Analysis of multi-periodic investment strategies

András Urbán
urban@finance.bme.hu

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advisor: Dr. Mihály Ormos

Budapest University of Technology and Economics,
PhD School in Business and Management

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Contents

1   The aim of the research

In the thesis the theoretical and empirical analysis of the so-called multi-periodic, growth optimal investment strategies were performed, with distinguished emphasis on the differences between the traditional one period model (Markowitz, 1952, Sharpe, 1964, Lintner, 1965, Mossin, 1966) and the multi-periodic approach. In the thesis I propose a multi-period investment framework following Györfi, Lugosi and Udina (2007), Györfi, Urbán and Vajda (2007), Györfi, Udina and Walk (2008), Urbán and Ormos (2010) and Ormos and Urbán (2010). This makes it feasible to mathematically formalize the differences between the two approach. The key to the novel approach lays in the recognition that the traditional models follow the mean-variance approach, which is unable to capture the nature of the long-term growth as presented in the thesis. A multi-periodic investment is the sum of consecutive decisions, where capital gains are reinvested after each trading period. Despite the fact that maximizing the expected one-period return in each period mathematically promises the highest expected wealth level, the investor is able to approach this expected value with zero probability (Györfi, Ottucsák and Urbán, 2007). Based on this statement I show that the expected return is inapplicable to measure the long-term growth rate, while the expected log return is a proper metric. In the thesis I propose a new condition, which establishes a mathematical relationship between the classical mean-variance approach and the long-term wealth maximization (Ormos and Urbán, 2010), that is, using the parameters of the classical models I derive a condition which has to be satisfied for positive growth.

The mathematical consequences are illustrated with empirical analysis performed on the dataset of S&P 500 historical constituents from 1970 until 2008. The results confirm the practical applicability of the proposed theory.

I also determine which investments are rational to hold in the case of multiple periods, where the relation of risk and growth rate differs from the classical approach. Since the growth rate, which is the function of the variance of returns, has a well-defined maximum, the investor is not compensated for taking higher risk than that at the growth-optimal point. This result was empirically approved, which shows at the most serious difference between the one-periodic and multi-periodic approach, since the Modern Portfolio Theory (Markowitz, 1952) and the follower models are based on the assumption that the investors are always compensated for risk taking. This assumption fails in the multi-periodic approach.

I introduce the so-called log-optimal and semi-log-optimal strategies (Györfi, Urbán and Vajda, 2007), which are the sequential extensions of the growth-optimal approach. The proposed semi-log-optimal method promises a semi-optimal asymptotic growth rate for all processes in the class of stationary and ergodic processes. The procedure is the approximation of the kernel based estimator of Györfi, Lugosi and Udina (2005). I perform tests on a benchmark timeseries of NYSE stocks. The dataset contains 36 stocks

and covers a 22 year-long interval. The aim of the analysis is the establishment of a comparison between the new method and the one of Györfi, Lugosi and Udina (2005). My empirical results confirms that the semi-log-optimal strategies are able to achieve outstanding returns similarly to the log-optimal method. This means that the second order Taylor approximation used in the semi-log-optimal portfolios performs as well as the pure log-optimum portfolio. Since our model (Ormos and Urbán, 2011 and Ormos, Urbán and Zoltán, 2009) assumes daily portfolio rebalancing I could not ignore the effect of transaction costs. Thus I amended the portfolio optimizer with a cost management module.

My third contribution to the field is the assessment of the empirical log-optimal trading strategy (Ormos and Urbán, 2011 and Ormos, Urbán and Zoltán, 2009) which utilizes the concept introduced in the first part of the thesis. Since these strategies were not analysed in the literature with equilibrium models, the outstanding performance of the strategies has not been proven before. The previous researches only dealt with growth rate of the investments neglecting necessary adjustments for risk. My investigation is based on four equilibrium models: the CAPM (Treynor, 1962, Sharpe, 1964, Lintner, 1965, Mossin, 1966), Fama-French 3-factor model (Fama-French 1992,1993) and the CAPM and the 3-factor model amended with momentum factor (Carhart, 1997). The historical tests run on the dataset of the Dow Jones Industrial Average components. According to the results I draw the conclusion that the introduced growth-optimal, active investment strategy, which takes transaction costs into account, was able to achieve significant, positive abnormal returns during the investigation period. This outstanding performance leads back to a kind of market inefficiency, since the proposed methods are able to make better investment decisions than the pure random selection of assets.

2   Traditional mean-variance approach versus growth-optimality

Investigating a market of long-term investments where capital gains are reinvested after each trading period. Evolution of the multi-period investment is easier to characterize using price relatives rather than rates of return. Price relative $\mathbf{x}_i$ is a stochastic variable representing the relative change in the value of an individual asset or portfolio on trading period i such that

$$\mathbf{x}_i = \frac{p_i}{p_{i-1}},$$

where prices on day i and $i-1$ are denoted by p_i and p_{i-1}, respectively. The relative price corresponds to rate of return since

$$\mathbf{x}_i \equiv \mathbf{r}_i + 1,$$

where $\mathbf{r}_i$ denotes the rate of return from trading period $i - 1$ to period i.

