# Asymptotic properties of the Lorentz process and some closely related models

**SYNOPSIS**
**of PhD Thesis**

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# Introduction

Chaotic, "stochastic" behavior of deterministic systems is much interesting from both theoretical and applied points of view. An archetype of such systems is the *Sinai billiard* - or equivalently, its periodic extension, the *periodic Lorentz process.* The definition of Sinai billiard is the following. Fix some strictly convex subsets $B_1, \dots B_k$ of the $d$ dimensional torus, whose boundaries fulfill some regularity conditions. These sets are thought of as scatterers. The continuous dynamics, called the *billiard flow* is the free flight of a point particle among the scatterers and its specular reflection on their boundaries. The speed of the particle is constant (equals to one, say), thus the phase space of the billiard flow consists of a spatial component ($d$ dimensional torus minus the scatterers) and a velocity vector, which is an element of the $d-1$ dimensional unit sphere. Lebesgue measure is a natural invariant measure in both coordinates. The same motion is described by the *billiard ball map*, which is the Poincaré section of the flow on the boundaries of the scatterers. Hence its phase space has spatial dimension is $d-1$ and full dimension $2d-2$. The simplest case is of course the planar one ($d=2$), where the phase space of the billiard ball map is two dimensional.
The development of the theory of planar Sinai billiard in the last decades is miraculous (consult [6] for lot of details). Besides ergodicity and hyperbolicity, the most interesting statistical properties are the decay of correlation and the central limit theorem (CLT), or diffusion. For an abstract dynamical system $(\mathcal{M}, \mathcal{F}, \mu)$, the former means that $\int f(g \circ \mathcal{F}^n) d\mu$ is exponentially small in $n$ if $\int f d\mu = \int g d\mu = 0$ and $f$ and $g$ are chosen from a nice set of functions (definitely containing the free flight function for the billiard ball map). With this terminology, CLT means that $\frac{1}{\sqrt{n}} \sum_{k=1}^{n} f \circ \mathcal{F}^k$, as a random variable with respect to $\mu$, weakly converges to a Gaussian distribution.
CLT for Sinai billiards (or equivalently, for Lorentz processes) was first proven in [1]. It turned out that a nontrivial condition for CLT is that the free flight function should be bounded.

**Definition 1.** *We say that a Sinai billiard (or periodic Lorentz process) has* finite horizon*, if the free flight function is bounded. Equivalently, it has* infinite horizon*, if there is an infinite line which is disjoint from the interior of all scatterers.*

In 1998, Young [27] introduced the tower technique which was strong enough to prove exponential decay of correlation for the billiard ball map and also provided a new, transparent proof of the CLT. Her method was successfully applied by Szász and Varjú in the case of infinite horizon [26]. According to their most interesting result, the presence of infinite horizon yields a slightly super-diffusive behavior. The displacement of the particle in $n$ steps, rescaled

by $\sqrt{n \log n}$, converges to some Gaussian distribution. In fact, they proved a local version of this limit theorem and also that of the CLT in case of finite horizon [25].
Chernov and Dolgopyat managed to further simplify the proof of the CLT with their method of "standard pairs". This technique is also applicable to many other problems. Besides the proof of the limit theorem in both finite and infinite horizons ([3, 4]), it also yielded more delicate statistical properties (e.g. convergence to Brownian motion, law of iterated logarithm [3]) and also limit theorems for related models (e.g. for a systems with two particles colliding with each other, too, [5], or for billiards under external fields). Further, several arguments from probability theory has been successfully reapplied to Sinai billiards, thanks to this technique. Dolgopyat, Szász and Varjú [9] proved delicate recurrence properties of the periodic Lorentz process with finite horizon. They also proved CLT for a non-periodic Lorentz process with finite horizon [10], where periodicity is spoiled in a compact domain. A related conjecture is the following.

**Conjecture 1.** *Modify the scatterer configuration of a periodic Lorentz process with infinite horizon on a compact subset (the modification still satisfies the assumptions of the Sinai billiard). Then, the super-diffusive limit theorem remains valid.*

In the last few years, some other non-homogeneous modifications of the periodic Lorentz process were also considered, see for instance [24] for a very recent one. As both the delicate statistical properties of the periodic Lorentz process and the basic statistical properties of some non-homogeneous versions are current active research fields, there are plenty of interesting, challenging questions, a few of which we are discussing in the thesis.
Roughly speaking, Chapters 2 and 3 are about delicate properties of periodic case, mainly in the framework of some stochastic models designed for better understanding of the Lorentz process. Chapters 4 and 5 address some inhomogeneity (in Lorentz process, and one dimensional expanding maps, respectively). Chapters 6 and 7 deal with two and higher dimensional periodic Lorentz process with infinite horizon. Chapters 3 and 7 were prepared in the hope that they might be useful by attacking Conjecture 1.
The thesis itself is based on four published journal papers [15, 16, 17, 18], and a preprint [19]. It contains 6 Chapters (and an Introduction). The results presented in Chapters 5,6 and 7 are joint with Tamás Varjú (besides my adviser prof. Domokos Szász). That is why I found it more appropriate to summarize the half of these joint results (namely, Chapter 5 and the half of Chapter 6) in this synopsis.

