding to an excitation between $-A = \min H(t)$ and $A = \max H(t)$, and the fixed point of the Poincare-map can be calculated from $G(-A)$ and $G(A)$. This map can be defined as $z_n = P_n(z_0, \mu)$, where $\mu = [a, b]$ is the vector of parameters defined above, which can be interpreted as $a$ is the starting point parameter and $b$ is the width parameter of the loop. Only these parameters are required for the calculation of the fixed-point of the map. In the case of symmetrical loops parameter $a$ implies\(^1\) parameter $b$ thus the two parameters can not be handled independently. The resulting fixed-point

\[^1\text{Since the minor loops are symmetric and } a = G(-A), a + b = G(A) \text{ holds, given the width of the loop } 2A, \text{ parameter } a \text{ and } b \text{ can be calculated.}\]
The above fixed point equation for the periodic orbits has been constructed on the basis of the symbolic solution of (3.1). The solution of the fixed point equation for various parameter configurations of symmetrical minor loops can be seen in Fig. 3.4.
It is clear from the figure that the smaller the width of the minor loop the higher the accommodation is. The fixed point value \( P_\infty(b) = 0.5 \) corresponds to a tiny minor loop in the center of the major hysteresis curve with large transient accommodation, while the value \( P_\infty(b) = 0 \) represents the major loop itself with no accommodation at all. The value of \( P_\infty(b) = 0 \) corresponds to \( b = 1 \), which means, that \( G(-A) = 0 \), and since \( G(.) \) is a CDF, \( G(-A) = 0 \) implies, that \( A = \infty \), thus it is the ‘largest possible’ major loop.

### 3.2.3 The role of \( q \) in symbolic solution

The shape of the reversal curves and minor loops depends mainly on the shape of the major loop from which the minors are derived, and on an additional parameter \( q \). This parameter appears as the exponent of the expression \( \gamma \), which is unity on the major loop and \( 0 < \gamma < 1 \) in the minor loops. The symbolic solution of the ODE problem can be obtained for general \( q \), but the structure of the symbolic solution becomes extremely complicated for large values of \( q \), thus for the sake of simplicity in the recent investigation the \( q = 2 \) case has been described. In Fig. 3.5 two stable loops can be seen with different values of parameter \( q \), starting from the same initial condition.

![Fig. 3.5. Stable loops after transient accommodation for different values of parameter \( q \)](image)

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3.3 A numerical field calculation problem with hysteresis

The diffusion of electromagnetic fields in nonlinear media has significant practical importance in many areas, like magnetic recording, design of magnetic components, eddy current analysis in hysteretic conductors, just to mention a few of them. The problem of electromagnetic diffusion in hysteretic media represented by second order Nonlinear Partial Differential Equation (NPDE) with a hysteresis-type nonlinearity. The numerical solution of nonlinear, diffusion-type PDEs requires sophisticated algorithms in order to get the correct solution. One of the main problems is the instability of the numerical procedures under certain conditions [37]. The situation is even more difficult when the nonlinearity has the form of a hysteresis-operator. Implementation of a hysteresis-operator into a numerical solution method of a PDE can result in a numerical model with very wide varieties of behavior including chaos (the model contains a fairly large number of parameters depending on each other). The main aim of this section is to investigate this type of model, and on the basis of the investigation I want to predict the conditions of stable and unstable behavior.

Fig. 3.6. Infinite halfspace filled with hysteretic media
3.3.1 Formulation of the problem

Configuration of the considered one dimensional diffusion problem can be seen in Fig. 3.6. The hysteretic media fills the halfspace and it is fed with sinusoidally alternating magnetic field. The governing diffusion equation can be derived from the Maxwell’s equations (assuming quasi-static case)

\[ \nabla \times \mathbf{H} = \sigma \mathbf{E}, \quad (3.4) \]

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (3.5) \]

where \( \mathbf{H} \) is the magnetic field strength, \( \mathbf{E} \) is the electric field intensity, \( \mathbf{B} \) is the magnetic flux density, and \( \sigma \) is the electrical conductivity, which is in the actual field calculation problem has been set to \( \sigma = 2.5 \cdot 10^5 \, [\text{S/m}] \).

Due to the presence of the nonlinear material, the above system of equations, has to be completed with the following constitutive relation

\[ \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}), \quad (3.6) \]

where \( \mathbf{M} = \mathcal{H}\{\mathbf{H}\} \), \( \mathcal{H}\{\cdot\} \) is a scalar nonlinear hysteresis operator\(^2\), and \( \mu_0 \) is the permeability of vacuum, and has the value \( \mu_0 = 4\pi \cdot 10^{-7} \, [\text{H/m}] \).

From (A.1), (A.2) and (3.6) the diffusion equation for the one dimensional case can be derived, taking into account that

\[ \frac{\partial}{\partial y} = \frac{\partial}{\partial z} \equiv 0, \quad (3.7) \]

and

\[ H_x(r, t) = H_y(r, t) \equiv 0, \quad r = (x, y, z), \quad (3.8) \]

which means that \( H \) field has only \( z \) component (\( H_z \)) and this component changes only in direction \( x \).

From the above assumptions on the derivatives, the one dimensional diffusion equation in Descartes-coordinates takes the following form for the field variable \( H_z(x, t) \)

\[ -\frac{\partial^2 H_z(x, t)}{\partial x^2} + \sigma \mu_0 \frac{\partial H_z(x, t)}{\partial t} = -\sigma \mu_0 \frac{\partial M_z(x, t)}{\partial t}, \quad (3.9) \]

where \( M \) depends on \( H \) through the hysteresis operator \( \mathcal{H}\{\cdot\} \).

As it can be seen from (3.9), the solution of the diffusion problem means the solution of a 2nd order nonlinear PDE, which can not be solved analytically due to the nonlinear hysteresis operator.

\(^2\)Since in this case the problem to be solved is essentially one-dimensional, a scalar hysteresis operator is applied.
3.3.2 The hysteresis models applied

In the recent examination a symbolically defined hysteresis model is applied, the Limiting Loop Proximity scalar Hysteresis operator (LLPH), which is an analytical, phenomenological model of hysteresis [21]. The model is analytical in nature having a closed form representation. One of the main differences between the LLPH model and the model developed in Thesis 1, is that in the case of the LLPH model the sequence of reversal points \((H_r, M_r)\) has to be handled separately as they appear explicitly in the model, while the operator proposed in Thesis 1 has a more compact representation, since the ODE of the model implicitly contains these reversal values, thus they do not have to be concerned about. The reason behind the selection of the LLPH model for this study is to investigate the explicit reversal point dependency of the solutions of the nonlinear problem.

The LLPH operator can be described as

\[
\mathcal{H}\{H\} = \frac{2 M_s}{\pi} \arctan \left( \frac{H_{pr} P(x) + H - \delta H_c}{h_0} \right),
\]

where \(M_s\) is the saturation value of the magnetization, \(H_r\) is the last reversal point of the \(H\) field, \(M_r\) is the value of magnetization corresponding to \(H_r\), \(\delta \in \{-1, 1\}\) gives the direction (descending/ascending) of the hysteresis curves, \(H_c\) is the coercive field, \(H_{pr}\) is the proximity field, \(P(x)\) is the proximity function, \(\zeta\) and \(h_0\) are loop-shaping parameters with values \(\zeta = 1\) and \(h_0 = 1 \times 10^3\) respectively. Besides the loop-shaping parameters, the value of the coercive field \(H_c = 1 \times 10^3 \text{ [A/m]}\) and the saturation magnetization \(M_s = 1.6 \times 10^6 \text{ [A/m]}\) can be obtained from the identification procedure of the hysteresis model [21].

All of the numerical values are given in SI units, and the value of each fixed parameter will be given at the place of their first appearance.

3.4 The numerical solution

There are several methods to solve a PDE numerically, like Finite Element Method (FEM) [6, 55], Boundary Element Method (BEM), Finite Difference Time Domain (FDTD), etc. In this investigation the Yee-algorithm is introduced, which basically is an FDTD method [94], and the FEM formulation of the problem is given as well in order to compare the nonlinear iteration procedure resulting from the different solution methods. In the Yee-algorithm a special double grid
system is specified to define a numerical scheme for Maxwell’s curl equations [35]. Finally the numerical scheme can be transformed into a fixed-point problem, which can be solved directly (i.e. Picard iteration method [87, 90]).

### 3.4.1 The Yee-algorithm

Applying the Stokes-theorem for a surface $A$ bounded by a closed curve $l$, (A.1) and (A.2) take the following form:

\[
\oint_l \mathbf{H} \, dl = \sigma \int_A \mathbf{E} \, dA, \tag{3.13}
\]

\[
\oint_l \mathbf{E} \, dl = -\frac{\partial}{\partial t} \int_A \mathbf{B} \, dA. \tag{3.14}
\]

The numerical solution of (3.13) and (3.14) on a properly defined Yee-grid [94, 35, 36] yields the following iteration scheme:

\[
E_{n+1/2}^{j+1} = \frac{1}{\sigma \theta \Delta x} (H_j^n - H_{j+1}^n) + \frac{1 - \theta}{\theta} E_{n-1/2}^{j+1}, \tag{3.15}
\]

\[
H_{j+1}^{n+1} = H_j^n + M_j^n - M_{j+1}^{n+1} - \frac{\Delta t}{\mu_0 \Delta x} (E_{j+1/2}^{n+1/2} - E_{j-1/2}^{n+1/2}), \tag{3.16}
\]

where the $\theta$ in the difference scheme in this case is $\theta = 2/3$, the indexes denoted by $j$ correspond to the discretized space and indexes denoted by $n$ to the discretized time steps.

The above equations in a matrix notation are

\[
E^{n+1/2} = P_H H^n + P_E E^{n-1/2}, \tag{3.17}
\]

\[
H^{n+1} = S_H (H^n + M^n - M^{n+1}) + S_E E^{n+1/2}, \tag{3.18}
\]

where $P_H, P_E, S_H, S_E$ are band matrices defined as

\[
P_H = \frac{1}{\sigma \theta \Delta x} \begin{pmatrix} 1 & -1 & 0 & \ldots & \ldots \\ 0 & 1 & -1 & 0 & \ldots \\ \ldots & 0 & 1 & -1 & \ldots \\ \ldots & \ldots & 0 & 1 & -1 \end{pmatrix}, \quad P_E = \frac{1 - \theta}{\theta} I, \tag{3.19}
\]

\[
S_H = I, \quad S_E = -\frac{\Delta t}{\mu_0 \Delta x} \begin{pmatrix} 1 & 0 & \ldots & \ldots \\ -1 & 1 & 0 & \ldots \\ 0 & -1 & 1 & 0 & \ldots \\ \ldots & 0 & -1 & 1 & 0 \end{pmatrix}. \tag{3.20}
\]

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I is the identity matrix and M is defined by the elements of H through the $\mathcal{H}\{,\}$ nonlinear scalar hysteresis-operator.

Since (3.18) is implicit (in order to calculate $H^{n+1}$, value of $M^{n+1}$ has to be known) an inner iteration has to be carried out to determine $M^{n+1}$ from an initial estimation of $H^{n+1}$.

### 3.4.2 The fixed-point problem in the Yee iteration scheme

The inner iteration in the numerical solution scheme is a point in the whole iteration process when $H^n, M^n$ and $E^{n+\frac{1}{2}}$ supposed to be known, thus the expressions containing these known values can be represented by a vector ($R$). Using this vector, the inner iteration with the unknown value $H^{n+1}$ can be formulated as follows from (3.18)

\[
H^{n+1} = -\frac{M^{n+1} + R}{G(H^{n+1})}.
\]  
(3.21)

where $R = (H^n + M^n) + S_E E^{n+\frac{1}{2}}$. In view of

\[
M^{n+1} = \mathcal{H}\{H^{n+1}\},
\]  
(3.22)

and using the actual $H^n$ as a starting point of the iteration $H^n \rightarrow H^j (j = 0)$ yields

\[
H^{j+1} = -\frac{\mathcal{H}\{H^j\} + R}{G(H^j)}.
\]  
(3.23)

The iteration scheme above is a pure fixed-point problem with $H^{n+1}$ as the fixed point, so if $\exists \hat{H} : \hat{H} = -\mathcal{H}\{\hat{H}\} + R$, then $\hat{H}$ is the exact solution of inner iteration (3.23) and $\mathcal{H}\{\hat{H}\}$ will be used as $M^{n+1}$ in (3.18). As it can be shown the existence and stability of this fixed point strongly depends on the parameters of the numerical scheme. The stability of this iteration is a crucial point in the solution of (3.9) and it will be the topic of the following sections. Function $G$ in the iteration scheme contains the $\mathcal{H}\{H, H_r, M_r, \delta\}$ hysteresis operator, which is a scalar nonlinear operator depending on its four arguments. These arguments are the actual iterated value $H$, the $H$ an $M$ field reversal points $(H_r, M_r)$ and the direction $\delta$ of the applied field $H$.

### 3.5 Examination of stability properties

The stability analysis discussed here is focusing on the implicit inner iteration, where (3.23) governs the behavior of the solution. Since (3.23) is usually not convergent in its pure form, an appropriate fixed-point method has to be applied to solve.
3.5.1 Solution of the inner iteration

The solution of a (3.23) type iteration can be approached in many ways. In this case the Picard fixed-point iteration method is introduced to find the solution of (3.23), since based on the literature the fixed-point iteration technique is found to be very effective and advantageous for the solution of hysteretic problems [7, 8, 15, 90]. Applying the Picard iteration scheme the iteration takes the form

\[ H^{j+1} = H^j - \eta \left( H^j - G(H^j) \right) \]  

(3.24)

where \( \eta \) is responsible for the contraction property of the map, the speed of convergence and the stability of the solution. Equation (3.24) can also be formulated as follows:

\[ H^{j+1} = (1 - \eta)H^j + \eta G(H^j) \]  

(3.25)

Applying the numerical scheme for the solution of the diffusion problem with \( \eta = 0.005 \) the evaluation process will stop when the amplitude of the exciting field \( H \) reaches \( H \approx 950 \). The forthcoming examination of the behavior of the iterative map around this value shows the reasons behind this phenomenon.

In this approach I have developed, the iteration problem (3.25) will be treated as a discrete dynamical system, so in order to examine its stability, some basic concepts are incorporated from the theory of dynamical systems described briefly in Appendix C.

3.6 Handling the instability

Since the inner iteration (3.25) defines element-wise operations on the vector \( H \) of the discretized space, in the following examinations I focus on one particular element of this vector to demonstrate the behavior of the iteration, thus from now on an arbitrary coordinate function of \( G_{PB} \) is analyzed, hence the scalar notation of quantities.

3.6.1 The unstable case

The stability of possible fixed points of (3.24) can be visualized using the \( G_{PB}(H) \) map of the iteration. Fig. 3.7a shows the \( G_{PB}(H) \) function. At the point where \( G_{PB}(H) \) intersects the identity function a fixed point can be seen at \( \hat{H} \approx 950 \). It is known [32] that \( \hat{H} \) is a stable fixed point of (3.25) if

\[ \left| \frac{\partial G_{PB}(H)}{\partial H} \right|_{H=\hat{H}} < 1. \]  

(3.26)
From this condition a global stability criterion can be obtained for $\eta$ in the fixed-point scheme defined by (3.24) (with notation $G'_P(B)(H) = \frac{\partial G_{PB}(H)}{\partial H}$)

$$|G'_P(B)(H)| < 1 \rightarrow |1 - \eta(1 + H'\{H\})| < 1 \rightarrow 0 < \eta(1 + H'\{H\}) < 2,$$  \hspace{1cm} (3.27)

from here

$$\eta < \frac{2}{1 + H'\{H\}}, \quad \forall H,$$  \hspace{1cm} (3.28)

where $H'\{H\}$ can be bounded and substituted by $\chi_{max}$ maximal differential susceptibility$^3$. From the above condition an upper bound of $\eta$ can be calculated, which in this case for the introduced hysteresis operator is $\eta < 1.96 \cdot 10^{-3}$. In order to examine an unstable solution, a larger $\eta = 5 \cdot 10^{-3}$ was chosen at this time.