When price relatives $\mathbf{x}_1, \mathbf{x}_2, \ldots$ are realizations of *independent and identically distributed* $\mathbf{X}_1, \mathbf{X}_2, \ldots$, we can take the following expectation of the wealth W_n after n investment periods:

$$\mathbb{E}\{W_n\} = \mathbb{E}\{W_0 \cdot \mathbf{x}_1 \cdot \mathbf{x}_2 \cdot \ldots \cdot \mathbf{x}_n\} = W_0 \prod_{i=1}^{n} \mathbb{E}\{\mathbf{x}_1\} = W_0 e^{n \ln \mathbb{E}\{\mathbf{x}_1\}}, \tag{1}$$

where W_0 is the initial wealth. Considering (1) an investor may guess that an asset promising maximum expected return will also gain the maximum wealth after several investment periods, where, under expected value I mean statistical expectation. Moreover, since on i.i.d. markets where the expected return and the variance exists

$$\left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \right) \to \mathbb{E}\{\mathbf{X}_1\} \text{ as } n \to \infty$$

and

$$\mathbf{Var} \left\{ \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \right\} \to 0 \text{ as } n \to \infty,$$

an investor who has the opportunity to fund his wealth several times, has better a chance to reach his expectation on average than one who simply invests for a single period. This interpretation of the law of large numbers may suggest that if the investment horizon is very long, the investor should take only the expected return into account since the variance of the average relative converges to zero. We may accept this approach if we assume traditional conditions when the investor's utility function is based only on the mean and variance of the price relatives. This simple logic could lead to the argument that in the long-run it is worth holding investments with high return expectations, even if they are risky, since the risk disperses over multiple periods. Note, however, the variance of the average relative decreases as the length of the investment horizon grows, while the variance of the expected wealth does not.

In lieu of the common mean-variance approach, I capture the nature of the capital growth by another way. One which characterizes how multiple-period growth works. In a multi-period investment W_n is the product of the former wealth W_{n-1} and the actual price relative such as

$$W_n = \mathbf{x}_n \cdot W_{n-1},$$

recursively

$$W_n = \mathbf{x}_n \cdot \mathbf{x}_{n-1} \cdot \ldots \cdot \mathbf{x}_1 \cdot W_0.$$

With more manipulation we can write

$$W_n = W_0 \prod_{i=1}^{n} \mathbf{x}_i = W_0 e^{\sum_{i=1}^{n} \ln \mathbf{x}_i} = W_0 e^{n G_n}, \tag{2}$$

where G_n denotes the average rate of growth

$$G_n = \frac{1}{n} \sum_{i=1}^{n} \ln \mathbf{x}_i.$$

Taking the logarithm in (2) gives

$$\ln \left(\frac{W_n}{W_0} \right)^{1/n} = G_n,$$

which means that maximization of W_n and maximization of G_n are equivalent. Additionally, if $\mathbf{x}_i$ relatives are i.i.d., $\ln \mathbf{x}_i$ log returns are i.i.d. too. Thus, the law of large numbers comes into play again with the following form:

$$G_n \to \mathbb{E}\{\ln \mathbf{X}_1\} \text{ as } n \to \infty, \tag{3}$$

where $\mathbb{E}\{\ln \mathbf{X}_1\}$ is called expected rate of growth or simply expected log return. If n is large enough, the average rate of growth captures the typical value of W_n since

$$W_n \approx W_0 e^{n\mathbb{E}\{\ln \mathbf{X}_1\}}$$

by (2) and (3). Without loss of generality I assume $W_0 = 1$ initial investment.

2.1 Requisites for long-term growth

If an investor's goal is to achieve positive growth rate, he has to hold investments with $\mathbb{E}\{\ln \mathbf{X}_1\}$ higher than zero. Since the typical price relatives are around 1 Györfi, Urbán and Vajda (2007) suggested an approximation of the function $\ln z$ such as

$$\ln z \approx h(z) = z - 1 - \frac{1}{2}(z-1)^2, \tag{4}$$

which is the second order Taylor expansion at $z = 1$. In order to compute $\mathbb{E}\{\ln \mathbf{X}_1\}$ I use the approximation

$$\mathbb{E}\{\ln \mathbf{X}_1\} \approx \mathbb{E}\{\mathbf{X}_1 - 1\} - \frac{1}{2}\mathbb{E}\left\{ (\mathbf{X}_1 - 1)^2 \right\}. \tag{5}$$