# 1 Range of a random walk with internal states

An interesting question for random walks is that how many sites are visited by the walker up to time $n$, where $n$ is large. The first results for this question was given by Dvoretzky and Erdős, in their celebrated paper [11]. This work answers many relevant questions for the simplest model, namely, for simple symmetric random walk. In Chapter 2 of the thesis we answer the corresponding questions for random walks with internal states (RWwIS). These walks are thought of as some stochastic models of the Lorentz process with finite horizon (see [23]), thus, in particular, we assume the existence of third of fourth moments, if necessary. It is also important to mention that the analogous problem for planar Lorentz process with finite horizon was solved by Pène in [20]. This Chapter is based on the article [15].

**Definition 2.** *Let $E$ be a finite set. On the set $H = \mathbb{Z}^d \times E$ $(d = 1, 2, ...)$, the Markov chain $\xi_n = (\eta_n, \varepsilon_n)$ is a random walk with internal states (RWwIS), if for $\forall x_n, x_{n+1} \in \mathbb{Z}^d, \quad j_n, j_{n+1} \in E$*

$$P(\xi_{n+1} = (x_{n+1}, j_{n+1}) | \xi_n = (x_n, j_n)) = p_{x_{n+1}-x_n, j_n, j_{n+1}}.$$

There are some basic assumptions which will throughout be supposed. These are the following:

(i) $(\varepsilon_0, \varepsilon_1, ...)$ - obviously a Markov chain - is irreducible and aperiodic (its stationary distribution will be denoted by $\mu$)

(ii) the arithmetics are trivial, with the notation in [13], $L = \mathbb{Z}^d$

(iii) the expectation of one step is zero provided that $\varepsilon_0$ is distributed according to its unique stationary measure

(iv) the covariance matrix, $\sigma$ (see the exact definition in [13]), exists and is nonsingular.

Let $L_d(n)$ denote the number of visited sites until time $n$, that is

$$L_d(n) = |\{v \in \mathbb{Z}^d : \exists k \leq n, \eta_k = v\}|.$$

Let us denote the expectation of $L_d(n)$ by $E_d(n)$ and the variance by $V_d(n)$. The main results can be summarized as follows.

**Theorem 2.** *Independently of the distribution of* $\varepsilon_0$*, the following estimates are true*

$$\begin{aligned} E_3(n) &= n\gamma_3 + O(\sqrt{n}) \\ E_4(n) &= n\gamma_4 + O(\log n) \\ E_d(n) &= n\gamma_d + \beta_d + O(n^{2-d/2}) \qquad \text{for } d \geq 5 \\ V_d(n) &= O\left(n^{1+\frac{2}{d}}\right) \end{aligned}$$

*with some constants* $\gamma_d, \beta_d$*, depending on the RWwIS.*

**Theorem 3.** *For RWwIS in* $d \geq 3$ *strong law of large numbers holds, namely*

$$P\left(\lim_{n\to\infty} \frac{L_d(n)}{E_d(n)} = 1\right) = 1.$$

**Theorem 4.** *Let* $d = 2$*. For arbitrary* $\nu$ *distribution of* $\varepsilon_0$*,*

$$E_2(n) = \frac{2\pi\sqrt{|\sigma|}n}{\log n} + O\left(\frac{n \log\log n}{\log^2 n}\right).$$

**Theorem 5.** *For arbitrary* $\nu$ *distribution of* $\varepsilon_0$

$$V_2(n) = O\left(\frac{n^2 \log\log n}{\log^3 n}\right).$$

*Moreover, the great order is uniform in* $\nu$*.*

**Theorem 6.** *For any RWwIS in* $d = 2$*, strong law of large numbers holds, namely*

$$P\left(\lim_{n\to\infty} \frac{L_2(n)}{E_2(n)} = 1\right) = 1.$$

**Proposition 7.** *With arbitrary distribution of* $\varepsilon_0$ *the following holds*

$$E_1(n) \sim \sqrt{\frac{8\,|\sigma|}{\pi}} n^{1/2}.$$

As the simple symmetric random walk, the RWwIS is also recurrent in dimensions 1 and 2 and transient in dimension $d \geq 3$. In fact, the planar recurrence is in some sense weak, meaning that it takes very long time to return. This phenomenon makes the treatment of the planar case much more complicated than that of other dimensions. In particular, one needs to estimate the error term of the local limit theorem for RWwIS (the leading term was computed in [13]). Thus we provide some refinements of this local limit theorem, some of which might deserve some independent interest. In the synopsis, we formulate one of these refinements, the reader is referred to the thesis for more details.