Besides the magnitude of the derivative of the map, there is another, more visual way of analyzing the stability, using the second iterate of $G_{PB}(H)$. In Fig. 3.7b the mapfunction $G_{PB}(H)$ can be seen composed with itself$^4$. From the figure it is clear that the fixed point at $\tilde{H} \approx 950$ is unstable, and there is a stable period two solution which is undesirable in this case.

My further goal is to examine the iteration as a function of a parameter of the nonlinear hysteresis-operator. In Fig. 3.8 the attractor of the fixed-point iteration (3.24) can be seen with respect to parameter $M_r$ (Fig. 3.8a), and parameter $H^0$ (Fig. 3.8b).

The attractor has been calculated by successive iterations of the map, ignoring two hundred iterations as transients, and plotting the last twenty points. The figure shows a typical parameter setting when the numerical scheme is used in field calculation. The magnetic field strength $H$ starts from an initial condition $H(x, 0) = 0$, and the corresponding magnetization is also zero, thus $H_r = 0$, $M_r = 0$. Value of $R$ is calculated from (3.21) and $\eta$ is intentionally selected a bit large to point out how it causes loosing of stability in inner iteration. As Fig. 3.8 shows, the attractor bifurcates near the fixed point and the starting set of parameters leads to a periodic behavior of the map, so the iteration becomes unstable at $H \approx 950$ (Fig. 3.8b), and there is a stable, period-two orbit at $M_r = 0$ (Fig. 3.8a). It practically means that the fixed-point iteration will never stop for these parameter values, but it will jump back and forth between the stable period-two fixed points. The solution in this case is a periodic one, which makes it impossible to finish the inner iteration under the condition $\|M^{j+1} - M^j\| < \varepsilon$, with $\varepsilon$ as a predefined maximum error. Furthermore - as it can be seen in the figure - the numeric iteration can exhibit fully chaotic behavior at certain parameter settings.

To regain stability the damping constant has to be decreased toward the value obtained from global stability condition (3.28). Fig. 3.9a and 3.10a shows similar

$^3$The efficiency of selecting $\eta$ under this condition will be discussed later.

$^4$G$_{PB}^{(2)}(.)$ denotes ($G_{PB} \circ G_{PB})(.)$
attractors with a bit smaller values of $\eta$. It is noticeable that the bifurcation point and the chaotic region have been shifted toward higher magnitude parameters (Fig. 3.9a). As $\eta$ approaches the desired value according to condition (3.28), the chaotic/periodic behavior disappears from the parameter-space. Since the parameter-space is multi-dimensional ($H^0, H_r, M_r, \delta, \eta$) and the attractor exhibits chaotic behavior with bifurcations, the method of Lyapunov exponents [32, 86] can be introduced to gain stability information on the whole parameter-space.

As it was shown above, a fixed-point iteration can be handled as a discrete
Fig. 3.8. The attractor of map (3.24) as function of (a) $M_r$ and (b) $H^0$ ($\eta = 0.005$)

dynamical system, governed by a map. A dynamical system can be stable or unstable under certain conditions and parameter configurations. It has been shown in the preceding section, that the origin of unstable behavior during the inner iteration is the bifurcation of the attractor of the map $G_{PB}$. As the most straightforward way of examining stability of a system with bifurcations I have introduced the calculation of the Lyapunov exponents of the system. As it is shown, the Lyapunov parameter-space will contain the stability information of the
\[ H^0 = 940.3, H_r = 0, \eta = 0.0045 \]

a) attractor with \( M_r \) as control parameter

b) Lyapunov exponents with \( M_r \) as control parameter

Fig. 3.9. The attractor of iteration (3.24) as function of \( M_r \) (a) and corresponding Lyapunov exponents (b) (\( \eta = 0.0045 \))

nonlinear iterative map, because it shows the bifurcation points\(^5\), the boundary between stable and unstable regions.

Fig. 3.9b and 3.10b shows the Lyapunov exponents of iteration (3.24) with respect to parameter \( M_r \) and \( H^0 \) (\( \eta = 0.0045 \)). Along with the attractors it can be clearly seen, that Lyapunov exponents contain information about the behavior

\(^5\)Iso-lines and iso-surfaces in higher dimensional cases.
Fig. 3.10. The attractor of iteration (3.24) as function of $H^0$ (a) and corresponding Lyapunov exponents (b) ($\eta = 0.0045$) of the iteration.

A Lyapunov exponent corresponding to a parameter value shows the behavior of the map of the iteration\textsuperscript{6}. In stable regions the exponent is negative, at bifurcation points it is zero, and in chaotic regions the exponent is positive. It is important to notice, that $\Lambda(\mu) < 0$ in parameter regions corresponding to

\textsuperscript{6}See Appendix C
3.6.2 The Lyapunov space

Calculating the Lyapunov exponents on the whole parameter-space makes possible to examine the stability properties of the iteration procedure. The parameter space in the case of (3.24) is a five dimensional space since the iteration contains $H^0, H_r, M_r, \delta,$ and $\eta$ as parameters. These parameters form the $\mu \in \Gamma$ parameter vector. Since the hysteresis operator shows central symmetry, it is enough to examine one direction in the operator (e.g. $\delta = 1$). Furthermore $\eta$ must be handled in a special way since the goal is to examine the changes in structure of the Lyapunov space according to changes of $\eta$, thus the parameter vector becomes $\mu = (H^0, H_r, M_r)$, which is only three dimensional.

Fig. 3.11 shows a slice from the Lyapunov space $\Lambda(H^0, H_r, M_r)$ at $H^0 = 940.3$. The stable and unstable region is separated by bifurcation points. These points form a band on the surface of the graph. As it can be seen in Fig. 3.11 the structure of the Lyapunov space is fairly complicated and the unstable region in the parameter-space is comparable in size or even smaller than the stable region even with an intentionally large value of $\eta$. In addition I have proven, that decreasing $\eta$ results in shrinking of the unstable region in the parameter-space. This encourages for further examinations about the proper value of this damping constant, because if $\eta$ is unnecessarily small the fixed-point iteration converges more slowly and field calculation takes much more time.

3.6.3 The speed of convergence

As it was discussed in the previous points, there is a parameter $\eta$ in the iteration scheme (3.24), which the rate of convergence and the stability of the solution depends on. In order to solve the fixed-point problem effectively (to find the solution of the inner iteration in a few\(^7\) steps), the proper value of $\eta$ has to be selected.

As it has been pointed out before, the sufficient condition for a map to be a contraction is $|G_{PB}(H)| < 1$. From this condition the following stability criterion can be obtained

$$\eta < \frac{2}{1 + \mathcal{H}\{\hat{H}\}}, \quad \forall H \in \Omega,$$

where $\Omega$ is the domain which includes $H^0$ and $\hat{H}$, and $\mathcal{H}\{\cdot\}$ is the hysteresis operator. From criterion (3.29) an upper bound can be given for $\eta$

$$\eta < \frac{2}{1 + \chi_{\text{max}}},$$

\(^7\)In practice as few as possible.
where $\chi_{\text{max}}$ is the maximum differential susceptibility, formally

$$\chi_{\text{max}} = \max_H \mathcal{H}' \{H\},$$

Eq. (3.31)

with $\mathcal{H}' \{\cdot\}$ denoting the derivative of the scalar hysteresis operator $\mathcal{H} \{\cdot\}$, or in the case of a discrete operator, it can be substituted by the local differential susceptibility. In any case, the maximum susceptibility $\chi_{\text{max}}$ can be found at the coercive field $H_c$, thus

$$\chi_{\text{max}} = \mathcal{H}' \{H_c; H_r = -H_s, M_r = -M_s, \delta = 1\}.$$  

Eq. (3.32)
The stability of the fixed-point iteration (3.24) is guaranteed if (3.30) holds, but the iteration can be very slow because of the selection of $\eta$ according to condition (3.30) is too strict for most of the timesteps. The Lyapunov space of the iteration shows that the unstable region is rather small compared to the whole space of the possible parameter configurations. This means that iteration (3.24) could be much faster if $\eta$ was larger. On the other hand selecting a larger $\eta$ causes a bifurcation in the attractor (Fig. 3.8), and makes the iteration unstable (periodic).

To overcome this problem I have established a method for calculating the proper value of $\eta$ for the iteration. As it has been stated before, the map has to be a contraction in order to converge. Pushing this concept a little bit further it can be seen that closer the derivative of the map to zero at the fixed point, the faster the iteration is. The goal is to transform the map to fulfill the condition $G_{PB}(\hat{H}) = 0$, since in this way it can be achieved, that the order of convergence of the iteration to be locally ‘almost quadratic’ according to the theorem corresponding to the rate of convergence of iterative maps having zero derivatives [72]. Applying this concept it can be derived that

$$\eta_j = \frac{1}{1 + \mathcal{H}'\{\hat{H}_j\}}, \quad \forall j$$

(3.33)

where $\hat{H} = [\hat{H}_1, \hat{H}_2, \ldots, \hat{H}_j, \ldots]$ is the fixed point of (3.24). Since an appropriate $\eta$ has to be calculated for each component of $H$ in the discretized space, hence the indexed notation. Because of this fixed point is not known at this stage, some kind of approximation of it has to be substituted into (3.33). The most straightforward approach is using $H^n$ to approximate $\hat{H}$.

This method for selecting $\eta$ using the local approximation of differential susceptibility dramatically decreases the required number of steps in the iteration\(^8\). In the field calculation problem (3.9) by selecting $\eta$ according to

\(^8\)Certainly calculating a new $\eta$ at each time-step consumes some extra time.

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(3.33), the number of required iteration steps became more than an order of magnitude fewer in each time step compared to the case of selecting $\eta$ according to the global stability condition (3.30). In Fig. 3.12 the number of inner iteration cycles can be seen for each time-step $n$ of the solution procedure of the nonlinear problem. It is clear from the figure that the local selection of the parameter $\eta$ (Fig. 3.12.b) is far more efficient than the application of the global stability condition throughout the whole calculation.

### 3.7 FEM formulation of the nonlinear diffusion with hysteresis

In this section I develop the finite element numerical solution procedure of equation (3.9), which can be written as

$$-rac{\partial^2 H}{\partial x^2} + \mu_0 \sigma (\dot{H} + \dot{M}) = 0,$$

where $M = \mathcal{H}\{H\}$ is a hysteretic function (operator) of $H = H_z$, and overdot denotes time derivatives. The boundary conditions are $H(0, t) = A \sin(\omega t)$, and $H(l, t) = 0$ in the case of sinusoidal excitation (the length of the domain $l$ is assumed to be at least five times larger than the linear penetration depth $\delta_l$).

The solution is obtained by handling the equation as a stationary problem, applying backward Euler scheme for the time derivatives as follows

$$-rac{\partial^2 H^n}{\partial x^2} + \mu_0 \sigma \left( \frac{H^n - H^{n-1}}{\Delta t} + \frac{M^n - M^{n-1}}{\Delta t} \right) = 0,$$

with the exact solution $H_0$. Treating $f$ as a source term of the stationary problem and denoting the approximate solution of (3.35) by $H$, the weighted residual form [35, 55] of the above equation on domain $\Omega$ considering that there are no residuals on the boundaries (weighted by arbitrary weighting function $N_\Omega$) is

$$\int_{\Omega} R_\Omega N_\Omega dx = 0,$$

substituting the residual $R_\Omega = \mathcal{L}\{H\} - f$, with the differential operator $\mathcal{L} = \partial^2 / \partial x^2$ we have the weighted residual form of (3.35)

$$\int_{\Omega} \frac{\partial^2 H}{\partial x^2} N_\Omega dx - \int_{\Omega} f N_\Omega dx = 0.$$

---

9Since the FEM solution is constructed for the stationary problem corresponding to each of the time-steps, from now on the index $n$ of time step is omitted in the weighted residual equations.
It is advantageous at this point to use the weak form for better (weaker) admissibility conditions. The weak form can be constructed by integrating the domain integral by parts which yields

$$- \int_{\Omega} \frac{\partial H}{\partial x} \frac{\partial N_\Omega}{\partial x} dx - \int_{\Omega} f N_\Omega dx + \left[ \frac{\partial H}{\partial x} N_\Omega \right]_0^l = 0. \tag{3.38}$$

This way first order polynomial approximation is satisfactory for both the solution and the weighting function as follows

$$H = \sum_{i=1}^L \phi_i H_i, \quad N_\Omega = \sum_{i=1}^L \phi_i \delta H_i, \tag{3.39}$$

with locally defined approximation functions $\phi_i$, and an arbitrary $\delta H_i$ according to the Galerkin method, where the weighting function is $N_\Omega = \delta H$.

The approximation functions $\phi_i$ have to fulfill the followings

$$\phi_i(\xi_j) = \delta_{ij}, \quad \sum_{i=1}^L \phi_i(\xi) = 1 \tag{3.40}$$

for any local $\xi$ in an element, so they are mutually orthogonal and form a complete set.

Substituting the approximation of $N_\Omega$ into (3.38) yields

$$- \int_{\Omega} \frac{\partial H}{\partial x} \sum_{i=1}^L \frac{\partial \phi_i}{\partial x} \delta H_i dx - \int_{\Omega} f \sum_{i=1}^L \phi_i \delta H_i dx + \left[ \frac{\partial H}{\partial x} \sum_{i=1}^L \phi_i \delta H_i \right]_0^l = 0, \tag{3.41}$$

which can be written as, considering integration of sums, and that $\delta H_i = 0$ on the boundaries (the third term is zero)

$$- \sum_{i=1}^L \delta H_i \int_{\Omega} \frac{\partial H}{\partial x} \frac{\partial \phi_i}{\partial x} dx - \sum_{i=1}^L \delta H_i \int_{\Omega} f \phi_i dx = 0, \tag{3.42}$$

substituting the approximation of $H$ results in

$$- \sum_{i=1}^L \delta H_i \int_{\Omega} \sum_{j=1}^L \frac{\partial \phi_i}{\partial x} H_j \frac{\partial \phi_i}{\partial x} dx - \sum_{i=1}^L \delta H_i \int_{\Omega} f \phi_i dx = 0, \tag{3.43}$$

which can be written in the following form

$$- \sum_{i=1}^L \delta H_i \sum_{j=1}^L \int_{\Omega} \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} dx H_j - \sum_{i=1}^L \delta H_i \int_{\Omega} f \phi_i dx = 0. \tag{3.44}$$

Since the above equality has to hold for arbitrary $\delta H_i$ if follows that

$$- \sum_{j=1}^L \int_{\Omega} \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} dx H_j = \int_{\Omega} f \phi_i dx, \quad \forall i = 1 \ldots L, \tag{3.45}$$
which can be written in matrix form as

\[ KH = -P \]  

(3.46)

where

\[ K_{ij} = \int_{\Omega} \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} \, dx, \quad P_i = \int_{\Omega} f \phi_i \, dx, \]  

(3.47)

with \( H \) as the solution vector.

Considering linear shape functions defined as

\[ \phi_i(x) = \begin{cases} 
\frac{1}{\Delta x} (x - x_{i-1}), & \text{if } x_{i-1} < x < x_i \\
-\frac{1}{\Delta x} (x - x_{i+1}), & \text{if } x_i < x < x_{i+1} \\
0, & \text{otherwise},
\end{cases} \]  

(3.48)

the elements of matrix \( K \) after the integration can be written as

\[ K_{ij} = \begin{cases} 
2/\Delta x, & \text{if } i = j \\
-1/\Delta x, & \text{if } |i - j| = 1 \\
0, & \text{otherwise},
\end{cases} \]  

(3.49)

which is a tri-diagonal band matrix.