In the model of Györfi, Urbán and Vajda (2007), Ottucsák and Vajda (2006,2007) proposed to use the following variance identity:

$$\mathbf{Var}\{s\} = \mathbb{E}\{s^2\} - \mathbb{E}^2\{s\}.$$

Thus, we obtain the implicit condition on the single-period mean and variance for positive growth (Ormos and Urbán, 2010):

$$
\begin{aligned}
\mathbb{E}\{\ln \mathbf{X}_1\} &\approx 2\mathbb{E}\{\mathbf{X}_1\} - \frac{1}{2}\mathbb{E}\left\{\mathbf{X}_1^2\right\} - \frac{3}{2} \\
&= -\frac{1}{2}\mathbb{E}^2\{\mathbf{X}_1\} + 2\mathbb{E}\{\mathbf{X}_1\} - \frac{3}{2} - \frac{1}{2}\left(\mathbb{E}\left\{\mathbf{X}_1^2\right\} - \mathbb{E}^2\{\mathbf{X}_1\}\right) \\
&= -\frac{1}{2}\mathbb{E}^2\{\mathbf{X}_1\} + 2\mathbb{E}\{\mathbf{X}_1\} - \left(\frac{3}{2} + \frac{1}{2}\mathbf{Var}\{\mathbf{X}_1\}\right) \\
&> 0.
\end{aligned}
\tag{6}
$$

with subject to $0 < \mathbb{E}\{\mathbf{X}_1\} < 2$. Based on this derivation I declare the following statement (Ormos and Urbán, 2010):

1. **A necessary condition of the long term growth using the parameters of Markowitz's approach**

$$
\mathbb{E}\{\mathbf{X}_1\} > 2 - \sqrt{1 - \mathbf{Var}\{\mathbf{X}_1\}}
$$

   subject to

$$
2 > \mathbb{E}\{\mathbf{X}_1\} \text{ and } 1 > \mathbf{Var}\{\mathbf{X}_1\}.
$$

That is, if the mean does not exceed the risk increment considerably, the investor cannot rely on long-term growth.

2.2 Growth optimal investments

To illustrate the optimal portfolio choice problem in a multi-period framework I performed empirical analysis. The test (Ormos and Urbán, 2010) was performed on the dataset of S&P 500 historical constituents from 1970 until 2008. Figure 1 presents the $\mathbb{E}\{\mathbf{X}_1\}$ expected relative price (return) and the $\mathbb{E}\{\ln \mathbf{X}_1\}$ expected rate of growth as the function of standard deviation in two panels. The expected returns and growth rates of the investment opportunities are located under and on the frontiers with the solid curves. On the right side of the dashed vertical line which assigns the position of the max $\mathbb{E}\{\ln \mathbf{X}_1\}$ portfolio, the risk increments are not compensated by higher rates of growth and result in lower level of wealth after the 20 year-long period. Although the market compensates by higher expected returns. In our expected growth rate-standard deviation framework, since (6) is concave, the growth rate curve has a well-defined maximum, which is equal to the growth-optimal choice in the multi-period framework.
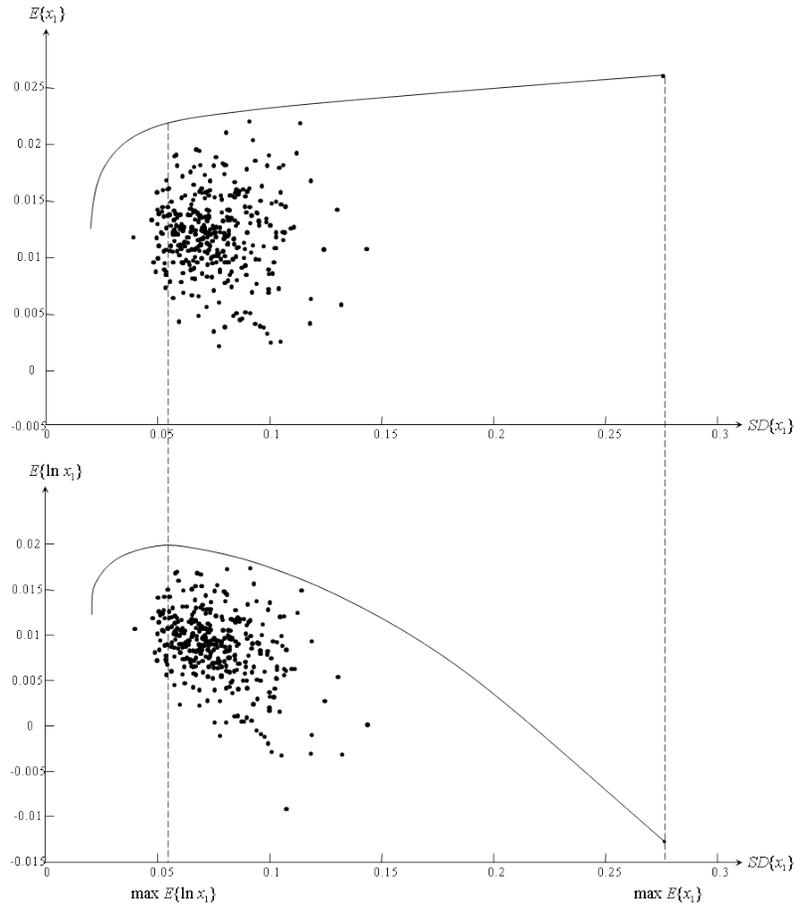
Figure 1: Localization of efficient portfolios.