**Theorem 8.** *For a one dimensional RWwIS the existence of appropriately defined third moment imply*

$$\begin{aligned} P(\xi_n &= (x,k)|\xi_0 = (0,j)) - \\ &- \mu_k \frac{1}{\sqrt{2\pi n}\sigma} \exp\left(-\frac{x^2}{2n\sigma^2}\right)\left[1 - \frac{ir_3}{6} x\left(3\sigma^2 n - x^2\right)\frac{1}{\sigma^6}\frac{1}{n^2}\right] = o\left(\frac{1}{n}\right), \end{aligned}$$

*where the convergence is uniform in $x$.*

# 2 Recurrence properties of a Heavy Tailed Random Walk

Chapter 3 of the thesis is about delicate recurrence properties of a random walk, the step distribution of which has the same tail asymptotics as the planar Lorentz process with infinite horizon. We address exactly the same questions Dolgopyat, Szász and Varjú were dealing with in [9] (which were also important by the proof of the convergence to the Brownian motion in locally perturbed periodic Lorentz process with finite horizon [10]). These questions include the tail asymptotics of the distribution of the first return time to the starting position (origin), limit theorem for the local time at the origin and for the hitting time of the origin as started from far away. The consequence is that in case of the infinite horizon, the recurrence properties are weaker, as expected - for instance, the local time up to $n$ is scaled by $\log\log n$ in compare to $\log n$ in finite horizon. Some of these results can be proven for the Lorentz process too, but some of them are open. These results have appeared in [16].
Define independent random variables $X_i$, such that

$$\mathbb{P}\left(X_i = n\right) = c_1 |n|^{-3},$$

if $n \neq 0$, and $E_i$ to be uniformly distributed on the 4 unit vectors in $\mathbb{Z}^2$. Now put $\xi_i = X_i E_i$. (Here, of course, $c_1 = \frac{1}{2\zeta(3)}$, but this will not be important for us.) Define the Heavy-Tailed Random Walk (HTRW) by $S_n := \sum_{i=1}^n \xi_i$.
This distribution is the same, as the one of the free flight vector of the Lorentz process with infinite horizon (see [26]). However, one could think that our choice is rather special, as the walker can only step along the $x$ and $y$ axis. But this is not the case, as a particle performing Lorentz process can have arbitrary long steps only in finitely many directions, too. Here, we choose that two particular directions, but this is not essential.
Further, define the one dimensional HTRW as

$$Q_n := \sum_{i=1}^n X_i.$$

**Definition 3.** *Let $\tau_2$ be the first return to the origin in two dimensions, i.e.*

$$\tau_2 = \min\{n > 0 : S_n = (0,0)\}.$$

**Theorem 9.** $\mathbb{P}(\tau_2 > n) \sim \frac{4\pi c_1}{\log\log n}$.

**Theorem 10.** *Let $N_2^n = \#\{k \le n : S_k = (0,0)\}$. Then*

$$\frac{N_2^n}{\log\log n}$$

*converges to an exponential random variable with expected value $\frac{1}{4\pi c_1}$.*

**Definition 4.** *Let $t_v$ be the hitting time of the origin, starting from the site $v \in \mathbb{Z}^2$, i.e.*

$$t_v = \min\{k \ge 0 : S_k = (0,0) | S_0 = v\}.$$

**Theorem 11.**

$$\frac{\log\log t_v}{\log\log |v|} \Rightarrow \frac{1}{U}$$

*as $|v| \to \infty$, where $U$ is uniformly distributed on $[0,1]$ and $\Rightarrow$ stands for weak convergence.*

**Definition 5.** *Let $\tau_1$ be the first return to the origin in one dimension, i.e.*

$$\tau_1 = \min\{n > 0 : Q_n = 0\}.$$

**Theorem 12.** $\mathbb{P}(\tau_1 > n) \sim \frac{2\sqrt{c_1}}{\sqrt{\pi}}\sqrt{\frac{\log n}{n}}$.

**Theorem 13.** *Let $N_1^n = \#\{k \le n : Q_k = 0\}$. Then*

$$\frac{N_1^n \sqrt{\log n}}{\sqrt{n}}$$

*converges to a Mittag-Leffler distribution with parameters $1/2$ and $(2\sqrt{c_1})^{-1}$, i.e. to the distribution, the $k^{th}$ moment of which is*

$$\frac{1}{(2\sqrt{c_1})^k} \frac{k!}{\Gamma\left(\frac{k}{2}+1\right)}.$$

The first similar results were, of course, for simple symmetric random walk (see [12] and [7]). In order to prove Theorem 11 - just like in Chapter 2 -, a refinement of the local limit theorem is also needed here. In particular, we prove

**Theorem 14.** *For the one dimensional HTRW the following estimation holds uniformly in x*

$$\mathbb{P}(Q_n = x) - \frac{1}{\sqrt{2\pi}\sqrt{2c_1}\sqrt{n \log n}} \exp\left(-\frac{x^2}{4c_1 n \log n}\right) = O\left(\frac{\log\log n}{\sqrt{n \log^3 n}}\right).$$

*For the two dimensional HTRW the following estimation holds uniformly for* $x \in \mathbb{R}^2$

$$\mathbb{P}(S_n = x) - \frac{1}{2\pi 2c_1 n \log n} \exp\left(-\frac{|x|^2}{4c_1 n \log n}\right) = O\left(\frac{\log\log n}{n \log^2 n}\right)$$

Finally, we extend the results of this Chapter to such walks, where the probability of a step of length $n$ is asymptotically equal to $c/n^3$.