Applying a quadrature (trapezoidal formula) for the approximation of \( P_i \) yields

\[ P_i = f(x_i)\Delta x. \]  

(3.50)

From here the final linear system can be constructed based on (3.46) applying (3.35) as

\[ \frac{1}{\Delta t} (2H^n_i - H^n_{i-1} - H^n_{i+1}) = -\frac{\Delta x \mu_0 \sigma}{\Delta t} (H^n_i + H \{ H^n \} - R), \]  

(3.51)

where \( R = H^{n-1}_i + M_i^{n-1} \), which can be written in matrix notation as

\[ AH^n = -H \{ H^n \} + R, \]  

(3.52)

where

\[ A_{ij} = \begin{cases} 
2\lambda + 1, & \text{if } i = j \\
-\lambda, & \text{if } |i - j| = 1, \\
0, & \text{otherwise},
\end{cases} \quad \text{with } \lambda = \frac{\Delta t}{\mu_0 \sigma (\Delta x)^2}. \]  

(3.53)

It can be seen, that in every time-step \( n \) a system of equations has to be solved. System (3.52) is generally nonlinear (unless \( H \) is linear), and can be solved by some fixed point or Newton iteration. If the matrix \( A \) is invertible, the fixed point equation to be solved is

\[ H^n = A^{-1}(-H \{ H^n \} + R). \]  

(3.54)

Nevertheless it is usually not desirable to use \( A^{-1} \) instead of the matrix \( A \) especially in the case of large problems, because \( A^{-1} \) is a full matrix and \( A \) is a sparse band matrix as it can be seen in (3.53).
3.8 General frame for the Picard-iteration of diffusion-type PDE with hysteresis

Considering a one dimensional diffusion problem (the one depicted in Fig. 3.6), the governing equations can be deduced from the following Maxwell’s equations (with the assumption of quasi-stationarity and having no external current sources)

\[ \nabla \times H = \sigma E, \quad (3.55) \]
\[ \nabla \times E = -\frac{\partial B}{\partial t}, \quad (3.56) \]

completed with the constitutive relation \( B = \mu_0(H + M) \), \( M = \mathcal{H}\{H\} \), where \( \mathcal{H}\{\cdot\} \) can be a general scalar hysteresis operator.

By taking the curl of (3.55), and substituting the appearing \( \nabla \times E \) by the right hand side of (3.56) taking into account the constitutive relation we have

\[ \nabla \times (\nabla \times H) = -\mu_0\sigma \left( \frac{\partial H}{\partial t} + \frac{\partial M}{\partial t} \right). \quad (3.57) \]

The above equation can be simplified further assuming that the divergence of \( H \) is zero\(^{10}\) (which is actually the case here)

\[ \nabla^2 H = \mu_0\sigma \left( \frac{\partial H}{\partial t} + \frac{\partial M}{\partial t} \right). \quad (3.58) \]

Introducing a linear differential operator \( \mathcal{L}\{\cdot\} \) for the space discretization of the Laplacian on the left hand side of (3.58) and introducing the time-step index \( n \) corresponding to the Backward-Euler difference scheme results in

\[ \mathcal{L}\{H^n\} = \frac{\mu_0\sigma}{\Delta t}(H^n - H^{n-1} + M^n - M^{n-1}), \quad (3.59) \]

substituting \( M^n \) by the application of the hysteresis operator, and introducing some simplifying notations, the above equation can be written as

\[ H^n = \theta \mathcal{L}\{H^n\} - \mathcal{H}\{H^n\} + R, \quad (3.60) \]

where \( \theta = \Delta t/(\mu_0\sigma) \) and \( R = H^{n-1} + M^{n-1} \).

At this point a more general notation will be introduced, thus the symbols referring to actual physical quantities are omitted, so the equation I want to solve is the following:

\[ u^n = \theta \mathcal{L}\{u^n\} - \mathcal{H}\{u^n\} + R. \quad (3.61) \]

\(^{10}\)In the actual 1D diffusion problem examined each component of \( \nabla \cdot H \) is zero.
Since $\mathcal{L}\{\cdot\}$ is a linear operator it can be represented with a matrix $L$, and introducing this matrix (3.61) takes the form
\[(I - \theta L)u^n = -\mathcal{H}\{u^n\} + R,\] (3.62)
where $I$ is the identity matrix, $\theta \in \mathbb{R}$, $R \in \mathbb{R}^p$ constants, and the matrix $I - \theta L$ will be denoted by $A$ resulting in
\[u^n = -A^{-1}\mathcal{H}\{u^n\} - r,\] (3.63)
with solution vector $u \in D \subset \mathbb{R}^p$, and $r = -A^{-1}R$.

Equation system (3.63) is the basis of the upcoming stability analysis, where the Picard-method will be applied for the solution of the nonlinear system.\textsuperscript{11}

### 3.8.1 Stability analysis

In order to solve (3.63), due to its nonlinear nature an iteration procedure has to be applied. Formulating the iteration by the Picard-method leads to the following scheme:
\[u^{n+1} = u^n - \eta (u^n - g(u^n)),\] (3.64)
where $g(u^n)$ represents the right hand side of (3.63), and $\eta > 0$ is the damping constant of the iteration. With the substitution of $g(u^n)$ it takes the form
\[u^{n+1} = u^n - \eta (u^n + A^{-1}\cdot \mathcal{H}\{u^n\} + r),\] (3.65)
containing the hysteresis operator $\mathcal{H}\{\cdot\}$ which is applied element-wise (in discretized space) to the elements of $u^n$.

As a starting point it can be stated, that if $||g_{PB}(u^n)|| < 1$ holds for all $u^n \in D \subset \mathbb{R}^p$, then (3.64) is convergent, since in that case $g(.)$ is surely a contraction mapping on the whole domain $D$.

Writing (3.65) by coordinates
\[
\begin{align*}
u_1^{n+1} &= u_1^n - \eta(u_1^n + (A^{-1}\cdot \mathcal{H}\{u^n\})_1 + r_1) \\
u_2^{n+1} &= u_2^n - \eta(u_2^n + (A^{-1}\cdot \mathcal{H}\{u^n\})_2 + r_2) \\
&\vdots \\
u_i^{n+1} &= u_i^n - \eta(u_i^n + (A^{-1}\cdot \mathcal{H}\{u^n\})_i + r_i) \\
&\vdots \\
u_p^{n+1} &= u_p^n - \eta(u_p^n + (A^{-1}\cdot \mathcal{H}\{u^n\})_p + r_p),
\end{align*}
\] (3.66)
\textsuperscript{11}Due to the presence of the nonlinear hysteresis operator, basically this system results from the above finite difference scheme, the FEM discretization of the problem, and the Yee-algorithm as well.
where the scalar hysteresis operator applies to the input vector \( \mathbf{u}^n \). At this point it will be assumed that the operator \( \mathcal{H}(\cdot) \) can be substituted by a function \( \mathbf{h} : \mathbb{R}^p \rightarrow \mathbb{R}^p \), which (at least locally) represents the behavior of the hysteresis operator, and the partial derivatives \( \partial h_i(u)/\partial u_j \) are exists and bounded for all \( i, j = 1, 2, \ldots, p \).

The coordinate functions of \( \mathbf{h}(\mathbf{u}) \) are the followings

\[
\begin{align*}
h_1(\mathbf{u}) &= f(u_1) \\
h_2(\mathbf{u}) &= f(u_2) \\
&\vdots \\
h_i(\mathbf{u}) &= f(u_i) \\
&\vdots \\
h_p(\mathbf{u}) &= f(u_p),
\end{align*}
\]

(3.67)

thus the \( i \)-th component \( h_i(\mathbf{u}) \) of the vector-valued function \( \mathbf{h}(\mathbf{u}) \) depends only on the \( i \)-th component of the input vector \( \mathbf{u} \). Introducing the above defined function in (3.66) has the following form

\[
\begin{align*}
u_1^{n+1} &= u_1^n - \eta(u_1^n + (A^{-1} \cdot \mathbf{h}(\mathbf{u}^n))_1 + r_1) \\
u_2^{n+1} &= u_2^n - \eta(u_2^n + (A^{-1} \cdot \mathbf{h}(\mathbf{u}^n))_2 + r_1) \\
&\vdots \\
u_i^{n+1} &= u_i^n - \eta(u_i^n + (A^{-1} \cdot \mathbf{h}(\mathbf{u}^n))_i + r_1) \\
&\vdots \\
u_p^{n+1} &= u_p^n - \eta(u_p^n + (A^{-1} \cdot \mathbf{h}(\mathbf{u}^n))_p + r_1),
\end{align*}
\]

(3.68)

and this system defines the mapping of the Picard-iteration scheme with the substituted function \( \mathbf{h}(\cdot) \), furthermore by taking the derivative of the right hand side, the \( \mathbf{g}_{PB}(\mathbf{u}) \) Jacobian of the system can be obtained.

It can be seen, that on the right hand side of the system, the same matrix-vector product can be found in each row. Taking a closer look to that operation it can be found, that it is advantageous to write the elements of
\( A^{-1} \cdot h(u) \) row-wise as dot-products (the indices \( n \) of time step are omitted)

\[
(A^{-1} \cdot h(u))_1 = a_1 \cdot h(u) = \sum_{j=1}^{p} a_{1j} h_j(u)
\]

\[
(A^{-1} \cdot h(u))_2 = a_2 \cdot h(u) = \sum_{j=1}^{p} a_{2j} h_j(u)
\]

\[
\vdots
\]

\[
(A^{-1} \cdot h(u))_i = a_i \cdot h(u) = \sum_{j=1}^{p} a_{ij} h_j(u)
\]

\[
\vdots
\]

\[
(A^{-1} \cdot h(u))_p = a_p \cdot h(u) = \sum_{j=1}^{p} a_{pj} h_j(u),
\]

where \( a_i \) denotes the \( i \)-th row of \( A^{-1} \), \( a_{ij} \) is the \((i, j)\) element of \( A^{-1} \) and \( h_j(u) = f(u_j) \) according to the definition of \( h(.) \).

Now the general form of the \( i \)-th row of \( g_{PB}(u) \) can be written as

\[
g_{PB,i}(u) = u_i^n - \eta(u_i^n + \sum_{j=1}^{p} a_{ij} h_j(u) + r_1) = \\
u_i^n - \eta(u_i^n + \sum_{j=1}^{p} a_{ij} f(u_j) + r_1),
\]

where \( u_j \) is the \( j \)-th element of vector \( u \), and from which the \( j \)-th element of the \( i \)-th gradient vector corresponding to the \( i \)-th row of the Jacobian is

\[
\frac{\partial g_{PB,i}(u)}{\partial u_j} = \frac{\partial u_i}{\partial u_j} - \eta \left( \frac{\partial u_i}{\partial u_j} + \frac{\partial}{\partial u_j} \sum_{k=1}^{p} a_{ik} f(u_k) \right),
\]

from where, regarding that

\[
\frac{\partial}{\partial u_j} \sum_{k=1}^{p} a_{ik} f(u_k) = \sum_{k=1}^{p} a_{ik} \frac{\partial f(u_k)}{\partial u_j} = a_{ij} \frac{\partial f(u_j)}{\partial u_j},
\]

since \( \frac{\partial f(u_k)}{\partial u_j} = 0 \) if \( k \neq j \) follows, that the \((i, j)\) component of the Jacobian has the form

\[
g'_{PB,(i,j)}(u) = \frac{\partial u_i}{\partial u_j} - \eta \left( \frac{\partial u_i}{\partial u_j} + a_{ij} \frac{\partial f(u_j)}{\partial u_j} \right),
\]

from where the diagonal elements \((i = j)\) of the Jacobian are

\[
g'_{PB,(i,i)}(u) = 1 - \eta \left( 1 + a_{ii} \frac{\partial f(u_i)}{\partial u_i} \right),
\]
and the off-diagonal elements \((i \neq j)\) have the form

\[
g'_{P_B,(i,j)}(u) = -\eta \left(a_{ij} \frac{\partial f(u_j)}{\partial u_j}\right). \tag{3.75}
\]

During the application of a numerical method the above Jacobian can be determined in runtime (since \(a_{ij}\) are known a priori) with local approximations of the actual derivatives of the hysteresis operator, and by the aid of the calculated Jacobian the stability properties of the iteration can be reliably estimated, and the damping constant \(\eta\) can be adjusted in runtime if needed in order to achieve faster convergence.

Another approach of obtaining information from map (3.65) is to drop the spatial dependence resulting from the hysteresis operator by assuming the worst case, i.e. that the derivative of the operator is the possible absolute maximum \(\chi_{\text{max}}\) resulting from the construction of the model, and the matrix \(A^{-1}\) is bounded by the supremum \(\lambda^*\) of the magnitudes of the elements of its spectrum. In this case the derivative of the right hand side of (3.65) can be written as

\[
|1 - \eta(1 + \chi_{\text{max}}\lambda^*)| < 1, \tag{3.76}
\]

constructing the ‘worst case’ scenario for the stability investigation.

From the condition above the following upper bound can be obtained for \(\eta\)

\[
\eta < \frac{2}{1 + \chi_{\text{max}}\lambda^*}, \tag{3.77}
\]

in which \(\lambda^*\) is the spectral radius of \(A^{-1}\). Since \(A^{-1}\) is a full matrix, instead of calculating the eigenvalues, an estimation of \(\lambda^*\) can be given based on the tri-diagonal band matrix \(A\) defined in (3.62) if the actual implementation of the discrete laplacian is known. It is known, that the eigenvalues \(\lambda^*_i\) of \(A^{-1}\) are the reciprocal of the eigenvalues \(\lambda_i\) of \(A\), thus

\[
\max_i |\lambda^*_i| = \frac{1}{\min_i |\lambda_i|}. \tag{3.78}
\]

Since matrix \(A\) is a tri-diagonal band matrix with well-defined structure and elements, the spectral radius \(\rho(A)\) can be estimated by the Gersgorin-theorem [12]. The theorem states that each of the \(\lambda_i\) eigenvalues of a matrix \(A\) lying on the complex plane is contained by one of the \(D_i\) disks defined by

\[
D_i = \{ z : |a_{ii} - z| \leq R_i \},
\]

where \(z \in \mathbb{C}\) is a complex number, \(a_{ii}\) is the \(i\)-th diagonal element of the matrix, and \(R_i = \sum_{j=1,j\neq i}^p |a_{ij}|\) is the absolute sum of the off-diagonal elements on the \(i\)-th row.

In this case let the diagonal elements of \(\theta L\) (defined in (3.62)) denoted by \(-d^*\), then each of the diagonal elements of \(A\) are \(a_{ii} = 1 + d^*\) and the \(i\)-th absolute row sum is \(R_i = r^*\). Since \(A \in \mathbb{R}^{p \times p}\) is symmetrical, its eigenvalues are real,
and from the Gersgorin-theorem it follows, that each of the eigenvalues are in the interval \([1 + d^* - r^*, 1 + d^* + r^*]\). Assuming that \(L\) defines a 2nd-order scheme,

\[
L_{ij} = \begin{cases} 
-2/(\Delta x)^2, & \text{if } i = j \\
1/(\Delta x)^2, & \text{if } |i - j| = 1 \\
0, & \text{otherwise}, 
\end{cases}
\]  

(3.79)

it follows, that \(d^* = r^*\), thus \(\lambda_i \in [1, 1 + 2d^*], \forall i\) and the lower bound for the eigenvalues \(\lambda_i \geq 1\), \(\forall i\) holds.

From this result and (3.78) it follows, that \(0 < \lambda^* < 1\), thus substituting \(\lambda^*\) by the upper bound in (3.77) results in

\[
\eta < \frac{2}{1 + \chi_{\max}},
\]

(3.80)

which gives the ‘worst case’ upper bound\(^{13}\) for the damping constant \(\eta\) ensuring the stability of iteration (3.64). As it is apparent from (3.64), lacking the examination of the inner structure of the iteration, the global stability criterion has the form of the same upper bound for the damping constant \(\eta\), that had been derived before (3.28) in the case of FDTD discretization.

\(^{12}\)In the first and last row \(r^* = d^*/2\), but that results in an even smaller interval, thus it is irrelevant concerning the current investigation.

\(^{13}\)This upper bound is essentially the same that has been derived in (3.28) for the case of the inner iteration of the problem solved by the Yee-algorithm, with no inherent spatial interdependence of the hysteresis operator.
3.9 Thesis 2.