Thus, I conclude (Ormos and Urbán, 2010):

2. **The Markowitz approach captures the one periodic behaviour of investments. In the case of multiple periods the growth rate has a maximum level. Beyond the growth-optimal point, holding mean-variance efficient portfolios is not rational for a long-term compounder. The growth-optimal portfolio satisfies**

$$\arg\max_{\mathbf{X}_1} \mathbb{E}\{\ln \mathbf{X}_1\} \approx \arg\max_{\mathbf{X}_1} -\frac{1}{2}\mathbb{E}^2\{\mathbf{X}_1\} + 2\mathbb{E}\{\mathbf{X}_1\} - \left(\frac{3}{2} + \frac{1}{2}\mathbf{Var}\{\mathbf{X}_1\}\right)$$

subject to

$$2 > \mathbb{E}\{\mathbf{X}_1\} \text{ and } 1 > \mathbf{Var}\{\mathbf{X}_1\}.$$

3 Sequential investment strategies

In the second part of the thesis I introduce an empirical portfolio strategy which is based on the concept of long-term growth proposed before. My goal is to implement a trading model and algorithm similarly to Györfi, Lugosi and Udina (2007) and Györfi, Urbán and Vajda (2007). This model amended with a cost management tool can be used to empirically prove, whether the log-optimal portfolios are able to achieve abnormal returns or not.

3.1 Log-optimal portfolio strategy

The model of stock market investigated in the section is the one considered, among others, Breiman (1961), Algoet and Cover (1988), Cover (1991), Gerencsér, Rásonyi, Szepesvári and Vágó (2005), Györfi, Lugosi and Udina (2007), Györfi, Urbán and Vajda (2007), Györfi, Udina and Walk (2008), Urbán and Ormos (2010) and Ormos and Urbán (2011).

Consider a market of d assets. A *market vector* $\mathbf{x} = (x^{(1)}, \dots x^{(d)})^\mathsf{T} \in \mathbb{R}^d_+$ is a vector of d nonnegative numbers representing price relatives for a given trading period. That is, the j-th component $x^{(j)} > 0$ of $\mathbf{x}$ expresses the ratio of the consecutive closing prices of asset j. In other words, $x^{(j)}$ is the factor by which capital invested in the j-th asset grows during the trading period.

The investor is allowed to diversify his capital at the beginning of each trading period according to a portfolio vector $\mathbf{b} = (b^{(1)}, \dots b^{(d)})^\mathsf{T}$. The j-th component $b^{(j)}$ of $\mathbf{b}$ denotes the proportion of the investor's capital invested in asset j. Throughout the paper we assume that the portfolio vector $\mathbf{b}$ has nonnegative components with $\sum_{j=1}^d b^{(j)} = 1$. The fact that $\sum_{j=1}^d b^{(j)} = 1$ means that the investment strategy is self financing and the consumption of capital is excluded. The non-negativity of the components of $\mathbf{b}$ means that short selling shares on margin are not permitted. Let S_0 denote the investor's initial capital. Then at the end of the trading period the investor's wealth becomes

$$S_1 = S_0 \sum_{j=1}^d b_1^{(j)} x_1^{(j)} = S_0 \langle \mathbf{b}_1 \, , \, \mathbf{x}_1 \rangle \, ,$$

where $\langle \cdot \, , \, \cdot \rangle$ denotes inner product.

The evolution of the market in time is represented by a sequence of market vectors $\mathbf{x}_1, \mathbf{x}_2, \dots \in \mathbb{R}^d_+$, where the j-th component $x_i^{(j)}$ of $\mathbf{x}_i$ denotes the amount obtained after investing a unit capital in the j-th asset on the i-th trading period. For $j \leqslant i$ we abbreviate by $\mathbf{x}_j^i$ the array of market vectors $(\mathbf{x}_j, \dots, \mathbf{x}_i)$ and denote by Δ_d the simplex of all vectors $\mathbf{b} \in \mathbb{R}^d_+$ with nonnegative components summing up to one. An *investment strategy* is a sequence $\mathbf{B}$ of functions

$$\mathbf{b}_i : \left(\mathbb{R}^d_+\right)^{i-1} \to \Delta_d \, , \qquad i = 1, 2, \dots$$

so that $\mathbf{b}_i(\mathbf{x}_1^{i-1})$ denotes the portfolio vector chosen by the investor in the i-th trading period, upon observing the past behavior of the market. We write $\mathbf{b}(\mathbf{x}_1^{i-1}) = \mathbf{b}_i(\mathbf{x}_1^{i-1})$ to ease the notation. Therefore we get by induction that

$$S_n = S_0 \prod_{i=1}^{n} \left\langle \mathbf{b}(\mathbf{x}_1^{i-1}), \mathbf{x}_i \right\rangle = S_0 e^{\sum_{i=1}^{n} \ln \left\langle \mathbf{b}(\mathbf{x}_1^{i-1}), \mathbf{x}_i \right\rangle} = S_0 e^{n W_n(\mathbf{B})} ,$$

where $W_n(\mathbf{B})$ denotes the *average growth rate* of the investment strategy $\mathbf{B} = \{\mathbf{b}_n\}_{n=1}^{\infty}$:

$$W_n(\mathbf{B}) = \frac{1}{n} \ln S_n = \frac{1}{n} \sum_{i=1}^{n} \ln \left\langle \mathbf{b}(\mathbf{x}_1^{i-1}), \mathbf{x}_i \right\rangle .$$

Obviously, maximization of $S_n = S_n(\mathbf{B})$ is equivalent to the maximization of $W_n(\mathbf{B})$.