# 3 Lorentz Process with shrinking holes in a wall

Consider a periodic planar Lorentz process with finite horizon, restricted to a horizontal strip. In this setting, the horizontal component of the diffusively rescaled trajectory converges to the one dimensional Brownian motion, which is an easy consequence of the same statement in the plane. Now, if one puts a vertical wall to the zeroth cell, then the trajectory converges to the reflected Brownian motion, but if there is a hole on the wall - thus the particle eventually get through it -, then the limit is again the Brownian motion (see [10]). In Chapter 4 (and in [17]), we prove that if one puts a hole of decreasing size to the wall, then the limit is the so-called quasi reflected Brownian motion, a joint generalization of Brownian motion and reflected Brownian motion. An interesting feature of this stochastic process is that it is Markovian but not strong Markovian.
To be more precise, let the configuration space in the absence of the wall be $\mathcal{D} := (\mathbb{R} \times [0,1]) \setminus \cup_{i=1}^{\infty} O_i$. Here, $\{O_i\}_i$ is a $\mathbb{Z}$-periodic extension of a finite scatterer configuration in the unit square, which consists of strictly convex, pairwise disjoint scatterers, with $C^3$ smooth boundaries, whose curvatures are bounded from below by a positive constant. Further, we assume that $\cup_{i=1}^{\infty} O_i$ is symmetric with respect to the $y$-axis. The wall without the hole is $W_\infty = \{(x,y) \in \mathcal{D} | \, x = 0\} = \cup_{k=1}^{K} [\mathcal{J}_{k,l}, \mathcal{J}_{k,r}]$ where the subintervals of the $y$-axis, denoted by $[\mathcal{J}_{k,l}, \mathcal{J}_{k,r}]$, are the connected components of $W_\infty$.
The *holes* will be subintervals $I_n \subset W_\infty$, thus we will be considering a sequence $\{W_n = W_\infty \setminus I_n\}_n$ of walls. Now, the $n$-th configuration space of the *billiard flow* is $\mathcal{D}_n := (\mathbb{R} \times [0,1]) \setminus (W_n \cup \cup_{i=1}^{\infty} O_i)$. A massless point particle moves inside $\mathcal{D}_n$ (at time $t = 0$ the first hole is present, i.e. $n = 1$) with unit speed until it

hits the boundary $\partial\mathcal{D}_n$. Then it is reflected by the classical laws of mechanics (the angle of incidence equals to the angle of reflection) and continues free movement (or free flight) in $\mathcal{D}_{n+1}$. Thus, at the time instant of each reflection, the hole is replaced by an other one (meaning that the shrinking rate of the hole corresponds to real time). We also mention that the reflections on the horizontal boundaries of the strip does not play any role in our study. Thus one could define the vertical direction to be periodic (formally replace $[0,1]$ by $S^1$ in the definitions of $\mathcal{D}$ and $\mathcal{D}_n$) yielding the same results (with some different limiting variance).

Since we change the configuration space in the moment of the reflection, it is more convenient to use the discretized version of the billiard flow (the usual Poincaré section, the billiard ball map). Thus define the *phase spaces*

$$\mathcal{M}_n = \{x = (q, v), q \in \partial\mathcal{D}_n, v \in S^1, \langle v, u\rangle \geq 0 \text{ if } q \in \partial\mathcal{D} \},$$

where $u$ denotes the inward unit normal vector to $\partial\mathcal{D}$ at the point $q \in \partial\mathcal{D}$. Here, $q$ denotes the position of the particle at a collision and $v$ is the post-collisional velocity vector. If $q \in \partial\mathcal{D}$, $v$ can be naturally parameterized by the angle between $u$ and $v$ which is in the interval $[-\pi/2, \pi/2]$. If $q \in \partial W_n = W_n$, one can parameterize $v$ by its angle to the horizontal axis. Thus, if this angle is in the interval $[-\pi/2, \pi/2]$, then the particle is on the right-hand side of the wall, while it is on the left-hand side if this angle is either in the interval $[\pi/2, \pi]$ or in $(-\pi, -\pi/2]$.

Thus, the discretized version of the previously described billiard flow can be defined by the *billiard ball maps* $\mathcal{F}_n : \mathcal{M}_n \to \mathcal{M}_{n+1}$. Further, denote by $\kappa_n : \mathcal{M}_n \to \mathbb{R}$ the projection to the horizontal direction of the *free flight* vector from $\mathcal{M}_n$ to $\mathcal{M}_{n+1}$ (that is, if $x = (q, v) \in \mathcal{M}_n$ and $\mathcal{F}_n(x) = (\tilde{q}, \tilde{v})$, then $\kappa_n(x)$ is the projection to the horizontal axis of the vector $\tilde{q} - q$). We also assume that the billiard has finite horizon. Further, write $\mathcal{I}_n = \{I_k\}_{1\leq k\leq n}$ for the collection of the first $n$ holes, and

$$S_n(x, \mathcal{I}_n) = S_n(x) = \sum_{k=1}^{n} \kappa_k \mathcal{F}_{k-1} \ldots \mathcal{F}_1(x),$$

where $x \in \mathcal{M}_1$.