I have carried out the thorough analysis of the hysteresis operator concerning
its stability properties by the method of Poincare-sections in an appropriately
constructed phase space, and as a result of the analysis I was able to derive a
symbolic expression for the exact stabilizing location of the accommodating minor
hysteresis loops. I have also analyzed the stability properties of the numerical
solution of a diffusion problem with hysteresis, and after the examination of
the Lyapunov space of the parameters of the iteration corresponding to the
numerical solution I have suggested a method for the cost-effective calculation of
an appropriate damping constant ensuring the convergence of the iteration [80, 81, 75, 74].

2.1. I have investigated the stability of the ODE-based scalar hysteresis model
developed in Thesis 1, concerning the behavior of the accommodating minor
loops. I have constructed an appropriate three-dimensional phase space for
the representation of periodic solutions and by the aid of this representation
I have derived a Poincare-map, from which the exact location of the stable
minor loop can be obtained [80].

2.2. I have examined a nonlinear diffusion problem with hysteresis by the
Lyapunov method and based on the results I have developed a method
for the selection of the damping constant of the nonlinear iteration applied.
A one dimensional diffusion problem and its numerical solution has been
analyzed, paying particular attention to the fixed-point iteration resulting
from the implicit nature of the iteration scheme in nonlinear material. I
have proven, that the unstable behavior of the inner fixed-point iteration
originates from bifurcation or - in extreme cases - chaotic behavior of the
attractor of the iterative map [81, 75, 74]. Based on the examination of
the parameter space of the iteration I have developed a damping constant
calculation method to speed up the iteration significantly, thus the average
number of iterations needed is more than an order of magnitude lower then
in the case of the original damping constant [81].

2.3. I have analyzed the general structure of Picard-type iterations for
diffusion-type PDE containing a hysteresis operator, and I have derived
a general approach for the estimation and runtime control of the damping
constant of the iteration. I have shown for the one-dimensional case, that
the global stability condition for the damping constant of the iteration is
the same, regardless of the hysteresis model for both of the discretization
methods (FDTD Yee-algorithm, FEM) applied, and I have pointed out, that
the local selection of the damping constant during the nonlinear iteration
is more efficient, than applying a global stability condition throughout the
whole numerical solution of the problem.
Chapter 4

Application of the Statistical Hysteresis Model in Field Problems

4.1 Aim of this research

In this part of my research the possible applications of the developed hysteresis model is introduced in a few field calculation problems, and through these case studies it is proven, that the model is widely applicable and versatile enough to apply in various engineering problems. Furthermore in this chapter an actual industrial application of the model is shown in the frame of a two-phase flow network calculation environment developed by our department for the study of a steam network in a suburban region.

4.2 Penetration depth in one-dimensional magnetic diffusion with hysteresis

As a first case study of implementation, results are presented from the application of the model in a magnetic diffusion problem in conducting media. The geometry and the boundary conditions of the problem are the same as in the case of the problem discussed in Section 3.3.

The goal of the study is the investigation of the penetration depth by comparison of the numerical results obtained with the analytical results from the literature [62]. Since the scope of the analytical results are limited to the linear, the single-valued abruptly changing nonlinear, and a very simple kind of hysteretic case, the results and discussion goes along this way as well, with an outlook towards the numerical investigation of a fully hysteretic case with minor hysteresis loops in the solution.
The governing equation of the problem can be derived from the quasi-static Maxwell’s equations
\[
\nabla \times \mathbf{H} = \sigma \mathbf{E}, \quad \nabla \times \mathbf{E} = -\dot{\mathbf{B}},
\]
and the corresponding constitutive relation according to the nature of the medium investigated. The sinusoidal excitation on the boundary of the halfspace is defined as \(A \sin(\omega t)\).

### 4.2.1 Linear medium

In the case of linear medium the constitutive relation is \(\mathbf{B} = \mu \mathbf{H}\), where \(\mu = \mu_0 \mu_r\), \(\mu_0 = 4\pi \cdot 10^{-7} \text{ H/m}\) is the permeability of the vacuum, and \(\mu_r\) is the relative permeability of the medium.

By the substitution of the above relation into (4.1) and taking the curl of the first equation, it takes the form
\[
\nabla \times (\nabla \times \mathbf{H}) = -\sigma \mu \dot{\mathbf{H}},
\]
where rewriting the left hand side results in
\[
\nabla (\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H} = -\sigma \mu \dot{\mathbf{H}},
\]
from which, in view of \(\nabla \cdot \mathbf{H} \equiv 0\) we arrive at
\[
\sigma \mu \dot{\mathbf{H}} - \nabla^2 \mathbf{H} = 0, \quad (4.4)
\]
which is the governing equation of linear magnetic diffusion.

The analytic formula for the penetration depth for the linear case is
\[
\delta_l = \sqrt{\frac{2}{\mu \sigma \omega}}, \quad (4.5)
\]
where \(\omega\) is the natural frequency of the sinusoidal excitation, and \(\delta_l\) is the depth where the amplitude of the signal drops to \((1/e)\)-times of the original excitation amplitude at the boundary of the medium.

The actual parameters applied in the simulation are the followings: \(\sigma = 2 \times 10^6 \text{ S/m}\), \(\omega = 2\pi \times 50 \text{ rad/s}\), \(\mu_r = 1000\), \(A = 5 \text{ A/m}\), and the result of the finite element simulation shows a perfect match with the analytic expression as it can be seen in Fig. 4.1 showing the sinusoidal field at the linear penetration depth calculated by (4.5). The maximum value of the excitation at the boundary in the simulation is \(A = 5\), which is decreased to \((1/e) \cdot A = 5/e \approx 1.839\) according to the definition of the penetration depth in linear medium.

\(^1\)Since according to the problem definition the only nonzero field component is \(H_z\), which has zero derivative in direction \(z\), all terms of \(\nabla \cdot \mathbf{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z}\) are zero.
Fig. 4.1. Cross-sectional sinusoidal signal at $x = 1.592 \cdot 10^{-3}$ ($\delta_l$) in the case of linear medium.

The result suggests, that the finite element model and simulation gives very accurately the same result as the analytical calculation, encouraging the application of the same model structure extended with additional terms containing nonlinear dependency of the flux density on the magnetic field.

### 4.2.2 Nonlinear medium

As a consequence of introducing a nonlinear term in the magnetic diffusion equation, the analytic comparison has to be restricted to a very limited set of applicable nonlinearities due to the difficulties of the analytical solution of the resulting nonlinear partial differential equation [62]. As it is shown in this section, the numerical results obtained are approaching the analytical result given for the case of sharp (step-like) change of magnetization, as the steepness of the actual nonlinear magnetization curve applied in the simulation becomes higher and higher, approaching infinity (abrupt change in magnetization). Nevertheless it has to be emphasized, that the actual solution of the nonlinear PDE still shows a very significant deviation from the analytical solution given in [62] due to the major simplifications applied during the derivation of the solution analytically.

The constitutive relation applied for nonlinear media is $B = \mu_0 (H + M)$, where $M$ is the magnetization in the medium, and the function $M(H)$ defining the nonlinear material behavior is a sigmoid-like single-valued nonlinear function exhibiting saturation. Substituting this relation into (4.1), and applying the same rules as through equations (4.2) - (4.4), the governing equation takes the form

$$\dot{H} - \frac{1}{\sigma \mu_0} \nabla^2 H = -M.$$  \hspace{1cm} (4.6)
According to the one-dimensional problem, and with the application of the nonlinear relationship

$$M(H) = \frac{2M_s}{\pi} \arctan \left( \frac{H}{H_0} \right),$$  \hspace{1cm} (4.7)

where $M_s$ is the saturation level of magnetization and $H_0$ is a shape-parameter responsible for the steepness of the nonlinear (magnetization) curve; (4.6) takes the form

$$\frac{\partial H}{\partial t} - \frac{1}{\sigma \mu_0} \nabla \cdot (\nabla H) = -\chi_d \frac{\partial H}{\partial t},$$ \hspace{1cm} (4.8)

with differential susceptibility $\chi_d$, defined as the derivative of magnetization $M$ with respect to magnetic field strength $H$, which can be also handled as the ratio of the time derivatives of the magnetization $M$ and field strength $H$ as follows

$$\frac{\partial M}{\partial t} = \frac{2M_s}{\pi H_0} \frac{1}{1 + (H/H_0)^2} \frac{\partial H}{\partial t} \frac{1}{\chi_d},$$ \hspace{1cm} (4.9)

and finally, collecting the time derivatives, (4.8) can be given in the following, more compact form

$$\left(1 + \chi_d\right) \frac{\partial H}{\partial t} - \frac{1}{\sigma \mu_0} \nabla \cdot (\nabla H) = 0,$$ \hspace{1cm} (4.10)

defining the nonlinear diffusion equation to be solved.

The formula for the penetration depth can be given for the case of abruptly changing flux density according to Mayergoyz [62] as follows,

$$\delta_{nl} = \sqrt{\frac{2}{\mu_m \sigma \omega}},$$ \hspace{1cm} (4.11)

where $\mu_m = B_m/H_m$ is defined as the quotient of the maximum flux density and the maximum field strength. Since $B_m$ is an ‘ever increasing’ function of $H_m$, it is advantageous to rewrite $\mu_m$ as a function of $M_m$ which approaches a constant value since $\lim_{H \to \infty} M(H) = M_s$. Applying $B = \mu_0(H + M)$, $\mu_m$ can be rewritten as

$$\mu_m = \frac{\mu_0(H_m + M_m)}{H_m},$$ \hspace{1cm} (4.12)

where the maximum magnetization $M_m$ can be substituted$^{2}$ by the saturation value of magnetization $M_s$ resulting in

$$\mu_m = \mu_0 \left( 1 + \frac{M_s}{H_m} \right).$$ \hspace{1cm} (4.13)

$^{2}$Under the assumption, that $H_m$ is high enough to saturate the medium.
It can be shown, that (4.13) is valid for all possible values of $H_m$ only in the case of abruptly changing magnetization\(^3\). The maximum value of magnetization $M_m$ can be obtained from (4.7) taking $H = H_m$, and using this $M_m$ in (4.12) and substituting the resulting $\mu_m$ into (4.11), it takes the form

$$\delta_{nl} = \sqrt{\frac{2\pi H_m}{\sigma \mu_0 \omega (\pi H_m + 2M_s \arctan(H_m/H_0))}},$$

(4.14)

which is the actual formula of the penetration depth $\delta_{nl}$ containing the implementation of the magnetization curve.

Taking the limit, approaching sharp transition of magnetization curve

$$\lim_{H_0 \to 0^+} \sqrt{\frac{2\pi H_m}{\sigma \mu_0 \omega (\pi H_m + 2M_s \arctan(H_m/H_0))}} = \sqrt{\frac{2H_m}{\sigma \mu_0 \omega (H_m + M_s)}},$$

(4.15)

results in the same expression as (4.11) with $\mu_m$ defined as (4.13).

It should be noted, that as the nonlinear magnetization curve $M(H)$ approaches the step function (as $H_0 \to 0$), the actual implementation of the magnetization curve ($\tan^{-1}$ in this case) becomes irrelevant. Indeed, substituting (4.13) into (4.11) immediately results in the right hand side of (4.15).

![Fig. 4.2. The magnetic field $H(t)$ at the calculated penetration depth $\delta_{nl} = 8.9 \cdot 10^{-5}$ for various values of shape parameter $H_0$.](image)

In Fig. 4.2 a parametric solution of the diffusion problem can be seen at the calculated penetration depth $\delta_{nl} = 8.9 \cdot 10^{-5}$, displaying the effect of changing

\(^3\)It can be also interpreted by intuition, that introducing $M_s$ instead of $M_m$ for any $H_m$ assumes a sharp magnetization curve
the $H_0$ shape-parameter. It is clear from the figure, that the sharper the magnetization curve, the smaller the penetration depth is. Furthermore though the amplitude of the magnetic field $H(t)$ is less than 5% (corresponding to sharp transition $H_0 = 0.05$) of the amplitude of excitation $(H(t, x = 0) = 5 \sin(\omega t))$, it is still far from being zero.

It has to be emphasized at this point, that the smallest deviation of the magnetization curve from sharp step function (Heaviside) results in very significant difference between the analytically calculated penetration depth (assuming abrupt, sharp transition) and the real penetration depth corresponding to smooth (but fast) transition of magnetization calculated by finite element method. In the case of real world applications and in numerical simulations as well, the function describing the magnetization curve has a finite derivative, thus the actual steepness of the curve cannot be neglected in the calculation of penetration depth.

The figures Fig. 4.3 and Fig. 4.4 show the surface- and contour-plots of the penetration depth $\delta_{nl}$ and the differential susceptibility $\chi_d$ as functions of $H$ and $H_0$, according to their defining formulae (4.14) and (4.9) respectively. The values
of the rest of the constants $\sigma, \mu_0$ and $\omega$ had been defined above. The figures - similarly to Fig. 4.2 - also suggest, that there are quite large differences in $\delta_{nl}$ depending on the steepness parameter $H_0$ of the magnetization curve.

### 4.2.3 Nonlinear medium with hysteresis

According to the assumptions above of having an infinite half space filled with conductive, ferromagnetic medium, the governing equation

\[
\dot{H} - \frac{1}{\sigma \mu_0} \nabla^2 H = -\dot{M}, \tag{4.16}
\]

is the same as it has been derived in the case of nonlinear medium, but in the case of ferromagnetic material the constitutive relation contains a hysteresis operator.

Coupling of (4.16) with (2.38) results in the following system of PDEs to be solved

\[
\dot{H} - \frac{1}{\sigma \mu_0} \nabla^2 H = -\dot{M}, \tag{4.17}
\]

\[
\dot{M}^* = \gamma q g (\delta H) \dot{H},
\]

where the second equation defines the hysteresis operator. The coupling between the two equations is realized by a simple relation between the normalized magnetization $M^*$ of the hysteresis operator and the real magnetization $M$ defined as $M = 2M_s M^* - M_s$, thus substituting the time derivative of $M$ into the first equation of (4.17) takes the form

\[
\dot{H} - \frac{1}{\sigma \mu_0} \nabla^2 H = -2M_s \dot{M}^*, \tag{4.18}
\]

\[
\dot{M}^* = \gamma q g (\delta H) \dot{H}.
\]

For the hysteretic case, the formula of penetration depth can be defined in a similar way as the nonlinear penetration depth (4.11), with a slight difference in the interpretation of $\mu_m$, which is in this case can be defined as

\[
\mu_m = \mu_0 (1 + \chi_{d|H_m}), \tag{4.19}
\]

where $\chi_{d|H_m}$ is the differential susceptibility at the amplitude $H_m$ of sinusoidal excitation, thus the penetration depth for the ferromagnetic case can be written as

\[
\delta_h = \sqrt{\frac{2}{\sigma \omega \mu_0 (1 + \chi_{d|H_m})}}. \tag{4.20}
\]

The following cross section plot (Fig. 4.5) shows the magnetic field strength and flux density at the point of penetration depth $\delta_h$. 

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Fig. 4.5. Cross section plot at penetration depth $\delta_h = 4.47 \cdot 10^{-4}$ in the case of hysteretic medium

It can be seen in the figure, that the flux $B(t)$ is a very small amplitude harmonic function above zero, suggesting that the penetration depth $\delta_h$ defined for the hysteretic case corresponds to a spatial location, where the flux becomes very small, but still nonzero. Comparing the penetration depths of the single valued nonlinear and the hysteretic cases, it is clear, that $\delta_h \gg \delta_{nl}$, since in the case of hysteresis the penetrating magnetic field ‘wave’ lives much longer due to the remanent magnetization.

Fig. 4.6. Cross section plot and hysteresis curve at $x = 7.1 \cdot 10^{-5}$

For contrast, Fig. 4.6 shows the values of magnetic field strength $H(t)$, the flux $B(t)$, and the corresponding hysteresis curve at a cross sectional point ($x = 7.1 \cdot 10^{-5}$) closer to the boundary with sinusoidal excitation, where the magnitudes of field strength and flux density are much higher. It can be clearly seen from the figures, that the field values at the calculated penetration depth $\delta_h$ are negligible.
compared to the magnitude of the field values closer to the boundary.

The applicability of the presented hysteresis model is not restricted to the field of magnetics, it can be applied to other fields of physics as well for example to isobaric, temperature induced vapor-liquid phase transition as it is shown in the next sections.