Constraints. To make the analysis feasible, some simplifying assumptions are used that need to be taken into account in the usual model of log-optimal portfolio theory. Assume that

- the assets are arbitrarily divisible,

- the assets are available in unlimited quantities at the current price at any given trading period,

- there are no transaction costs,

- the behavior of the market is not affected by the actions of the investor using the strategy under investigation.

Assume that $\mathbf{x}_1, \mathbf{x}_2, \ldots$ is the realization of $\mathbf{X}_1, \mathbf{X}_2, \ldots$, which is a vector valued stationary and ergodic process in $\{\mathbf{X}_n\}_{-\infty}^{\infty}$.

The log-optimal choice. Without transaction costs, Algoet and Cover (1988) reveal that on stationary and ergodic markets the so-called *log-optimum portfolio* $\mathbf{B}^* = \{\mathbf{b}^*(\cdot)\}$ is the best possible choice. More precisely, in trading period n let $\mathbf{b}^*(\cdot)$ be such that

$$\mathbf{b}_n^*(\mathbf{X}_1^{n-1}) = \arg\max_{\mathbf{b}(\cdot)} \mathbb{E} \left\{ \ln \left\langle \mathbf{b}(\mathbf{X}_1^{n-1}), \mathbf{X}_n \right\rangle \big| \mathbf{X}_1^{n-1} \right\}. \tag{7}$$

If $S_n^* = S_n(\mathbf{B}^*)$ denotes the capital achieved by a log-optimum portfolio strategy $\mathbf{B}^*$, after n trading periods, then for any other investment strategy $\mathbf{B}$ with capital $S_n = S_n(\mathbf{B})$ and for any stationary and ergodic return process $\{\mathbf{X}_n\}_{-\infty}^{\infty}$,

$$\limsup_{n \to \infty} \frac{1}{n} \log \frac{S_n}{S_n^*} \leqslant 0 \quad \text{almost surely}$$

and

$$\lim_{n \to \infty} \frac{1}{n} \ln S_n^* = W^* \quad \text{almost surely,}$$

where

$$W^* = \mathbb{E} \left\{ \log \left\langle \mathbf{b}^*(\mathbf{X}_{-\infty}^{-1}), \mathbf{X}_0 \right\rangle \right\}$$

is the maximum possible growth rate of any investment strategy.

3.2 The semi-log-optimal choice

To determine $\mathbf{b}^*$ one has to know the whole distribution of $\mathbf{X}_n$ conditional on $\mathbf{X}_1^{n-1}$, meanwhile according to Markowitz's approach (1952) $\mathbf{b}$ is optimized on the $\mathbb{E}\{\langle \mathbf{b}, \mathbf{X}_n \rangle\}$ and $\mathbf{Var}\{\langle \mathbf{b}, \mathbf{X}_n \rangle\}$ framework and only requires the first two momentums of $\mathbf{X}_n$ (Francis, 1980). That is, in the traditional Markowitz type framework $\mathbb{E}\{\langle \mathbf{b}, \mathbf{X}_n \rangle\}$ is maximized with a constraint on $\mathbf{Var}\{\langle \mathbf{b}, \mathbf{X}_n \rangle\}$. I show a new approach introduced by Györfi, Urbán and Vajda (2007), which only uses the first two momentums of the distribution, and has almost optimal performance in terms of growth rate.

Let's take again the second order Taylor approximation of ln in $z = 1$ according to (4). Using the approximation we have that

$$\mathbb{E} \left\{ \ln \left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i \right\rangle \big| \mathbf{X}_1^{i-1} \right\} \approx$$
$$\approx \mathbb{E} \left\{ \left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i \right\rangle - 1 \big| \mathbf{X}_1^{i-1} \right\} - \frac{1}{2} \mathbb{E} \left\{ \left(\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i \right\rangle - 1 \right)^2 \big| \mathbf{X}_1^{i-1} \right\} .$$

Semi-log-optimal portfolio. Hence, the semi-log-optimal portfolio choice is

$$\bar{\mathbf{b}}(\mathbf{X}_1^{n-1}) = \arg\max_{\mathbf{b}(\cdot)} \mathbb{E} \left\{ h \left(\left\langle \mathbf{b}(\mathbf{X}_1^{n-1}), \mathbf{X}_n \right\rangle \right) \big| \mathbf{X}_1^{n-1} \right\}, \tag{8}$$

where $h(z)$ is the approaching function according to (4).

The $\bar{\mathbf{B}} = \{\bar{\mathbf{b}}_1, \bar{\mathbf{b}}_2, \ldots\}$ *semi-log-optimal investment strategy* is a series of functions, where $\bar{\mathbf{b}}_n = \bar{\mathbf{b}}(\mathbf{X}_1^{n-1})$. Regarding the performance of this method Vajda (2006) proved for $\bar{S}_n = S_n(\bar{\mathbf{B}})$, that if

$$0,5 \leqslant X^{(j)} \text{ and } \mathbb{E}\{|X^{(j)} - 1|^3\} < \infty,$$

then

$$\liminf_{n \to \infty} \frac{1}{n} \ln \bar{S}_n \geqslant W^* - \frac{5}{6} \mathbb{E}\{\max_m |X^{(m)} - 1|^3\} \quad \text{almost surely.}$$

Thus, the performance of a semi-log-optimal strategy is hardly worse than that for a pure log-optimal strategy.