What remains is the definition of the holes $I_n$. For this, fix some sequence $\underline{\alpha} = (\alpha_n)_{n\geq 1}$ with $\alpha_n \to 0$ and, independently of each other, choose uniformly distributed points $\xi_n$, $n \geq 1$ on $\cup_{i=1}^{K}[\mathcal{J}_{i,l}, \mathcal{J}_{i,r}]$. We will use the following three special choices:

1. Assume that $\xi_n \in [\mathcal{J}_{i,l}, \mathcal{J}_{i,r}]$, and denote $l_n = \mathcal{J}_{i,r} - \xi_n$. If $l_n > \alpha_n$, then put $I_n = (\xi_n, \xi_n + \alpha_n)$, otherwise put $I_n = (\xi_n, \mathcal{J}_{i,r}) \cup (\mathcal{J}_{i,l}, \mathcal{J}_{i,l} + \alpha_n - l_n)$, which is a subset of $W_\infty$ for $n$ large enough. With this particular choice, write

$$S_n^{\searrow}(x, \underline{\alpha}) = S_n^{\searrow}(x) = S_n(x, \mathcal{I}_n)$$

and

$$\mathcal{F}_n^{\searrow} = \mathcal{F}_n.$$

2. For each $1 \le k \le n$, let the random variables $\xi_n^{(k)}$ be independent and distributed like $\xi_n$. Assume that $\xi_n^{(k)} \in [\mathcal{J}_{i,l}, \mathcal{J}_{i,r}]$, and denote $l_n^{(k)} = \mathcal{J}_{i,r} - \xi_n^{(k)}$. If $l_n^{(k)} > \alpha_n$, then put $I_n^{(k)} = (\xi_n^{(k)}, \xi_n^{(k)} + \alpha_n)$, otherwise put $I_n^{(k)} = (\xi_n^{(k)}, \mathcal{J}_{i,r}) \cup (\mathcal{J}_{i,l}, \mathcal{J}_{i,l} + \alpha_n - l_n^{(k)})$, and finally $\mathcal{I}_n = (I_n^{(k)})_{1 \le k \le n}$. With this particular choice, write

$$S_n^{\equiv}(x, \underline{\alpha}) = S_n^{\equiv}(x) = S_n(x, \mathcal{I}_n).$$

3. Let $I_n = W_\infty$. With this particular choice, write

$$S_n^{(per)}(x) = S_n(x, \mathcal{I}_n),$$

and for a fixed $x$, define $S_t^{(per)}(x)$ for $t \ge 0$ as the piecewise linear, continuous extension of $S_n^{(per)}(x)$. Finally, write

$$\begin{aligned} \mathcal{F}^{(per)} &= \mathcal{F}_1, \\ \mathcal{M}^{(per)} &= \mathcal{M}_1. \end{aligned}$$

Here the first choice - the only really time dependent - is the most interesting one. In the second case, one has to redefine the whole trajectory segment $S_1^{\equiv}, \dots S_n^{\equiv}$ for each $n$, thus we have a sequence of billiards (in other words, the increments of $S_n^{\equiv}$ form a double array), while the third one is just a usual periodic Lorentz process.
There is a natural measure - the projection of the *Liouville measure* of the periodic billiard flow - on $\mathcal{M}^{(per)}$ which is invariant under $\mathcal{F}^{(per)}$. Denote the restriction of this measure to the two neighboring tori to the origin by $\mathbf{P}$. Note that $\mathbf{P}$ is finite, so normalize it to be a probability measure.
Finally, define $\mathcal{J} \subset \mathcal{M}^{(per)}$ as such points on the discrete phase space without any wall, from which before the forthcoming collision, the particle crosses $\cup_{i=1}^{K} (\mathcal{J}_{i,l}, \mathcal{J}_{i,r})$. Note that the finite horizon condition implies that $\mathcal{J}$ is bounded.

Now, we define the limiting processes. In fact, there going to be two very similar processes, we call both quasi-reflected Brownian motion and distinguish between them only in the abbreviation.
Consider a Brownian motion (BM) $\mathfrak{B} = (\mathfrak{B}_t)_{t \in [0,1]}$ with parameter $\sigma$ on $[0,1]$. Its local time at the origin is denoted by $\mathfrak{L} = (\mathfrak{L}_t)_{t \in [0,1]}$. That is,

$$\mathfrak{L}_t = \lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{\{|\mathfrak{B}_S| < \varepsilon\}} ds.$$

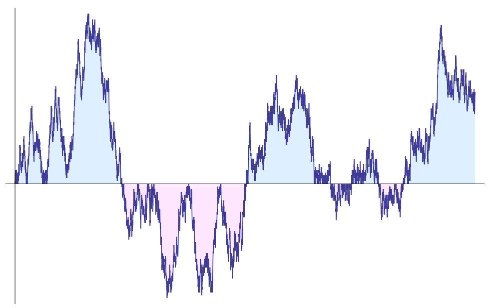

Figure 1: qRBM

Now, given $\mathfrak{B}$, consider a Poisson Point Process $\Pi$ with intensity measure $cd\mathfrak{L}$ with some positive constant $c$. With probability one, the support of the measure $cd\mathfrak{L}$ is $\mathfrak{Z}$, where $\mathfrak{Z} = \{s : 0 \le s \le 1 : \mathfrak{B}_s = 0\}$ is the zero set of $\mathfrak{B}$. Denote the points of $\Pi$ by $P_1, P_2, ...$ in decreasing order. In fact, $\Pi$ has finitely many points. If it has $m$ points, then put $P_{m+1} = P_{m+2} = ... = 0$. Further, write $P_0 = 1$ and introduce a Bernoulli distributed random variable $\eta$ with parameter $1/2$ (where the parameter means the probability of being equal to 1) which is independent of $\mathfrak{B}$ and $\Pi$.
Now, the process $\mathfrak{Q} = (\mathfrak{Q}_t)_{t\in[0,1]}$ with $\mathfrak{Q}_0 = 0$ and