4.3 Model implementation into a two-phase flow problem

In this section the developed hysteresis model is implemented into a two-phase flow problem, where the phase transition is modeled by an improved version of the apparent heat capacity method with the application of hysteresis [40, 41, 85].

As an introduction, a brief overview is given about the vapor-liquid phase transition, its modeling approaches, application areas, and the connection between hysteresis and first-order phase transition. It will be shown that hysteresis can be interpreted as an intrinsically appearing phenomenon during vapor-liquid phase transition and the hysteresis model developed is applicable for the improvement of the investigation of these types of physical problems as well.

4.3.1 The van der Waals theory of phase transition

As a starting point for modeling of first order phase transition, the van der Waals theory of phase transition is need to be taken into consideration as the most classical theory of liquid-gas phase change [13, 30, 71, 88, 105]. The simplest form of a description of a fluid system is based on the assumption that there is no (or negligibly weak) interaction between the particles of the system. This approach leads to the ideal gas law

\[ pV = NkT, \]  \hspace{1cm} (4.21)

where \( p \) is the pressure, \( V \) is the volume, \( N \) is the number of molecules in the system, \( T \) is the temperature, and \( k = 1.38 \times 10^{-23} \text{J/K} \) is the Boltzmann constant.

The ideal gas law (4.21) can also be written by the introduction of the number of moles \( n = N/N_A \) (\( N_A \) is Avogadro’s number) in the system as follows

\[ pV = nRT, \]  \hspace{1cm} (4.22)

where \( R = kN_A \) is the gas constant.

The above description inherently enables the unlimited compressibility of the gas and the presence of attractive forces between molecules is not taken into account either. A major improvement of van der Waals’ theory was the addressing of these two problems arising with the ideal gas law. First of all assuming a limit of compressibility by introducing a minimal volume \( V_{min} = (b/N_A)N \), which is
an extensive quantity (linear in $N$) with the proportionality constant $b/N_A$ and substituting $V$ by $V - V_{mm}$ in (4.22) results in

$$p(V - nb) = nRT,$$  \hspace{1cm} (4.23)

which is referred as the *Clausius equation of state* in which $b$ is a phenomenological parameter.

One further step to improve the state equation is to take into account the attractive forces between the molecules. The assumption here is that the attracting force will result in a decrease of the pressure of the gas, and this pressure decrease $\Delta p$ has two sources. One of them is the lowered momentum of each molecule, and the other one is the decreased number of molecules hitting the wall of the container. Assuming that both of these contributions are proportional to the density $N/V$ of the molecules, the pressure decrease is proportional to the square of the density and introducing the proportionality constant $a/N_A^2$ it can be written as $\Delta p = (a/N_A^2)(N/V)^2$ with another phenomenological parameter $a$.

Decreasing the pressure of (4.23) by this $\Delta p$ yields

$$p = \frac{nRT}{V - nb} - \frac{an^2}{V^2},$$  \hspace{1cm} (4.24)

which is the van der Waals equation of state. An other form of (4.24) can be derived by the introduction of specific volume $v = V/n$ (volume per mole), which results in

$$p = \frac{RT}{v - b} - \frac{a}{v^2}.$$  \hspace{1cm} (4.25)

By the aid of (4.25) the isotherms corresponding to the state transition can be calculated. A couple of transition curves can be seen in Fig. 4.7.

As it can be seen in the figure, the van der Waals theory has a serious problem for temperatures lower, than the critical temperature (corresponding to the uppermost isotherm in the figure), since the isotherms corresponding to these temperatures has a region where $(\partial p/\partial v)_T > 0$, which is clearly unphysical resulting in negative isothermal compressibility $\kappa = -\frac{T}{(\partial v/\partial p)_T}$.

In order to resolve this problem Maxwell suggested a simple solution known as Maxwell construction or Maxwell’s equal area rule, depicted in Fig. 4.8.

According to Maxwell’s approach, the region containing the positive derivative portion is replaced by a straight horizontal line in a way, that it makes the two areas (denoted by area A and area B in the figure) equal, ensuring the equality of the change in the corresponding internal energy function, which is defined as follows

$$\Delta u = T\Delta S - \int_{V_1}^{V_2} p(v)dv,$$  \hspace{1cm} (4.26)
where \( u \) is the internal energy, and \( S \) is the entropy of the system. Hence the change of internal energy along an isotherm based on the van der Waals equation can be written as

\[
\Delta u = T(S_2 - S_1) - \int_{V_1}^{V_2} p(v) dv,
\]  

(4.27)

applying the function \( p(v) \) from equation (4.25), and in the case of the substitution with a straight line the change of internal energy has the form

\[
\Delta u = T(S_2 - S_1) - p(V_2 - V_1),
\]  

(4.28)
Besides, that this heuristic approach provides a reasonable approximation of the procedure of phase transition, it is also clear, that there is a ‘metastable’ region around, where the actual phase transition can take place, and this is the point, where hysteresis phenomenon appears, under the assumption, that the phase transition has finite width enabling a certain amount of supersaturation. This concept allows a ‘smoother’ representation of the phase transition process, and enables the introduction of an appropriate hysteresis operator - based on a statistical approach - for the description of non-equilibrium phase change as it is discussed in the following sections.

4.3.2 The improved apparent heat capacity method for isobaric phase-transition

During the modeling and simulation of phase transitions, the application of a phase transition model having a ‘simple law’ of phase change can be really advantageous [17, 41, 64]. In these kind of models the phase transition criterion is usually defined by the simple rule, stating, that the actual phase transition occurs, when the temperature reaches the pressure defined equilibrium, and the order parameter of phase transition can be obtained directly from the physical quantities. These so called equation-of-state type models are easy to use, but they suffer from singularity problems due to the extremely high gradients caused by the step-like phase transition under the assumption, that the phase transition is instant, which is a serious difficulty, that can be overcome by assuming, that the region, where phase transition takes place has a finite width. Since the vapor-liquid phase transition processes appear on several, quite important and diverse areas of engineering [14, 31, 85, 49], and the modeling of the heat transfer and fluid flow problems coupled with dynamic phase transition can be extremely complex, any improvement of the equation-of-state type models improving their applicability is desirable.

One of the most preferred methods for modeling heat transfer problems with phase change is the apparent heat capacity method [17, 64], where latent heat effects due to phase change are incorporated in the so-called apparent heat capacity. The advantage of this approach is that temperature is the primary dependent variable, that derives directly from the solution of the energy-conservation equation. For purely conductive heat transfer, the heat conduction in the two-phase system can be described by

\[ C_p \frac{\partial T}{\partial t} + L \rho_1 \frac{\partial \xi}{\partial t} + \nabla \cdot (-\lambda \nabla T) = 0, \]

(4.29)

4Conductive heat transfer is described here, which will be extended to full convection-conduction with the introduction of hysteresis operator later.
where \( C_p \) denotes the effective volumetric heat capacity 
\[ C_p = c_{p1} \rho_1 \xi + c_{p2} \rho_2 (1 - \xi), \]
\( c_p \) is the specific heat capacity, \( \xi \) is the volume fraction of phase 1, \( L \) is the specific latent heat corresponding to phase transition, \( \lambda \) is the effective thermal conductivity of the mixture, and the numeric subscripts are the phase indices. At this point, with a subtle reformulation of the rate of latent heat release, it can be incorporated into the heat capacity and the equation takes the form
\[
\left( \frac{C_p + L \rho_1}{c_a} \right) \frac{\partial \xi}{\partial t} + \nabla \cdot (-\lambda \nabla T) = 0, \tag{4.30}
\]
where \( c_a \) is called the **apparent heat capacity**. Applying this method to the water vapor-liquid phase transition, by assuming constant heat capacity of the pure phases and constant latent heat \( (c_{pv} = 2.71 \text{ kJ/(kgK)}, c_{pl} = 4.4 \text{ kJ/(kgK)}, L = 2.26 \cdot 10^3 \text{ kJ/kg}) \), the apparent heat capacity versus relative temperature functions have a shape like the ones shown in Fig. 4.9.

The Dirac-\( \delta \) function (corresponding to impulse-like release/absorption of latent heat) can be approximated by Gaussian probability density function with various \( \Delta T \) according to
\[
\frac{d \xi}{dT} = \frac{\epsilon}{\sqrt{\pi}} \exp \left( -\epsilon^2 (T - T_s)^2 \right), \tag{4.31}
\]
where \( \epsilon \) is selected so, that \( \text{erf}(\epsilon \Delta T) = 1 - \delta_{\epsilon} \), where \( \Delta T \) is one-half of the prescribed phase change interval, \( T_s \) is the saturation temperature, \( \delta_{\epsilon} > 0 \) is a sufficiently small constant, and the function \( \text{erf} \) is defined as
\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt. \tag{4.32}
\]

The effective heat capacity, based on (4.30) can be defined as \( C_e = c_a \rho \), where \( \rho \) is the average density. The effective heat capacities with respect to the relative temperature for various values of \( \Delta T \) are shown in Fig. 4.10, representing the actual latent heat release/absorption as a local peak (gaussian-PDF) of specific heat capacity.

It can be seen that too small values of \( \Delta T \) would probably cause numerical difficulties due to very sharp ‘peaks’, but on the other hand, too large values cause release (or absorption) of latent heat at temperatures where phase transition is still unexpected. It appears from the numerical solutions presented in the literature [17], that the results obtained by approximating the phase change at a fixed temperature with a gradual change over a small temperature interval should be acceptable if
\[
\frac{2 \Delta T}{|T_{\text{init}} - T_{\text{end}}|} < 0.1, \tag{4.33}
\]
Fig. 4.9. Apparent heat capacity of water at $T_s = 182^\circ$C (10 bar) with various phase transition interval widths.

Fig. 4.10. Effective heat capacity of water at $T_s = 182^\circ$C (10 bar), with various phase transition interval widths, corresponding to vapor-liquid phase transition.

which practically means, that the width of the temperature interval of the smoothed phase transition region is at most the one tenth of the whole temperature range of the phase change process modeled.

In Fig. 4.10 it can be seen, that the maxima of the effective heat capacity functions are shifted from the saturation temperature towards the higher temperature regions, thus the latent heat is mainly released before the transition temperature is actually reached. The reason behind this is the large difference
between the densities of the pure phases (i.e. liquid water, and steam), and this difficulty can be addressed by choosing the vapor mass fraction $\varphi$ as the order parameter of the phase transition leading to a slightly different formalism and much more symmetrical effective heat capacities\(^5\).

### The statistical approach of non-equilibrium phase transitions

Statistical models of phase transition rely on the concept called the *local bistability hypothesis*, which originates from the works of Née on magnetic hysteresis. According to this hypothesis, a system can be handled as a collection of simple bistable units having only two free energy configurations [1, 30], and this way the overall behavior of the system during the phase transition can be derived on the basis of the ensemble consisting of these independent, elementary bistable entities.

This approach has been already applied to study hysteresis and relaxation effects [65, 4]. Basso et al. [4] proposed a hysteresis operator for thermally induced first-order phase transitions, and they found, that the switching rules of phase change are identical to those of the Preisach model of hysteresis [61, 33], therefore all the methods developed for Preisach model of hysteresis can be immediately applied to describe the thermally induced phase transformations in solids. The difficulty in this approach is, that the Gibbs free energy functions of the pure phases have to be known a priori. This feature is a serious drawback during the application of this method in actual engineering problems. Furthermore applying a Preisach-type hysteresis model to fluid systems, where the hysteresis memories have to be coupled to small volumes of fluid, which are in continuous movement, could lead to overwhelming complexity. On the other hand the proposed statistical model of hysteresis (see Thesis 1) fits well into these kind of application areas.

#### 4.3.3 A two-phase flow model with hysteresis

The system of conservation equations of a two-phase flow with heat transfer consists of three balance equations of mass, momentum and energy, and a fourth equation describing the evolution of the order parameter corresponding to the phase transition. In this case the evolution of the order parameter is described by a hysteresis operator which can be formalized as a PDE in a similar fashion to the nonlinear magnetic diffusion discussed above (see section 4.2.3). Coupling the operator equation to the three model equations of the flow, results in the system to be investigated.

Introduction of hysteresis into two-phase flow with phase transition can be really advantageous, since according to literature sources the equilibrium phase

\(^5\)In the actual FEM simulations in the case study, the vapor mass fraction $\varphi$ had been chosen as order parameter.
change models could not describe the experimentally observed supersaturation of vapor, therefore they overestimate the rate of condensate formation and the resulted heat transfer coefficient [46], thus it is reasonable to introduce a certain level of supersaturation and supercooling by the aid of the hysteresis operator. The particular value of the allowed supersaturation/supercooling is calculated on the basis of the hysteresis operator developed in Thesis 1. as it is outlined below, together with some basic assumptions on the structure of our physical fluid domain.

Since phase transition between water vapor and liquid belongs to the class of first-order phase transitions [11], it can be interpreted as a molecular exchange process that develops during a finite time interval, thus condensation/evaporation does not occur immediately, rather it has to be considered as a time dependent process leaving a narrow barrier to unstable and metastable states during the phase transition. A family of isotherms, and the spinodal decomposition curve of the liquid-vapor phase transition of water can be seen in Fig. 4.11 in the $p-v$ plane. From Fig. 4.11a it is apparent, that the spinodal curve (dashed line) is formed as the loci of local extrema of the van der Waals isotherms [2] (bounding the area of positive isothermal compressibility), while Fig. 4.11b shows the same phase diagram calculated by the IAPWS-95 equation of state [69]. The spinodal is considered as the collection of locus points surrounding the unstable region, separating the unstable and metastable areas, and it must be enveloped by the coexistence (binodal) curve as it can be seen in Fig. 4.11b. Metastable means in this context, that the vapor is stable concerning small disturbances, but in the case of significantly larger disturbances, the metastable single-phase state changes to a stable two-phase state [47].

In order to embed hysteresis into the context outlined here, the important concept of fluid cluster has to be introduced. The idea, that the model fluid consists of clusters of molecules [67] can be built on the analogy between the potential energy of shape memory alloys under shear load, and the specific Gibbs free energy of a van der Waals gas as it was shown by Vortmann et. al [99]. Each cluster consists of the same number of water molecules and, thus it has a constant mass. Changing of the pressure or temperature modifies the occupied volume of a cluster, leading to the change of the specific volume. The clusters are situated at different locations of the energetic potentials, and if they manage to pass the energetic barrier, they change their states. In this approach, the individual fluid clusters are characterized by a sharp existence function $x \in \{0, 1\}; \quad x = 1$ for the vapor phase, and $x = 0$ for the liquid phase respectively. In the ideal case of an isobaric temperature-induced phase transition, vapor-liquid phase change occurs if the temperature drops below the equilibrium saturation temperature determined by the local pressure. A non-equilibrium assumption (i.e. the finite speed of phase transition) enables some degree of supersaturation in the cluster without phase change. At this point, the limit of the acceptable

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6IAPWS - International Association for the Properties of Water and Steam
supersaturation $\Delta T$ has to be determined, which certainly has to be somewhere between the vapor spinodal and the equilibrium curve. In this work a $T/T_c$ ($T_c$ is the critical temperature) dependent supersaturation limit is applied according to the approach proposed by Jancskar [39]. The limit curve defined this way is sufficiently close to the equilibrium curve and vanishes at the critical point.

It means, that each individual cluster that is in vapor state can transfer into the liquid state in the temperature interval defined by $[T_s - \Delta T, T_s]$ and the
evaporation of a cluster that is in liquid form can occur in the temperature interval of $[T_s, T_s + \Delta T]$, consequently this switching process shows a hysteretic character considering a whole ensemble of fluid clusters, in a similar fashion as it was shown in the case of ferromagnetic ‘domain switching’ (Fig. 2.2, Fig. 2.3).

After introducing the role of hysteresis in phase transition, the governing equations of the flow can be established and extended with the proposed hysteresis model. The interface between the two phases is assumed to have a finite width and it is characterized by rapid but smooth transitions in the density, viscosity and other physical properties, and a phase function is introduced to represent the transition between the phases, and the set of conservation equations are solved for the entire domain allowing the properties vary across the phase separating interface characterized by this phase function.