3.3 Empirical estimation with experts

I show such methods which are responsible for estimating the conditional distribution. For empirical estimation let's define an infinite array of experts[1] $\mathbf{H}^{(k,\ell)} = \{\mathbf{h}^{(k,\ell)}(\cdot)\}$, where k, ℓ are positive integers. For fixed positive integers k, ℓ, choose the radius $r_{k,\ell} > 0$ in such a way[2]

$$\lim_{\ell \to \infty} r_{k,\ell} = 0.$$

Then, for $n > k + 1$, define the experts $\mathbf{h}^{(k,\ell)}$ as follows. Let J_n be the locations of matches:

$$J_n = \left\{ k < i < n : \|\mathbf{x}_{i-k}^{i-1} - \mathbf{x}_{n-k}^{n-1}\| \leqslant r_{k,\ell} \right\},$$

where $\| \cdot \|$ denotes the Euclidean norm.

Kernel-based portfolio selection. Put the kernel-based log-optimal portfolio determined by expert $\mathbf{h}^{(k,\ell)}$ for trading day n as follows[3]

$$\mathbf{h}^{(k,\ell)}(\mathbf{x}_1^{n-1}) = \arg\max_{\mathbf{b} \in \Delta_d} \sum_{\{i \in J_n\}} \ln \langle \mathbf{b}, \mathbf{x}_i \rangle, \tag{9}$$

if the product is non-void, and $\mathbf{b}_0 = (1/d, \dots, 1/d)$ otherwise. Thus, $\mathbf{h}^{(k,\ell)}$ seeks market vectors similar to $\mathbf{x}_{n-1}$, then determine the fix portfolio which maximizes the return on market vectors following the previously selected similar ones. The similarity is measured by Euclidean norm.

Combination of experts. These experts are mixed as follows: let $\{q_{k,\ell}\}$ be a probability distribution over the set of positive integers (k, ℓ) such that for all (k, ℓ), $\{q_{k,\ell}\} > 0$. If $S_n(\mathbf{H}^{(k,\ell)})$ is the capital accumulated by the elementary strategy $\mathbf{H}^{(k,\ell)}$ after n periods when starting with an initial capital $S_0 = 1$, then, after period n, the investor's capital becomes

$$S_n(\mathbf{B}) = \sum_{k,\ell} q_{k,\ell} S_n(\mathbf{H}^{(k,\ell)}). \tag{10}$$

3.4 Log-optimal strategies with transaction costs

At the set up of the initial model I assumes that the transactions cost is neglectable. This assumptions, however, does not coincide with the conditions of a real trading environment. To make the results of the empirical log-optimal models be comparable to the earlier published results, the use of a cost management model cannot be ignored

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[1]Experts are elementary strategies.

[2]This condition is necessary for mathematical proof (Györfi, Lugosi and Udina, 2006).

[3]The kernel-based portfolios are refereed by $\mathbf{b}$ instead of $\mathbf{h}$ for better understanding.

further. Since, my model assumes daily portfolio rebalancing the transaction costs may reduce the results in one or two magnitudes. Here I use proportional transaction costs according to Schäfer (2002) and ignore fix cost factors. Thus, we have the modified new condition:

- the level of proportional transaction costs is higher then 0 and the fix cost factors are neglectable.

Let S_n denote the wealth at the end of market day n, $n = 0, 1, 2, \cdots$, where without loss of generality let the investor's initial capital S_0 be 1 dollar. At the beginning of a new market day $n + 1$, the investor sets up his new portfolio, i.e. purchases/sells shares according to the actual portfolio vector $\mathbf{b}_{n+1}$. During this rearrangement, he has to pay transaction costs, therefore at the beginning of a new market day $n + 1$ the net wealth N_n in the portfolio $\mathbf{b}_{n+1}$ is not larger than the gross S_n. Using the above notations the gross wealth S_n at the end of market day n is

$$S_n = N_{n-1} \langle \mathbf{b}_n , \mathbf{x}_n \rangle. \tag{11}$$

Let w_n assign:

$$w_n = \frac{N_n}{S_n}.$$

Then

$$\frac{2c}{1+c} \sum_{j=1}^{d} \left(\frac{b_n^{(j)} x_n^{(j)}}{\langle \mathbf{b}_n , \mathbf{x}_n \rangle} - b_{n+1}^{(j)} w_n \right)^+ = 1 - w_n, \tag{12}$$

and

$$\frac{1-c}{1+c} \leqslant w_n \leqslant 1,$$

where c is the proportional cost factor (between 0 and 1).