$$\mathfrak{Q}_t = \begin{cases} (-1)^{\eta}|\mathfrak{B}_t| & \text{if } \exists n \in \mathbb{Z}_+ \cup \{0\} : t \in (P_{2n+1}, P_{2n}] \\ (-1)^{1-\eta}|\mathfrak{B}_t| & \text{otherwise} \end{cases}$$

is called the quasi-reflected Brownian motion with parameters $c$ and $\sigma$, and denoted by qRBM(c,$\sigma$), see Figure 1.

The definition of QRBM is similar to that of qRBM. The difference is that $c(d\mathfrak{L})$ now should be replaced by $c\frac{1}{\sqrt{t}}(d\mathfrak{L}_t)$. As a result, the Poisson process will have infinitely many points, which accumulate only at the origin. Now, denote by $P_1, P_2, \ldots$ these points in decreasing order (nota bene: there is no smallest one among them), put $P_0 = 1$ and define $\eta$ and QRBM(c,$\sigma$) as before.

As usual, $C[0,1]$ will denote the space of continuous functions on the interval $[0,1]$. Let the function $\mathbf{W}_n^{\searrow}$ be the following: $\mathbf{W}_n^{\searrow}(k/n) = S_k^{\searrow}/\sqrt{n}$ for $0 \le k \le n$ and define $\mathbf{W}_n^{\searrow}(t)$ for $t \in [0,1]$ as its piecewise linear, continuous extension. Let $\mu_n^{\searrow}$ denote the measure on $C[0,1]$ induced by $\mathbf{W}_n^{\searrow}$, where the initial distribution, i.e. the distribution of $x$, is given by $\mathbf{P}$. Analogously, define $\mu_n^{\equiv}$ with $\mathbf{W}_n^{\equiv}$, where $\mathbf{W}_n^{\equiv}(k/n) = S_k^{\equiv}/\sqrt{n}$.
Now, we can formulate the main result of Chapter 4.

**Theorem 15.** *There are positive constants $\sigma$ and $c_2$ depending only on the periodic scatterer configuration, such that*

1. *if $\exists c > 0 : \alpha_n\sqrt{n} \to c$, then $\mu_n^{\searrow}$ converges weakly to the measure induced by $QRBM(c_2 c, \sigma)$.*

2. *if* $\exists c > 0 : \alpha_n\sqrt{n} \to c$*, then* $\mu_n^{\equiv}$ *converges weakly to the measure induced by* $qRBM(c_2c, \sigma)$*.*

3. *if* $\alpha_n\sqrt{n} \to 0$*, then both* $\mu_n^{\searrow}$ *and* $\mu_n^{\equiv}$ *converge weakly to the convex combination of the measures induced by the reflected Brownian motion and the negative reflected Brownian motion, with weights* $1/2$*.*

4. *if* $\alpha_n\sqrt{n} \to \infty$*, then both* $\mu_n^{\searrow}$ *and* $\mu_n^{\equiv}$ *converge weakly to the Wiener measure.*

The most important ingredient of the poof is the local limit theorem for periodic Lorentz process (proved by Szász and Varjú, [26]). In fact, we also prove an extended version of the global limit theorem, which might deserve some independent interest. Let $L_{nt}$ denote the number of visits to a compact subset of the phase space in the time interval $[1, \lfloor nt \rfloor]$ (the compact subset is practically chosen to be $\mathcal{J}$). Then the following statement holds.

**Proposition 16.**

$$\left( \frac{S_{nt}^{(per)}}{\sqrt{n}}, \frac{L_{nt}}{\sqrt{n}} \right)_{t\in[0,1]} \Rightarrow (\mathfrak{B}_t, c_0\mathfrak{L}_t)_{t\in[0,1]},$$

*with some* $c_0 > 0$*, as* $n \to \infty$ *where the left hand side is understood as a random variable with respect to the probability measure* $\mathbf{P}$*, and* $\Rightarrow$ *stands for weak convergence in the Skorokhod space* $D_{\mathbb{R}^2}[0, 1]$*.*