The governing equations of the momentum transport are the Navier-Stokes equations with the $\kappa - \epsilon$ turbulence model [103] due to the fact, that steam flow is characterized by large Reynolds-numbers hence it is in the turbulent regime even in the case of extremely low average flow velocities, and the corresponding literature suggests, that particularly good results can be achieved by the application of $\kappa - \epsilon$ model [56]. The momentum- and mass-conservation equations are

$$\frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} = \nabla \cdot \mathbf{S}_u + \mathbf{F}_e,$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} = \nabla \cdot \mathbf{S}_u + \mathbf{F}_e,$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

where the first two equations of (4.34) represent the momentum conservation containing the velocity $\mathbf{u}$, the average density $\rho$, the source term $\nabla \cdot \mathbf{S}_u$, and the external forces $\mathbf{F}_e$, while the third equation stands for the conservation of mass. The source term corresponds to the sources due to pressure gradient, shear stresses, and turbulent kinetic energy respectively, with the notations; pressure $p$, kinematic viscosity $\mu$, turbulent viscosity $\mu_T = \rho C\kappa^2/\epsilon$, turbulent kinetic energy $\kappa$, and the identity tensor $\mathbf{I}$.

The equations representing the transport of turbulent kinetic energy $\kappa$ and the dissipation $\epsilon$ containing appropriate source terms according to [103] are the followings

$$\frac{\partial \kappa}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \kappa = \nabla \cdot \left[ \left( \mu + \mu_T \right) \left( \frac{\mu_T}{\sigma_{\kappa}} \right) \nabla \kappa \right] + P_\kappa + \rho \epsilon,$$

$$\frac{\partial \epsilon}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \epsilon = \nabla \cdot \left[ \left( \mu + \mu_T \right) \left( \frac{\mu_T}{\sigma_{\epsilon}} \right) \nabla \epsilon \right] + C_{\epsilon 1} \frac{\epsilon}{\kappa} P_\kappa - C_{\epsilon 2} \rho \epsilon^2 / \kappa,$$

where the $\kappa - \epsilon$ model constants can be obtained from experimental data [103, 56], and the values applied here are listed in Table 4.1.
Table 4.1. Model constants of the $\kappa - \epsilon$ turbulence model

<table>
<thead>
<tr>
<th>$\sigma_\kappa$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_\epsilon$</td>
<td>1.3</td>
</tr>
<tr>
<td>$C_{\epsilon 1}$</td>
<td>1.44</td>
</tr>
<tr>
<td>$C_{\epsilon 2}$</td>
<td>1.92</td>
</tr>
<tr>
<td>$C_\mu$</td>
<td>0.09</td>
</tr>
</tbody>
</table>

The energy balance of the system can be formulated as

$$\rho c_p \frac{\partial T}{\partial t} + \nabla \cdot (-\lambda \nabla T + \rho c_p T \mathbf{u}) = -L \rho \frac{\partial \phi}{\partial t},$$

(4.36)

where $L$ is the specific latent heat, $c_p$ is the heat capacity of the mixture, $\rho$ is the mixture density, $\lambda$ is the heat conductivity, and $\phi$ is the vapor mass fraction (order parameter of phase transition). The mixture averages can be calculated as follows

$$c_p = c_{pv} \phi + c_{pl} (1 - \phi),$$
$$\rho = \rho_v \xi + \rho_l (1 - \xi),$$
$$\lambda = \lambda_v \xi + \lambda_l (1 - \xi),$$

(4.37)

in view of the vapor mass fraction $\phi$, and the vapor volume fraction $\xi = \phi \rho / \rho_v$.

The indices $v$ and $l$ correspond to the vapor and liquid phases respectively.

The order parameter $\phi$ of the phase transition is assumed to be dependent on the temperature through the hysteresis operator developed in Thesis 1 according to the local non-equilibrium phase transition assumption. The hysteretic relation is described by the following equation, applying the general differential form (2.38) of the hysteresis model

$$\frac{\partial \phi}{\partial t} = \gamma^g g(\delta T) \frac{\partial T}{\partial t},$$

(4.38)

representing the time evolution of order parameter $\phi$, with the clear analogy between $\phi - T$ and $M - H$ relation. Nevertheless (4.38) in this form is only appropriate for implementation into heat transfer problems with diffusion only, because in the presence of convective transport of energy, the convective transport of $\phi$ itself has to be taken into account as well, which extends (4.38) as follows

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{u}) = \gamma^g g(\delta T) \left( \frac{\partial T}{\partial t} + \mathbf{u} \nabla T \right),$$

(4.39)

The time derivative of $T$ in the source term has also been changed, since it has to be calculated along a streamline in the velocity field in order to follow
the moving fluid cluster. In addition to this change, the parameter $\delta$ of the hysteresis model has to be changed as well, because it represents the sign of the time derivative of the input of the hysteresis operator, thus according to (4.39) the original definition (2.27) of $\delta$ changes to

$$\delta = \text{sgn} \left( \frac{\partial T}{\partial t} + u \nabla T \right). \quad (4.40)$$

With the substitution of $\partial \varphi / \partial t$ from (4.39) into (4.36) results in

$$\rho c_p \frac{\partial T}{\partial t} + \nabla \cdot (-\lambda \nabla T + \rho c_p T u) = -L\rho \left[ \gamma \delta g(\delta T) \left( \frac{\partial T}{\partial t} + u \nabla T \right) - \nabla \cdot (\varphi u) \right], \quad (4.41)$$

which can be rearranged into

$$\rho \left( c_p + L\gamma \delta g(\delta T) \right) \frac{\partial T}{\partial t} + \nabla \cdot (-\lambda \nabla T + \rho c_p T u) + L\rho \gamma \delta g(\delta T) u \nabla T = L\rho u \nabla \varphi, \quad (4.42)$$

assuming $\nabla \cdot u = 0$, and since under the same assumption the third term of (4.42) can be expressed as $\nabla \cdot (L\rho \gamma \delta g(\delta T) u T)$, it can be incorporated into the divergence operator on the left hand side of (4.41) containing the original diffusive and convective fluxes, resulting in

$$\rho \left( c_p + L\gamma \delta g(\delta T) \right) \frac{\partial T}{\partial t} + \nabla \cdot (-\lambda \nabla T + \rho c_p T u + L\rho \gamma \delta g(\delta T) u T) = L\rho u \nabla \varphi, \quad (4.43)$$

from which, after factoring the convective flux, the following form can be derived

$$\rho \left( c_p + L\gamma \delta g(\delta T) \right) \frac{\partial T}{\partial t} + \nabla \cdot \left( -\lambda \nabla T + \rho T u \left( c_p + L\gamma \delta g(\delta T) \right) \right) = L\rho u \nabla \varphi, \quad (4.44)$$

where the underbraced expression $C_e = c_p + L\gamma \delta g(\delta T)$ is the effective heat capacity corresponding to the phase transition. It is clear from this definition of the effective heat capacity, that it integrates seamlessly into the concepts depicted above about the frame of apparent heat capacity method. The first term of $C_e$ is the average heat capacity of the mixture, and the second term represents the 'apparent' increase of heat capacity due to release/absorption of latent heat during phase transition process. Furthermore the apparent heat capacity in this formulation is constructed by the application of a hysteresis operator, resulting in an allowable amount of supersaturation, thus ensuring the proper way\footnote{Proper in the sense, that according to the behavior of the hysteresis operator, no latent heat is released before the saturation temperature is actually reached.} of
release/absorption of latent heat, while having a smoothing effect on the large gradients arising, thus providing more stability for the numerical solution \[41, 85\]. Rewriting (4.44) with the simplified notation results in the final form of energy conservation as follows

\[
\rho C_e \frac{\partial T}{\partial t} + \nabla \cdot \left( -\lambda \nabla T + \rho C_e uT \right) = L \rho u \nabla \varphi. \tag{4.45}
\]

The source term on the right hand side of (4.45) reflects the energy gain/loss along a streamline. It is clear from the formulation of the source term, that the gradient of \( \varphi \) only affects the local source (local change of \( \varphi \) incorporated into effective heat capacity \( C_e \)) along the streamlines, since \( u \nabla \varphi \) is zero, when the two vectors are perpendicular, and has its maximum, when the two vectors are completely aligned.

The fact, that the hysteresis model developed has a formulation, which fits to the description of phase transition so perfectly, suggests, that the philosophy of the model has a strong connection to the fundamentals of phase change, thus it can be a really good tool for the modeling of these kind of phenomena, not restricted to the fields of ferro-magnetic and vapor liquid phase transition, but to describe and model other kind of phase transition and avalanche-like phenomena as well.

**Case study and simulation results**

In this section simulation results obtained by the finite element analysis of the two-phase flow problem described above is presented through a case study. The goal here is to present the behavior of the hysteresis model in an actual two-dimensional field computation problem involving two-phase flow with phase transition. The study describes a pipe segment with superheated steam on the inlet and heat loss towards the environment through the wall of the pipe. The inflow velocity is set to a very low value \( \mathbf{n} \cdot \mathbf{u}_0 \approx 5 \cdot 10^{-2} \text{ m/s} \) in order to avoid the modeling domain having very high aspect ratio. The geometry of the pipe segment with boundary conditions and the mesh structure can be seen in Fig. 4.12.

Quadrilateral mesh with boundary layers is chosen, because of the convection dominated nature of the problem. The sides of the quad elements are aligned to the dominant flow direction, and the boundary layer helps to resolve the large gradients close to the wall of the pipe. This way the problem can be solved quite efficiently\(^8\) without the application of adaptive mesh refinement techniques.

The boundary conditions are set to ensure the phase transition by low inlet velocity and high heat transfer coefficient corresponding to the heat loss through the wall of the pipe. The heat loss on boundary \( \Gamma_4 \) is defined as

\[
\mathbf{n} \cdot (\lambda \nabla T) = \alpha(z)(T - T_e) \tag{4.46}
\]

\(^8\)The number of quadrilateral elements in the mesh is 7600.
where \( \mathbf{n} \) is the boundary normal, \( \alpha \) is the heat transfer coefficient, and \( T_e \) is the external temperature around the pipe. In order to induce phase transition into the metastable region and back resulting in minor hysteresis loops of \( \varphi - T \) relation, the heat loss is defined by a function along the \( \Gamma_4 \) boundary as

\[
\alpha(z) = \alpha_0 (1 - 0.95 \varepsilon_s(z)) \tag{4.47}
\]

applying a smoothed Heaviside function \( \varepsilon_s(.) \) defining a drop of the heat transfer coefficient to 5% of its original value \( \alpha_0 \) at the half length \( z = 0 \) of the boundary \((z \in [-1, 1])\).

---

Fig. 4.12. Boundary conditions and mesh structure of the domain

(a) (b)

Fig. 4.13. Temperature field for various heat losses toward the environment

(a) \( \alpha_0 = 0.11 \), (b) \( \alpha_0 = 1.26 \)

Fig. 4.13 shows the temperature distribution on the domain for two different values of the heat transfer coefficient. In the figure the temperature field is given as a temperature difference from the saturation temperature \( T_s \), thus it is clear,
that in the case of high enough heat transfer coefficient the temperature drops below the saturation and phase transition occurs.

![Figure 4.14. Velocity field for various heat losses toward the environment](image)

(a) $\alpha_0 = 0.11$, (b) $\alpha_0 = 1.26$

As it can be seen in the figure, the temperature distributions are quite similar, the main deviation can be found in the region of the lowest temperature, since in Fig. 4.13b the system has to ‘warm up’ from the metastable region, which requires extra energy in the form of latent heat, resulting in a sharper temperature transition zone due to the ‘avalanche-like’ property of hysteretic phase transition.

The velocity fields also reflect the phase transition due to the increased density of the mixture along the pipe wall, which results in decreased velocity as it can be seen in Fig. 4.14.

In order to investigate the metastable region of the phase transition described by the hysteresis operator, Fig. 4.15 shows the hysteretic relation between the temperature, and the vapor mass- and volume fractions for various heat transfer coefficients indicated in the plot legends. It is apparent from the figure, that due to the high density differences between the pure phases, even a significant decrease of the vapor mass fraction implies only a very small decrease of the vapor volume fraction - which is the actual phase separator quantity - suggesting a very tiny amount of liquid formation on the wall of the pipe. As it can be seen in the $\varphi$ vs. $T - T_s$ plot in Fig. 4.15, the release/absorption of latent heat starts exactly at the saturation temperature $T_s$.

In Fig. 4.16 the relative temperature $T - T_s$ and the vapor mass fraction $\varphi$ can be tracked along the wall of the pipe (boundary $\Gamma_4$, Fig. 4.12), where the actual phase transition occurs due to the heat loss defined by (4.46) and (4.47). The result of changing the heat transfer coefficient according to (4.47) can be clearly seen in the figure, and it is also apparent, that the application of the hysteresis operator results in quite smooth behavior of the temperature in the metastable region, hence as the actual temperature passes the saturation value, the temperature gradient decreases significantly, but it does not change abruptly.
Fig. 4.15. Minor hysteresis loops representing the mixture behavior in the metastable region for various heat transfer coefficients.

Fig. 4.17 shows the average mixture density $\rho$, and the z-direction velocity $u_z$ along the boundary $\Gamma_4$ representing the pipe wall. As the figure suggests, due to the phase transition, the density of the mixture will change resulting in a change of the velocity of the flow.

The results shown above are really encouraging in a sense, that the implementation of the proposed hysteresis operator can be accomplished relatively easily into the FEM framework, since it is represented by a PDE (4.39), which enables a straightforward coupling to the other field equations, and furthermore besides, that the operator ‘behaves well’ in the simulation, the introduction of the hysteresis operator into this convection dominated two-phase
flow problem is really advantageous enabling more accurate modeling of phase transition phenomena \cite{85, 84, 40}.

In addition to the case study presented above, the hysteresis model developed had been implemented into a two-phase flow network simulation environment as a computational core in the form of a ‘reusable’ .NET component (see Appendix D).
Fig. 4.17. Average density and velocity along the wall of the pipe for various heat transfer coefficients

4.4 Thesis 3.

I have derived a formulation for the coupling of the developed hysteresis operator to the equations of a convection dominated fluid-flow problem, and I have implemented the developed hysteretic flow model into a two-phase flow network simulation software package as a computational core, which can be modularly embedded into higher level computational environments as a reusable component. I have demonstrated the applicability and versatility of the developed hysteresis model in two case studies, from the field of magnetics and two-phase flow with dynamic phase transition respectively. In the studies I have applied the finite
element method as primary numerical tool for the investigation of the behavior of the hysteresis operator developed [40, 85, 41, 82, 83].

3.1. I have implemented my model in a two-phase flow field calculation problem by extending the energy conservation equation of convective flow with a source term containing a hysteresis operator representing the release/absorption of energy in the form of latent heat, according to the principle of vapor/liquid phase transition. Based on this extended formulation, I have inserted the developed hysteresis model into a finite element calculation involving vapor-liquid first order phase transition [40, 85, 41] proving the applicability of the proposed model and demonstrating the hysteretic behavior in the metastable region.

3.2. I have implemented the model of two-phase flow with hysteretic phase transition as the computational core of an actual network flow calculation engineering software [82, 83] providing a simulation engine in the form of reusable component for all kinds of higher level software environments, which have access to interfaces offered by .NET based reusable components.

3.3. I have proved by a sequence of case studies regarding penetration depth of magnetic field in nonlinear media, and two-phase flow with dynamic phase transition, that both the developed hysteresis model and the method of FEM implementation are reasonable [85], and can be applied during actual engineering field-calculation problems.
Chapter 5

New scientific results

5.1 Thesis 1.

I have developed a general phenomenological scalar hysteresis model by the aid of a statistical approach based on simple statistical considerations of the behavior of simple bistable units (domains, clusters) of the media, which resulted in a versatile model of hysteresis having useful properties, furthermore I have also developed a possible vector extension of the model, and I have established an efficient measurement-based identification method validated by actual measurements, which proved to be very accurate [78, 73, 79, 77, 76].