Growth with costs. Starting with an initial wealth $S_0 = 1$ and $w_0 = 1$, wealth S_n at the closing time of the n-th market day becomes

$$S_n = N_{n-1} \langle \mathbf{b}_n , \mathbf{x}_n \rangle = w_{n-1} S_{n-1} \langle \mathbf{b}_n , \mathbf{x}_n \rangle = \prod_{i=1}^{n} [w(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{x}_{i-1}) \langle \mathbf{b}_i , \mathbf{x}_i \rangle],$$

that is reduction with costs means multiplying S with w. The average growth rate in the current case is

$$W_{n,c>0}(\mathbf{B}) = \frac{1}{n} \ln S_n .$$

The aim of the investor is maximizing this expression. If there are no transaction costs in the market the investor tends to maximize

$$W_{n,c=0}(\mathbf{B}) = \frac{1}{n} \sum_{i=1}^{n} \ln \left\langle \mathbf{b}(\mathbf{x}_1^{i-1}) , \mathbf{x}_i \right\rangle,$$

where $\mathbf{B}$ is a portfolio vector in a cost free market. Similarly, the W_n wealth achieved by a strategy in a non-cost-free market is

$$W_{n,c>0}(\mathbf{B}) = \frac{1}{n} \sum_{i=1}^{n} \ln\{w(\mathbf{b}(\mathbf{x}_1^{i-2}), \mathbf{b}(\mathbf{x}_1^{i-1}), \mathbf{x}_{i-1}) \langle \mathbf{b}(\mathbf{x}_1^{i-1}), \mathbf{x}_i \rangle\}.$$

Introduce the notation

$$g(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{x}_{i-1}, \mathbf{x}_i) = \ln(w(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{x}_{i-1}) \langle \mathbf{b}_i, \mathbf{x}_i \rangle),$$

then the average growth rate becomes

$$\begin{aligned} \frac{1}{n} \ln S_n &= \frac{1}{n} \sum_{i=1}^{n} \ln(w(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{x}_{i-1}) \langle \mathbf{b}_i, \mathbf{x}_i \rangle) \\ &= \frac{1}{n} \sum_{i=1}^{n} g(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{x}_{i-1}, \mathbf{x}_i). \end{aligned} \qquad (13)$$

Our aim is to maximize this average growth rate.

4   Equilibrium modeling

In this chapter I analize the problem, whether the previously proposed empirical log-optimal strategies are able to achieve significant positive abnormal returns which cannot be explain by major equilibrium models (Ormos and Urbán, 2011 and Ormos, Urbán and Zoltán, 2009). For the analysis I used a dataset other than the previously used benchmark. I applied the dataset of the Dow Jones Industrial Average (Dow Jones 30) historical constituents. Four equilibrium models were used: the CAPM (Treynor, 1962, Sharpe, 1964, Lintner, 1965, Mossin, 1966), Fama-French 3-factor model (Fama-French 1992,1993) and the CAPM and the 3-factor model amended with momentum factor (Carhart, 1997). According to the results I draw the conclusion that the introduced methods were able to achieve significant, positive abnormal returns during the investigation period.

I also formed a passively managed buy-and-hold strategy out of the same stocks as the Industrial Average. The advantage of this strategy is that it doesn't require transaction costs.

4.1   Methodology

To empirically test the log-optimal method, the return data of the Dow Jones Industrial Average (DJIA) components is applied for a 15-year-long period from January, 1991 through December, 2008. We perform tests on three distinct portfolios: (i) a portfolio

containing the components of the DJIA observed in December, 2005; (ii) a portfolio containing the components in January, 1991; and (iii) a portfolio which tracks the actual components. Each test portfolio contains 30 shares. Here I show the (iii). investigation.

The source of the asset returns is the database of *The Center for Research in Security Prices* (CRSP). The risk free rate is the yield of the 1 month U.S. Treasury bill collected also from the CRSP database. The CRSP value weighted return index including distributions, is made up of all New York Stock Exchange (NYSE), American Stock Exchange (AMEX) and NASDAQ shares served as market portfolio. The list of the DJIA components is obtained from Dow Jones' official webpage. The proportional transaction cost is set to 0.1% of the traded volume ($c = 0.001$) both for sale and purchase. We shorted of fix cost factor.

4.2 Equilibrium models

We built the most common equilibrium models. In order to estimate the regression coefficients, 4 models are applied, sequentially, the classical CAPM, the CAPM amended with momentum factor (cf. Carhart, 1997), the Fama-French three-factor model (cf. Fama and French 1993) and a four-factor model (cf. Carhart, 1997). More precisely, one can estimate the log-optimal premiums by the following ways sequentially:

$$r_l^t - r_f^t = \alpha_l + \beta_l(r_m^t - r_f^t) + \varepsilon_l^t \tag{14}$$

$$r_l^t - r_f^t = \alpha_l + \beta_l(r_m^t - r_f^t) + m_l MOM^t + \varepsilon_l^t \tag{15}$$

$$r_l^t - r_f^t = \alpha_l + \beta_l(r_m^t - r_f^t) + s_l SMB^t + h_l HML^t + \varepsilon_l^t \tag{16}$$

$$r_l^t - r_f^t = \alpha_l + \beta_l(r_m^t - r_f^t) + s_l SMB^t + h_l HML^t + m_l MOM^t + \varepsilon_l^t, \tag{17}$$

where l, t, r_f, $(r_m^t - r_f^t)$, ε stand for share l, time, risk free rate, market premium and estimation residuals, respectively. According to Fama and French (1993), SMB (small-minus-big) measures the average return difference between small and large capitalization assets, while HML (high-minus-low) is the average return difference between high and low book-to-market equity (B/M) companies. MOM is the momentum factor (cf. Jegadeesh and Titman, 1993, Carhart, 1997), which shows the average excess return of the past winners above the return of past loser securities. The regression coefficients α, β, s, h and m were estimated based on equation (14), (15), (16) and (17). For better understanding of the performance measure, the four estimations are performed on three distinct passive portfolios as well, which contain the December, 2005, the January, 1991 and the actual DJIA components, respectively. The results are compared to the results of the log-optimal portfolio. The data of the four risk factors was obtained from Kenneth French's website[4] and used up without any modifications.