# 4 Time dependent dynamics

In Chapter 5, we prove CLT for deterministic time-dependent dynamical systems. The result itself is applicable only in restricted settings - mainly for one dimensional expanding maps - but the time inhomogeneity is general. The latter means that instead of proving the central limit theorem for a typical sequence of some mappings, we can prove it for fixed sequences under some conditions. These are connected to the zero-cohomology condition in the autonomous case. This Chapter is based on [18].
Let $A$ be a set of numbers and $(X, \mathcal{F}, \mu)$ a probability space. For each $a \in A$ define $T_a : X \to X$. Suppose that $\mu$ is invariant for all $T_a$'s. Now consider a sequence of numbers from $A$, i.e. $\underline{a} : \mathbb{N} \to A$. Our aim is to prove some kind of central limit theorem for the sequence

$$f \circ T_{a_1}, f \circ T_{a_2} \circ T_{a_1}, ...$$

with some nice function $f : X \to \mathbb{R}$.
As usual,

$$\hat{T}_a g(x) = g(T_a x)$$

and $\hat{T}^*$ is the $L^2(\mu)$-adjoint of $\hat{T}$ (the so called Perron-Frobenius operator). Further, introduce the notation

$$\hat{T}_{[i..j]} = \begin{cases} \hat{T}_{a_i} \dots \hat{T}_{a_j} & \text{if } i \le j \\ Id & \text{otherwise} \end{cases}$$

and for simplicity write $\hat{T}_{[j]} = \hat{T}_{[1..j]}$.
Similarly, define

$$\hat{T}^*_{[i..j]} = \begin{cases} \hat{T}^*_{a_j} \dots \hat{T}^*_{a_i} & \text{if } i \le j \\ Id & \text{otherwise} \end{cases}$$

and $\hat{T}^*_{[j]} = \hat{T}^*_{[1..j]}$.
Further, define $\sigma$-algebras $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{F}_i = (T_{a_1})^{-1} \dots (T_{a_i})^{-1}\mathcal{F}_0$. We will need this sequence of $\sigma$-algebras to form a decreasing systems (cf. Assumption 2 of Theorem 17), restricting our approach to non-invertible maps. Let us assume that there is a Banach space $\mathcal{B}$ of $\mathcal{F}$-measurable functions on $X$ such that $\|g\| := \|g\|_{\mathcal{B}} \ge \|g\|_\infty$ for all $g \in \mathcal{B}$.
Finally, for the fixed function $f$, introduce the notation

$$u_k = \sum_{i=1}^{k} \hat{T}^*_{[i+1..k]} f.$$

With the above notation, our aim is to prove a limit theorem for $S_n(x) = \sum_{k=1}^n \hat{T}_{[k]} f(x)$.

**Theorem 17.** *Assume that $f$, $\underline{a}$ and $T_b$, $b \in A$ satisfy the following assumptions.*

1. $\int f d\mu = 0$.
2. *$T_b$ is onto but not invertible for all $b \in A$.*
3. *$f \in \mathcal{B}$ and there exist $K < \infty$ and $\tau < 1$ such that for all sequences $\underline{b}$ and for all $k$, $\|\hat{T}^*_{b_1} ... \hat{T}^*_{b_k} f\| < K\tau^k \|f\|$.*
4. (accumulated transversality) *Define $\chi_k$ as the $L^2$-angle between $u_k$ and the subspace of $(T_{a_{k+1}})^{-1}\mathcal{F}_0$-measurable functions. Then*

$$\sum_{k=1}^{N} \min_{j \in \{k,k+1\}} (1 - \cos^2(\chi_j))$$

*converges to $\infty$ as $N \to \infty$.*

*Then*

$$Var(S_n) \to \infty$$

*and*

$$\frac{S_n(x)}{\sqrt{Var(S_n)}}$$

*converges weakly to the standard normal distribution, where $x$ is distributed according to $\mu$.*

Assumption 3 roughly tells that there is an eventual spectral gap of the operators $\hat{T}^*_{a_j}$ which is quite a natural assumption. Assumption 4 guarantees that there is no much cancellation in $S_n$, for instance $f$ cannot be in the cohomology class of the zero function when $|A| = 1$.
We also give two examples, where the above conditions are fulfilled.

**Example 18.** *Define $(X, \mathcal{F}, \mu) = (S^1$, Borel, Leb$)$, $A = \{2, 3, \dots\}$, $T_a(x) = ax(\mod 1)$, $\mathcal{B} = C^1 = C^1(S^1)$,*

$$\|g\| := \sup_{x \in S^1} |g(x)| + \sup_{x \in S^1} |g'(x)|.$$

*Fix a non constant function $f \in C^1$ satisfying $\int f dx = 0$. Then there exists some integer $L = L(f)$ such that with all sequences $\underline{a}$ for which*

$$\#\{k : \min\{a_k, a_{k+1}, a_{k+2}\} > L\} = \infty$$

*the assumptions of Theorem 17 are fulfilled.*

**Example 19.** *Define $X, \mathcal{F}, \mu, A, T_b, \mathcal{B}$ as in Example 18. If $\underline{a}$ is a sequence for which there is a $b \in A$ such that for all integer $K$, one can find a $k$ for which*

$$a_k = a_{k+1} = \dots = a_{k+K-1} = b,$$

*and $f \in \mathcal{B}$, $\int f = 0$ is any function for which the equation $f = \hat{T}_b u - u$ has no solution $u$, then the assumptions of Theorem 17 are fulfilled.*