1.1. I have constructed an ODE-based scalar hysteresis model by the aid of statistical approach of hysteresis phenomena, which resulted in an improved variant of general Duhem-like models of hysteresis [78]. In the developed model I have introduced a mathematical description of the hysteresis curves, which enables of shaping the minor hysteresis loops independently from the major curve offering large versatility to the model, while preserving the advantageous properties of the closed form analytical formulation.

1.2. I have developed a two-step, measurement-based identification procedure of the model, applying linear combination of probability density functions for precisely shaping the major hysteresis curve by optimization, and a one-parameter minimization for approximating the shape of the minor loops [73, 79]. The identification of minor loops can be accomplished by either first order reversal curves or concentric minor loops.

1.3. I have proposed a possible vector extension of the established scalar model, which provides a reasonable description of vectorial hysteresis phenomena in two-dimensions [79, 77, 76]. The model captures the most important features of vectorial hysteresis, obeys the general fundamental physical principles of vector hysteresis, and shows good agreement with known experimental results.
5.2 Thesis 2.

I have carried out the thorough analysis of the hysteresis operator concerning its stability properties by the method of Poincare-sections in an appropriately constructed phase space, and as a result of the analysis I was able to derive a symbolic expression for the exact stabilizing location of the accommodating minor hysteresis loops. I have also analyzed the stability properties of the numerical solution of a diffusion problem with hysteresis, and after the examination of the Lyapunov-space of the parameters of the iteration corresponding to the numerical solution I have suggested a method for the cost-effective calculation of an appropriate damping constant ensuring the convergence of the iteration [80, 81, 75, 74].

2.1. I have investigated the stability of the ODE-based scalar hysteresis model developed in Thesis 1, concerning the behavior of the accommodating minor loops. I have constructed an appropriate three-dimensional phase space for the representation of periodic solutions and by the aid of this representation I have derived a Poincare-map, from which the exact location of the stable minor loop can be obtained [80].

2.2. I have examined a nonlinear diffusion problem with hysteresis by the Lyapunov-method and based on the results I have developed a method for the selection of the damping constant of the nonlinear iteration applied. A one dimensional diffusion problem and its numerical solution has been analyzed, paying particular attention to the fixed-point iteration resulting from the implicit nature of the iteration scheme in nonlinear material. I have proven, that the unstable behavior of the inner fixed-point iteration originates from bifurcation or - in extreme cases - chaotic behavior of the attractor of the iterative map [81, 75, 74]. Based on the examination of the parameter space of the iteration I have developed a damping constant calculation method to speed up the iteration significantly, thus the average number of iterations needed is more than an order of magnitude lower then in the case of the original damping constant [81].

2.3. I have analyzed the general structure of Picard-type iterations for diffusion-type PDE containing a hysteresis operator, and I have derived a general approach for the estimation and runtime control of the damping constant of the iteration. I have shown for the one-dimensional case, that the global stability condition for the damping constant of the iteration is the same, regardless of the hysteresis model for both of the discretization methods (FDTD Yee-algorithm, FEM) applied, and I have pointed out, that the local selection of the damping constant during the nonlinear iteration is more efficient, than applying a global stability condition throughout the whole numerical solution of the problem.
5.3 Thesis 3.

I have derived a formulation for the coupling of the developed hysteresis operator to the equations of a convection dominated fluid-flow problem, and I have implemented the developed hysteretic flow model into a two-phase flow network simulation software package as a computational core, which can be modularly embedded into higher level computational environments as a reusable component. I have demonstrated the applicability and versatility of the developed hysteresis model in two case studies, from the field of magnetics and two-phase flow with dynamic phase transition respectively. In the studies I have applied the finite element method as primary numerical tool for the investigation of the behavior of the hysteresis operator developed [40, 85, 41, 82, 83].

3.1. I have implemented my model in a two-phase flow field calculation problem by extending the energy conservation equation of convective flow with a source term containing a hysteresis operator representing the release/absorption of energy in the form of latent heat, according to the principle of vapor/liquid phase transition. Based on this extended formulation, I have inserted the developed hysteresis model into a finite element calculation involving vapor-liquid first order phase transition [40, 85, 41] proving the applicability of the proposed model and demonstrating the hysteretic behavior in the metastable region.

3.2. I have implemented the model of two-phase flow with hysteretic phase transition as the computational core of an actual network flow calculation engineering software [82, 83] providing a simulation engine in the form of reusable component for all kinds of higher level software environments, which have access to interfaces offered by .NET based reusable components.

3.3. I have proved by a sequence of case studies regarding penetration depth of magnetic field in nonlinear media, and two-phase flow with dynamic phase transition, that both the developed hysteresis model and the method of FEM implementation are reasonable [85], and can be applied during actual engineering field-calculation problems.
Chapter 6

Future plans and possible continuation of research

In the research work discussed in the previous chapters I have made some improvements concerning hysteresis modeling, stability analysis of hysteretic problems, and furthermore I have presented an actual application of the developed hysteresis model in a project regarding the analysis, modeling and simulation of a two-phase flow network. Since the research carried out is quite interdisciplinary, there are many areas, where the continuation of the research could be desirable and would possibly lead to promising further results.

The investigation of the role and applicability of the developed hysteresis model in the field of avalanches would be an interesting and promising research direction, since the fundamentals of the model are built on a statistical description of the collective behavior of bistable units resulting in an ‘average’ behavior in the form of a hysteresis curve. Nevertheless if the research could focus on a more general class of this ensemble of bistable units, it could lead to a possible connection to earthquake modeling [20], temperature induced brittle-ductile phase transitions in solids, etc.

Investigating and improving methods of measurement and identification of vectorial hysteresis [51, 52, 53] could also be a possible area of future research activity, since there are many complex field calculation problems (or complicated geometries), where the application of a scalar model of hysteresis is not satisfactory, and the intrinsically vectorial nature of hysteresis phenomenon has to be taken into account. However the measurement of vectorial hysteresis is a very resource-intensive procedure, requiring not only sophisticated measuring devices and methods, but carefully prepared samples and a precisely built single sheet tester, which can have a considerable investment demand. In spite of the difficulties, this direction of research is really useful not only from the practical point of view (efficient and reliable methods of measurement, validation of vectorial models), but it could also have a theoretical impact, enabling a deeper and more insightful understanding of hysteresis phenomena in general.
The implementation details of the various hysteresis operators into actual numerical computations is also an important part extending the recent research, since the practical application of a hysteresis operator in a numerical environment is far from trivial, and the design, the representation, and the programming aspects of the task are really under appreciated and worth to having considered as an independent research area connecting to the topic.

The identification procedure developed for the hysteresis model could be refined further by constructing an appropriate metric for measuring the distance between measured major hysteresis curves and the CDF of normal distribution, which would have the potency of providing more insight into the fundamental properties and origins of hysteretic behavior.

The stability investigation methods developed, in their actual form are valid for one dimensional discretizations. It would be a concern of further research to continue these investigations, and extend the scope of the methods established to higher dimensional cases, expecting even more general results, with potential applications in numerical computational software packages.
Appendix A

The applied method of scalar hysteresis measurement

This appendix contains a brief outline of the scalar hysteresis measurement method applied for the measurements carried out for the model validation described in Chapter 2.

A.1 The set-up of the hysteresis measurement

The schematic of the measurement setup is shown in Fig. A.1. In the figure it can be seen, that the specimen is equipped by two coils; a primary coil providing the excitation current, and a secondary coil for the measuring of the induced voltage. As Fig. A.1 shows, the whole measurement procedure is conducted by a PC equipped with National Instruments LabView 8.6 software and a NI ELVIS II board. The NI ELVIS II board is responsible for interfacing the signal flow between the computer and the actual measuring device. The ferromagnetic specimen measured is a Fe-Si 3.1%(wt) transformer sheet, which have thickness of 0.27mm. For the purpose of excitation there is a primary excitation coil driven by a KIKUSUI PBX 20-20 bipolar power supply as current source, where the excitation current is measured indirectly as a voltage across a reference resistor, which is assumed to be constant-valued regardless of current flow and eventual changes in ambient temperature. The secondary coil attached is responsible for measuring the induced voltage by the aid of the NI ELVIS II board.

This set-up enables to produce various excitations by creating nearly homogenous magnetic field strength aligned with the specimen. The actual excitation is generated in the form of a current signal by the aid of the KIKUSUI PBX 20-20 programmable bipolar power supply controlled by the measurement software developed in LabView.
Fig. A.1. The block diagram schematics of the measurement set-up.

A.2 The theory behind the measurement

The theory behind the measurement can be derived from the quasi-stationary Maxwell equations as follows [33], the set of equations needed is

\[ \nabla \times \mathbf{H} = \mathbf{J}_s + \sigma \mathbf{E}, \]  
(A.1)

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \]  
(A.2)

where \( \mathbf{H} \) is the magnetic field strength, \( \mathbf{E} \) is the electric field, \( \mathbf{B} \) is the magnetic flux density and \( \mathbf{J}_s \) is the external current density. The eddy current term \( \sigma \mathbf{E} \) in (A.1) can be omitted due to the thin (0.27 mm) sheet specimen and low (max. 50Hz) frequency applied during this measurement.

The relation of the \( \mathbf{H} \) field to the excitation current can be derived from (A.1) like

\[ \nabla \times \mathbf{H} = \mathbf{J}_s, \]  
(A.3)

which can be written in integral form on a properly selected surface \( \Omega \) in the investigated domain as follows

\[ \int_{\Omega} \nabla \times \mathbf{H} \, d\Omega = \int_{\Omega} \mathbf{J}_s \, d\Omega, \]  
(A.4)
and by the application of Stokes’ theorem, (A.4) has the form
\[ \oint_{l} \mathbf{H} \, dl = \int_{\Omega} \mathbf{J}_s \, d\Omega, \]  
\hspace{1cm} (A.5)

where the right hand side is basically the sum of the excitation currents in the coil-domain, and assuming that the field strength is constant along the path of the integral the left hand side integral can be substituted by $\mathbf{H}l$, thus
\[ \mathbf{H}l = N_p I, \]  
\hspace{1cm} (A.6)

where $N_p$ is the number of turns of the primary excitation coil, $l$ is the length of the frame around the specimen, and $I$ is the primary excitation current.

In a similar way the flux density $\mathbf{B}$ can be derived from the integral form of (A.2)
\[ \int_{\Omega} \nabla \times \mathbf{E} \, d\Omega = -\frac{\partial}{\partial t} \int_{\Omega} \mathbf{B} \, d\Omega, \]  
\hspace{1cm} (A.7)

applying Stokes’ theorem we have
\[ \int_{l} \mathbf{E} \, dl = -\frac{\partial}{\partial t} \int_{\Omega} \mathbf{B} \, d\Omega, \]  
\hspace{1cm} (A.8)

where the left hand side of the equation is the voltage $u_t$ of one turn of the measuring (secondary) coil, and the integral on the right hand side can be substituted by $\mathbf{B}A_s$ under the assumption that the flux density $\mathbf{B}$ is homogenous in the central region of the specimen, and spread flux can be neglected. Now (A.8) can be written as
\[ u_t = -\frac{\partial}{\partial t} \mathbf{B}A_s. \]  
\hspace{1cm} (A.9)

Since the measuring coil has $N_s$ turns, the actual measured voltage $u$ of the secondary coil is $N_s$ times higher than $u_t$, thus finally we have
\[ \frac{1}{N_s} u = -\frac{\partial}{\partial t} \mathbf{B}A_s, \]  
\hspace{1cm} (A.10)

where $A_s$ is the cross-sectional area of the the sheet that the specimen is made of, $N_s$ is the number of turns of the secondary coil. Writing (A.10) in integral form leads to
\[ \mathbf{B} = \frac{1}{N_s A_s} \int u \, dt. \]  
\hspace{1cm} (A.11)

At this point (A.6) and (A.11) is enough to calculate the field strength and flux density inside the specimen under homogenous assumptions.
Appendix B

Derivation of the Poincare-map

In this appendix the details of the derivation of the Poincare-map corresponding to the accommodating minor loops of the hysteresis model is described (see section 3.2).

B.1 The dynamical system

The governing equations of the periodically excited hysteresis operator are

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -2\alpha x_2 - (\alpha^2 + \omega^2)x_1 \\
\dot{x}_3 &= \gamma \dot{g}(\delta x_1) x_2
\end{align*}
\]  

(B.1)

introducing the notation \(x_1, x_2\) and \(x_3\) corresponding to field variables \(H, \dot{H}\) and \(M\) respectively, overdot denoting time derivatives, and function \(\gamma\) has the form

\[
\gamma = \frac{1 + \delta(1 - 2x_3)}{2(1 - G(\delta x_1))},
\]

(B.2)

according to the definition of the hysteresis operator (see section 2.7).

Assuming, that the oscillator provides a non-decaying input for the model \((\alpha = 0)\), and fixing the frequency\(^1\) at \(\omega = \omega_0 > 0\), the input can be handled as a sequence of piecewise monotone functions. In this case of periodic excitation with constant amplitude, the output of the model is a piecewise monotone function as well, increasing in interval \(t \in [n2\pi/\omega, (2n + 1)\pi/\omega)\) and decreasing in interval \(t \in [(2n + 1)\pi/\omega, (n + 1)2\pi/\omega), n \in \mathbb{N}\). The \(x_2 = 0\) Poincare-plane separates the increasing and the decreasing portions of the output curve in the phase space (see Fig.B.1), thus it enables to solve system (B.1) separately for increasing and decreasing branches (\(\delta = 1\) and \(\delta = -1\) respectively).

\(^1\)Since the model is rate independent it is invariant with respect to \(\omega\), thus the value of frequency can be chosen arbitrarily providing \(\omega > 0\).
The parameter $q$ is set to $q = 2$, since though the symbolic solution of the ODE problem can be obtained for general $q$, the structure of the symbolic solution becomes extremely complicated for large ($q > 2$) values of $q$, thus for the sake of simplicity in the recent investigation the $q = 2$ case has been described.

B.2 The symbolic solution

The symbolic solution\(^2\) of (B.1) under the assumption of $\omega = 1$, and $\delta = 1$ (increasing branch), applying initial conditions

$$[x_1(0) = -A, x_2(0) = 0, x_3(0) = z_0], \quad (B.3)$$

is

$$x_1(t) = -A \cos(t)$$
$$x_2(t) = A \sin(t)$$
$$x_3(t) = \frac{(1 - 2z_0 + z_0 G(-A)) G(-A \cos(t)) - G(-A) + z_0}{(G(-A) - z_0) G(-A \cos(t)) + 1 + (-2 + z_0) G(-A)}. \quad (B.4)$$

Since resulting from the periodic excitation the output $x_3(t)$ of the model is periodic with $2\pi$, the decreasing branch ($\delta = -1$) can be calculated by solving (B.1) with the following initial conditions derived from (B.4) at $t = \pi$

$$\left[\begin{array}{c}
 x_1(\pi) = A, \\
 x_2(\pi) = 0, \\
 x_3(\pi) = \frac{(1 - 2z_0 + z_0 G(-A)) G(A) - G(-A) + z_0}{(G(-A) - z_0) G(A) + 1 + (-2 + z_0) G(-A)}
\end{array}\right], \quad (B.5)$$

\(^2\)Calculated by Maple® computer algebra system.
from where the solution $x_3(t)$ at $t = 2\pi$ applying the substitutions $G(-A) = a, G(A) = a + b$ has the form

$$x_3(t)_{t=2\pi} = \frac{(-1 + a) \ ((1 - 2 z_0 + a z_0) b + z_0 \ (-1 + a)^2)}{(-a^2 + a - 1 + z_0) b^2 + (-2 a^3 + 5 a^2 - 4 a + 1) b - (-1 + a)^2}.$$  

(B.6)

Introducing the notation $\mu = [a, b]$ and $z_n = x_3(t)_{\theta=2n\pi}$, the $z_{n+1} = P(z_n, \mu)$ Poincare-map with

$$P(z_n, \mu) = \frac{(-1 + a) \ ((1 - 2 z_n + a z_n) b + z_n \ (-1 + a)^2)}{(-a^2 + a - 1 + z_n) b^2 + (-2 a^3 + 5 a^2 - 4 a + 1) b - (-1 + a)^2},$$  

(B.7)

can be constructed, where $\mu$ is the vector of parameters which the map depends on.