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[4]*http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html*

4.3   Empirical results

The portfolios investigated here are the blends of the previous investments as they always hold stocks from the set of the actual DJIA components. This means that the passive portfolio is not a pure buy-and-hold strategy since the portfolio is rebalanced when change occurs in the components list. At the break points the passive portfolio reallocates the capital according to the components' price. Since in this case the portfolio rebalancing is very rare, the transaction costs are negligible[5]. I summarize the results of the four regressions in Table 1. Through the lens of this fact it is more substantial that, except the three-factor model, the $\widehat{\alpha}$-s are significantly positive for the log-optimal strategy, while they do not differ significantly from zero for the passive investment[6]. It is also worthy to note that the R^2-s are the highest in this case as the strategies track a set of the most important stocks in each subperiod. The $\widehat{\beta}$-s are notably higher for the log-optimal strategy, which is also mirrored in the higher values of the standard deviation. In contrast to the other portfolios, the $\widehat{h}$-s are non-significant for the log-optimal strategy, while they are significantly positive for the passive investment. Both the log-optimal and the passive strategy have negative loading on the momentum factor, although $\widehat{m}$ is not significant for the log-optimal strategy regarding the strongest four-factor model.

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[5]Four changes occurred during the 15-year long period in the DJIA.
[6]The three-factor model also refers to positive $\widehat{\alpha}$ at 0.1 significance level.

	Log-opt	Passive		$\widehat{\alpha}_L$	$\widehat{\alpha}_P$	$\widehat{\beta}_L$	$\widehat{\beta}_P$	$\widehat{s}_L$	$\widehat{s}_P$	$\widehat{h}_L$	$\widehat{h}_P$	$\widehat{m}_L$	$\widehat{m}_P$
CAPM													
R^2	0.62	0.78	**Coeff**	0.53	0.03	1.12	0.93						
F-stat	287.6	622.3	**t-stat**	1.92	0.16	16.96	24.95						
p-val	0.00	0.00	**p-val**	0.06	0.87	0.00	0.00						
adj R^2	0.62	0.78											
CAPM+MOM													
R^2	0.63	0.81	**Coeff**	0.68	0.19	1.09	0.89					-0.14	-0.16
F-stat	151.6	376.7	**t-stat**	2.43	1.28	16.50	25.50					-2.57	-5.48
p-val	0.00	0.00	**p-val**	0.02	0.20	0.00	0.00					0.01	0.00
adj R^2	0.63	0.81											
THREE-FACTOR MODEL													
R^2	0.65	0.85	**Coeff**	0.47	-0.14	1.22	1.07	-0.21	-0.17	0.13	0.26		
F-stat	108.3	341.0	**t-stat**	1.67	-1.06	16.35	30.14	-2.72	-4.57	1.37	5.50		
p-val	0.00	0.00	**p-val**	0.10	0.29	0.00	0.00	0.01	0.00	0.17	0.00		
adj R^2	0.64	0.85											
FOUR-FACTOR MODEL													
R^2	0.66	0.87	**Coeff**	0.58	-0.01	1.18	1.03	-0.19	-0.15	0.11	0.23	-0.10	-0.11
F-stat	83.19	289.06	**t-stat**	2.05	-0.10	15.56	29.80	-2.48	-4.21	1.17	5.29	-1.84	-4.52
p-val	0.00	0.00	**p-val**	0.04	0.92	0.00	0.00	0.01	0.00	0.24	0.00	0.07	0.00
adj R^2	0.65	0.87											

Log-opt				**Passive**			
Avg Monthly prem	1.14%	**Std Dev**	5.93%	**Avg Monthly prem**	0.87%	**Std Dev**	4.35%
Avg Annual prem	13.68%	**Std Dev**	22.69%	**Avg Annual prem**	10.44%	**Std Dev**	16.24%

Table 1: Regression coefficients of portfolios formed of actual DJIA Components.

Based on the proposed model and trading algorithm, and the new empirical results of the analysis of the Dow Jones Industrial Average: the positive, significant values of the $\widehat{\alpha}$-s I conclude as follows (Ormos, Urbán and Zoltán, 2009, Ormos and Urbán, 2011)::

3. **The introduced growth-optimal, active investment strategy, which takes transaction costs into account, was able to achieve significant, positive abnormal returns during the investigation period.**

That is, the log-optimal portfolios are able to achieve outstanding returns in financial point of view.

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7 My other publications in the field

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