# 5 On Dettmann's Horizons Conjectures

Chapter 6 of the thesis is about periodic Lorentz process in dimension $d \geq 3$ and with infinite horizon. Note that the high dimensional case is much more difficult than the planar one. Even for finite horizon, much less is known than in $d = 2$, see [2] for the present state of the theory. The infinite horizon also makes the picture more difficult. Recall that in the planar case, the scaling of the trajectory is slightly super-diffusive. In $d \geq 3$, the first step is to ascertain the tail asymptotics of the free flight function. If it is the same as the one in the planar case (that is, $\sim C/t$), then it is reasonable to expect the same super-diffusive behavior. In [8], Dettmann formulated conjectures, which provide the tail asymptotics in quite a generality. The essence of the conjectures is that super-diffusion is expected if and only if there exists a non-degenerate horizon of maximal possible dimension (i.e. $d-1$). It is quite easy to prove that if there is such a horizon, then the distribution of the free flight function decays at least as $1/t$, thus super-diffusion is expected. However, if this $d-1$ dimensional horizon is degenerate, then the question is much less obvious and turns out to be an interesting problem is geometry on its own right. We discuss this problem, which gives the proof of Dettmann's second conjecture. It is worth mentioning that our proof uses results from the theory of the small scatterer size limit of Lorentz processes, see [14]. This Chapter is based on [19].

Let $\{O_i\}_{i=1,\dots,n}$ be some open subsets of the $d$ dimensional torus, whose boundaries are $\mathcal{C}^3$-smooth hypersurfaces. Notice that we do not require the scatterers to be disjoint (nota bene different scatterer configurations can lead to identical configuration spaces, if the differences are covered by other scatterers). Points in the boundary intersections $q \in \partial O_i \cap \partial O_j$ are called corner points. We also require that at any point of the boundary $\partial O_i$, the curvature operator $K$ is uniformly bounded from above: there exists a universal constant $\kappa_{\max}$, such that for every tangent vector $v$ of the hypersurface $\partial O_i$, the inequality $0 \le K(v,v) \le \kappa_{\max}\|v\|^2$ holds. Note that we do not require lower bound on the curvature. (This setup is in fact called *semi-dispersing billiard*, which is of crucial importance in the so-called Boltzmann-Sinai Ergodic Hypothesis, see [22], [21].)

The configuration space of the billiard flow is just $\tilde{Q} = \mathbb{R}^d \setminus \cup_{i=1}^{\infty} O_i$, where $\{O_i\}_{i=n+1,\dots}$ are the translated copies of $\{O_i\}_{i=1,\dots,n}$ with translations by integer vectors. The phase space is $\tilde{M} = \tilde{Q} \times S^{d-1}$ where $S^{d-1}$ is the set of possible velocity vectors. The billiard flow acts on $\tilde{M}$, and Lebesgue measure is invariant in both coordinates. Obviously, the billiard flow is analogously defined on $M = Q \times S^{d-1} = (\tilde{Q}/\mathbb{Z}^d) \times S^{d-1}$. Denote by $\mu$ the Lebesgue measure on $M$. Now, the tail distribution of the free flight function is given by

$$F(t) = \mu\{(q,v) : \tau(q,v) > t\},$$

where $\tau(q,v) = \inf_{s>0}\{q + sv \in \cup_{i=1,2,\dots} O_i\}$.

The "degenerate $d-1$ dimensional horizon" means the following. Assume that there is a $d-1$ dimensional affine subspace in $\mathbb{R}^d$, which is disjoint to all the scatterers. Further, assume that any such subspace touches some scatterers from both sides (or, in other words, if $q+V$ is a collision free affine subspace with $\dim V = d-1$, then for any small enough $v \notin V$, $q+v+V$ intersect with some scatterer). Under the above assumptions, we have the following theorem.

**Theorem 20.** *As $t \to \infty$, we have*

$$F(t) = \begin{cases} O(t^{-2}), & 3 \le d \le 5 \\ O(t^{-2} \log t), & d = 6 \\ O\left(t^{\frac{2+d}{2-d}}\right), & d > 6. \end{cases}$$

*Further, if we also assume that the curvature is bounded away from* 0 *(from below) uniformly at every point of $\partial Q$ (dispersing case), then*

$$F(t) \asymp \begin{cases} t^{-2}, & 3 \le d \le 5 \\ t^{-2} \log t, & d = 6 \\ t^{\frac{2+d}{2-d}}, & d > 6. \end{cases}$$

In fact, the power $\frac{2+d}{2-d}$ was not conjectured by Dettmann. Further, we prove slightly more, the periodicity of the billiard can be some arbitrary non-degenerate $d$ dimensional lattice $\mathcal{L}$ instead of $\mathbb{Z}^d$.
Finally, we mention a Conjecture of ours, which could be the first step in the possibly long way to prove super-diffusive limit theorem for a high dimensional periodic Lorentz process with a non-degenerate $d-1$ dimensional horizon.

**Conjecture 21.** *In a $d$ dimensional dispersing billiard with at least one $d-1$ dimensional non-degenerate horizon, if $\tau$ is large, then the length of the following free flight (i.e. just after the first collision) is typically of order $\tau^{1/d}$.*

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