This function can be applied to iteratively calculate the consecutive intersection points of a particular trajectory (defined by a particular initial condition) and the Poincare-plane $x_2 = 0$, thus by the aid of (B.7), the exact location of the fixed point of $z_{n+1} = P(z_n, \mu)$ can be calculated by solving

$$z = P(z, \mu).$$  

(B.8)

It is also interesting to note, that the location of the fixed point is invariant with respect to the shape of the actual function $G(.)$ describing the hysteresis curve, only the starting point parameter $a$, and the width parameter $b$ of the hysteresis loop has to be known a priori.
Appendix C

Dynamical systems and stability

The concept of stability can be approached from the field of nonlinear dynamical systems. The methods of stability analysis of dynamical systems can also be used effectively for analyzing numerical methods of nonlinear PDEs. The most fundamental definitions and concepts used from the theory of dynamical systems are the followings.

C.1 The concept of dynamical systems and phase spaces

The classical theory of dynamical systems \[102\] handles two distinct kinds of systems: discrete and continuous.

Continuous dynamical systems are usually defined in the form of systems of ordinary differential equations (ODEs),

\[
\dot{x} = f(x, t, \mu), \tag{C.1}
\]

where \(\mu\) is a parameter vector. If function \(f(.)\) does not depend explicitly on time \(t\) then the system of ODEs (C.1) is called autonomous.

Discrete dynamical systems are usually given in the form of an iteration (and this is the point where dynamical systems meet numerical methods of solution of nonlinear PDEs) as

\[
x^{n+1} = g(x^n, \mu). \tag{C.2}
\]

The phase space of system (C.1) and (C.2) is a \(k\)-dimensional space where the trajectories of the system lay. These trajectories are parametric curves in the form \(x(t) = [x_1(t), x_2(t), \ldots, x_k(t)]\) representing the solutions of the system.
C.1.1 Stability of a solution

Stability of a solution intuitively means that a solution evolving from an initial condition “near” the stable solution will remain in the “vicinity” of the stable solution, or approaches it asymptotically. For example the dynamics of a pendulum of a clock can lead to two different behaviors. One of them is the stopped clock, which is a stationary point or a fixed point in the phase space of the pendulum. The other behavior is a periodic solution which corresponds to the properly operating clock. Each initial condition (the initial position and velocity of pendulum) approaches one of these two stable solutions. The exact definitions formally are the followings [102]:

Lyapunov stability: An \( x^*(t) \) solution is stable if \( \forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \) such that, for any other \( y(t) \) solution, satisfying \( |x^*(t_0) - y(t_0)| < \delta \), then \( |x^*(t) - y(t)| < \varepsilon, t > t_0, t_0 \in \mathbb{R} \).

Asymptotic stability: An \( x^*(t) \) asymptotically stable if it is Lyapunov stable, and \( \exists \varrho > 0 \), that if \( |x^*(t_0) - y(t_0)| < \varrho \), then \( \lim_{t \to \infty} |x^*(t) - y(t)| = 0 \).

As a brief example Fig. C.1 shows a two dimensional phase-space with trajectories originating from different initial conditions, corresponding to the equations of motion of a point mass in a one-dimensional potential field \( \Psi(x) \)

\[
\ddot{x} + \mu \dot{x} + \Psi'(x) = 0, \tag{C.3}
\]

where \( \mu \) is the friction coefficient, \( x \) is the space coordinate, overdot denotes time derivatives, and prime denotes space derivative (gradient), thus \( \Psi'(x) \) is the force acting on the mass. Rewriting (C.3) as a system results in

\[
\begin{align*}
\dot{q} &= p \\
\dot{p} &= -\mu p - \Psi'(q),
\end{align*}
\tag{C.4}
\]

with the notation \( q = x, p = \dot{x} \).

As it can be seen in Fig.C.1, there are four different solutions corresponding to different initial conditions and all of them are approaching one of the two stable fixed-points defined by the potential function (dashed line) \( \Psi(x) = \cos(3x)/x \) in this region. It is also apparent from the figure, that there is an unstable fixed point (hill) in between the two stable ones. The exact locations of the fixed points can be obtained as the solutions of the system \( p = 0, \Psi'(q) = 0 \).

In a similar double-well potential field without friction \( (\mu = 0) \), all of the solutions starting from different initial conditions will wrap onto a limit cycle corresponding to periodic solutions of (C.3) as it can be seen in Fig.C.2.

C.1.2 Stability of fixed points

In the theory of dynamical systems the examination of stability is focusing on the fixed points where the iteration stops [18]. These points are very important
because the structure of the phase-space depends on the properties of the fixed points. A fixed point $\hat{x}$ in the time evolution of a dynamical system means that the system is in a stationary state

$$\hat{x} = g(\hat{x}, \mu).$$

The stability of a fixed point depends on the $g$ map. The $\hat{x}$ fixed point is stable if $g$ is a contraction mapping on a closed bounded domain $\Omega$. Formally a map $g : \Omega \to \Omega$ is a contraction if it is Lipschitz-continuous with a Lipschitz constant $\rho < 1$. A function $g : \Omega \to \Omega$ is Lipschitz-continuous if $\forall x, y \in \Omega, \exists \rho > 0$:

$$\| g(x) - g(y) \| < \rho \| x - y \|.$$  

(C.6)

If $g \in C^1$, then sufficient condition for $g$ to be contraction is $|g'(x)| < 1, \forall x \in \Omega$.
Concerning the stability of the fixed points, one of the most important results of nonlinear stability analysis is the Hartman-Grobman theorem which relates the linearized dynamics of the system to the local nonlinear dynamics [18]. The theorem states, that if \((\hat{x}, \hat{\mu})\) is a hyperbolic fixed point of the map (C.2), then there exists a homeomorphism \(\Psi : \mathbb{R}^k \rightarrow \mathbb{R}^k\) and a local neighborhood \(U \subset \mathbb{R}^k\) of \(\hat{x}\) where

\[
g(x, \mu) = \Psi^{-1}(g'(\hat{x}, \hat{\mu}) \cdot \Psi(x)), \tag{C.7}
\]

with \(x \in U\) and \(g(x, \mu) \in U\), which is a statement of that the nonlinear dynamics close to the fixed point \(\hat{x}\) is qualitatively the same as the linearized dynamics. It practically means, that close enough to a fixed point the asymptotic behavior of the linear and nonlinear system is the same\(^1\), and from the theorem it follows, that in the vicinity of a fixed point, the stability properties and asymptotic behavior can be derived from the Jacobian \(g'(\hat{x}, \hat{\mu})\) corresponding to the linearized map.

### C.2 The Lyapunov exponents

One of the characteristic properties of chaotic (strongly unstable) behavior is that nearby trajectories in the phase space of the system are diverge at exponential rate. It means that a mapfunction \(g : \Omega \times \Gamma \rightarrow \Omega\) depending on the variable of iteration \(x \in \Omega\) and a parameter vector \(\mu \in \Gamma\), shows chaotic behavior under certain parameter configurations, thus the parameter space contains domains where the behavior is chaotic [32].

The divergence of nearby trajectories formally means that

\[
|g^n(x^0 + \varepsilon, \mu) - g^n(x^0, \mu)| = \varepsilon e^{\lambda n}, \tag{C.8}
\]

where \(g^n\) denotes \(g \circ g \circ \cdots \circ g\) (composition of \(g\) with itself \(n\)-times) and \(\varepsilon\) is the initial distance between trajectories. Equation (C.8) states that the distance between two nearby trajectories (one travels through \(x^0\), the other travels through \(x^0 + \varepsilon\)) will grow exponentially with parameter \(\lambda\) under successive iterations. This divergence is a **local property** since the phase space is bounded. Taking the natural logarithm of both sides gives

\[
\ln \left( \frac{|g^n(x^0 + \varepsilon, \mu) - g^n(x^0, \mu)|}{\varepsilon} \right) = n\lambda, \tag{C.9}
\]

and assuming \(\varepsilon \to 0\), (C.9) takes the form

\[
\frac{1}{n} \ln \left| \frac{\partial g^n(x^0, \mu)}{\partial x} \right| = \lambda. \tag{C.10}
\]

\(^1\)Nevertheless the theorem tells nothing about the size of the local neighborhood, where the statement holds.
Applying the chain rule for (C.10) results (with notation $g'(x, \mu) = \frac{\partial g(x, \mu)}{\partial x}$)

$$\lambda = \frac{1}{n} \ln \left| g'(x^{n-1}, \mu) g'(x^{n-2}, \mu) \cdots g'(x^0, \mu) \right|, \quad (C.11)$$

from where the final form of the Lyapunov exponent of the scalar map $g : \Omega \times \Gamma \to \Omega \subset \mathbb{R}$ for a specified $\mu \in \Gamma$ is

$$\lambda = \frac{1}{n} \sum_{i=0}^{n-1} \ln |g'(x^i, \mu)|. \quad (C.12)$$

Calculating the Lyapunov exponents over the parameter-space provides information about the stability of the system under certain parameter configuration ($\mu$). If $\lambda > 0$ for a given $\mu$ the attractor is chaotic and if $\lambda = 0$ a bifurcation occurs. This calculation of Lyapunov exponents for various parameter configurations can be handled as a function $\Lambda : \Gamma \to \mathbb{R}$. 
Appendix D

Model Application in a two-phase flow network calculation environment

The model presented had been applied to study the behavior of a suburban steam network by implementing the hysteretic two-phase flow model of the pipe segments into an integrated network calculation software environment developed by our department\(^1\). My task was the implementation of the simulation core in the MATLAB environment, and integration of the core into the graphical user interfaces developed/applied for the network analysis system. The network investigation environment is capable of defining and modifying whole networks, handling network elements with various properties, carrying out simulations of the behavior of the network and visualization and exporting of the results. In this section an overview is given about the fundamental concepts and capabilities of the system developed.

D.1 Brief overview of the implementation

Modeling, simulation and visualization of steam pipe-line networks where phase change can occur requires the solving of quite difficult numerical problems even in the case of homogenous model assumption. An efficient tool for these types of problems is the MATLAB mathematical software package extended with the capabilities of the COMSOL finite element environment. In this approach an efficient hierarchical modeling and simulation environment is established, where the numerical routines of the MATLAB package can be accessed from a user friendly software developed in the .NET framework. In this way the computational core can be accessed from any high level programming environment.

\(^1\)University of Pécs, Pollack Mihály Faculty of Engineering, Department of information Technology
which is capable of using the services offered by the .NET framework. It is very advantageous in the case of these kind of tasks, since in this way the simulation core written in MATLAB can be easily attached to for example a geographic information system [38] like ArcGIS, and by means of this coupling the actual hysteretic two-phase flow model can be applied to various engineering fields and problems like water-, gas- and steam networks, or even for modeling of the blood flow in the cardiovascular system. The task of defining, modeling and simulating a complex network is accomplished by the cooperation of several software components. In order to use the simulation software efficiently, the network models developed in the MATLAB-COMSOL environment has to work together fluently with the C# .NET based user interface, on which the inputs can be defined and the results can be analyzed in a versatile and intuitive way.

D.1.1 Data interconnection

The various tasks corresponding to modeling, simulation, parametrization, visualization are accomplished by several software components (modules). For the efficient application of the simulation environment the simultaneous working of the MATLAB-based network models, the C#.NET-based network- and data manipulation user interface, and the ArcGIS geographic information system is required. The integration of the above mentioned systems can be realized effectively in the .NET framework. The MATLAB code based network model can be built into a .NET component by the aid of MATLAB Compiler and .NET Builder, and this component can be handled as a ‘class-object’ in any .NET compatible programming language.

The component obtained by the .NET Builder is built in a framework software which is made in the C# environment. The framework software had been compiled into two different .NET assemblies (Common Language Runtime code). One of them is a console version to be easily accessed from an ArcGIS VB macro, the other one has a graphical UI, and it can be used as a stand alone application for defining and simulating networks. In order to use the .NET component compiled from the MATLAB network model, the MATLAB runtime environment (MCR, Matlab Component Runtime), and the .NET framework need to be installed. The links and data connections between the software modules can be seen in Fig. D.1.

D.1.2 The MATLAB simulation module and the graphical user interface

The development of the network model had been accomplished entirely in the MATLAB environment, since the handling of the differential equations of the system and the arrays of parameters required for the description of the network, is much more efficient compared to a traditional programming language. The
MATLAB environment contains several useful tools for the solution of the problems arising during the development process of model formulation, and for the visualization of the detailed numerical results. The network simulation module consist of a hierarchic structure of MATLAB scripts and functions. The main part of the code is responsible for the reading of input data, the model simulation (solution of the model equations for the given input conditions), and the storing of the results in the corresponding output files. In this solution process many sub-routines and functions are called. One part of them accomplish administration tasks (i.e. sorting and ordering of adjacency matrices), and the other part of them define the structure of the system-equations and calculate the temperature- and pressure dependent physical properties. The system model can be used for simulation in the MATLAB environment, but without a graphical user interface (GUI) this environment is not flexible enough, the parametrization and the visualization of the results requires MATLAB programming knowledge.

In order to define, model, and simulate a freely configurable and parametrizable network topology, a graphical input-output interface is needed on which one can accomplish complex network modeling tasks without programming knowledge.

Two versions of user interfaces had been connected to the model-simulation core with the desired properties mentioned above. One of the user interfaces is a C#.NET based application with the primary task of defining the network, setting the model parameters, carrying out simulations on the model network and

Fig. D.1. Data connections between the software components
visualization of the results. All of the functions of the computational core can be accessed from the graphical user interface in an easy-to-use form. Besides the visualization of the simulation results, the software has an excel-export feature of all of the data corresponding to the modeling and simulation. The simulation software based on Matlab-.NET connection has a user friendly, easy-to-use user interface (Figure D.2), which the operations needed for modeling and simulation can be accessed from. The visualization of the results takes place in the same user interface, thus the user of the software is able to do the investigation of a certain network with various parameter settings, and various topologies.

Fig. D.2. The user interface of the network simulation software developed

The other GUI solution was developed by connecting the simulation core to the ArcVIEW geographical information system. In this way the user can carry out simulations on a network which is built on a map database in the context of the actual environment of the physical network. The connection had been established by a VB macro module implemented in the macro programming environment of the ArcVIEW system [82, 92]. The module serves as an interface between the network defined in the ArcVIEW system, and the simulation core, which can be accessed as a .NET component. Thus the user can carry out all of the modeling and simulation tasks through the graphical user interface of the ArcVIEW system, such as defining the network topology, setting the model parameters, simulating the network and visualize the results. The data
corresponding to the simulation results are stored in the map database together with the topology and parameter values. Both of the above tasks require that the simulation module developed in the MATLAB environment be accessible from the .NET development environment. This requirement can be fulfilled by the aid of the MATLAB Compiler and .NET Builder (toolboxes of the MATLAB package), which are capable of compiling executable files or .NET components from MATLAB functions.

This utilization of the connection possibilities between the high level software tools made it possible to solve a rather complex problem in a way, that each of the components of the final software architecture fulfills the role, which fits their certain application areas the most.

\section*{D.1.3 Simulation results}

In the following figures a few simulation results are shown corresponding to two different operating conditions (OC) of the network investigated. One of the operating conditions corresponds to a typical duty period of the consumers (OC-1), and the other operating mode represents a baseline configuration of the consumers (OC-2). (For various measurement results corresponding to the operating conditions of the network please refer to [91].) These kind of simulations enables the user to quickly and easily investigate the behavior of a network under different operating conditions, parameter configurations, with varying sources and topologies. All of the relevant physical parameters (steam quality, temperature, pressure, mixture density etc.) can be accessed from the result of the simulation and the data can be exported for archiving or further analysis.
Fig. D.3. Result of steam network simulation (OC-1) with average steam quality shown on the branches
Fig. D.4. Result of steam network simulation (OC-1) with average density shown on the branches
Fig. D.5. Result of steam network simulation (OC-2) with average steam quality shown on the branches
Fig. D.6. Result of steam network simulation (OC-2) with average density shown on the branches


### Bibliography


