HIGH–QUALITY AND EFFICIENT VOLUME RESAMPLING

Thesis points of PhD dissertation

BALÁZS DOMONKOS

Supervisor:

BALÁZS CSÉBFALVI, PHD

Budapest University of Technology and Economics
Faculty of Electrical Engineering and Informatics
Department of Control Engineering and Information Technology

Budapest
2012
HIGH–QUALITY AND EFFICIENT VOLUME RESAMPLING
Thesis points of PhD dissertation

BALÁZS DOMONKOS

Supervisor:
BALÁZS CSÉBFALVI, PHD
Synopsis

In many applications of engineering and computing science, a continuous phenomenon is often represented by its discrete samples. In order to operate on the underlying continuous function, first it has to be accurately reconstructed from its discrete representation.

The sampling process results in a replication of the continuous signal spectrum around the Fourier transform of the sampling lattice. Considering a bandlimited model, the perfect reconstruction filter preserves the primary spectrum without any distortion and prevents aliasing by removing all the replicas. According to the Fourier scaling property, the tightest arrangement of the replicas in the frequency domain yields the sparsest spacing in the spatial domain. Thus, to get an efficient sampling scheme, these replicas have to be packed together without any overlapping as densely as possible.

Reconstruction filters are originally designed for reconstructing one-dimensional signals. Planar and spatial filters can be easily derived for higher-dimensional Cartesian lattices as tensor product extensions of one-dimensional filters. However, Cartesian layout is not the optimal solution in terms of sampling efficiency. If the orthogonality constraint is relaxed, the continuous signal can be represented by fewer samples.

In two dimensions, when no prior assumptions are made on the shape of the spectrum, a circularly bounded spectrum is often considered. That is, for planar signals not just the sampling density but also the arrangement of the circularly bounded replicas is the free parameters of sampling. As the tightest packing of replicas corresponds to the sparsest sampling, the sampling task can be reformulated as a circle-packing problem. It is easy to see that the optimal solution is not the de facto standard square lattice, but the hexagonal lattice.

The benefits of non-Cartesian lattices are even more significant in higher dimensions. As it is optimal for sampling 3D signals with an isotropical bandwidth, the body-centered cubic (BCC) sampling received an increased attention from the perspective of signal reconstruction. This thesis work presents efficient representation and approximation techniques for such volumetric signals.

Volumetric Representation on BCC and FCC lattices

The Kepler conjecture proved in the end of the nineties justifies that spatial signals with spherically bounded spectrum can be most efficiently sampled on the BCC lattice. Such a BCC representation can be generated in several ways.

The first option is tomographic reconstruction. Medical imaging devices create projection images of spatially varying physical quantities such as density or contrast agent concentration. Such technologies are X-ray computed tomography (CT), single-photon emission computed tomography (SPECT), and positron emission tomography (PET). From the acquired projection images, tomographic reconstruction algorithms estimate the original spatial distribution of the captured physical quantity. One of the simplest tomo-
Fig. 1: Tomographic reconstruction of a mouse skeleton on a Cartesian grid (a) and on a BCC lattice.

The most common tomographic reconstruction algorithms is filtered back projection (FBP). FBP can be adapted to the BCC lattice by evaluating the back-projection formula in the BCC lattice points.

A mouse study acquired on a Mediso preclinical CT device is illustrated in Fig. 1. The projections are reconstructed on both traditional Cartesian lattice (a) and BCC lattice (b). For the BCC lattice, 30 percent less sample points are defined. Applying resampling filters of the same reconstruction capability result in similarly detailed images on both lattices but smoother isosurfaces on the BCC lattice. This can be explained by the fact that lattice points are located more isotropically in the BCC lattice.

In addition to the direct tomographic reconstruction, non-Cartesian data can be generated also from Cartesian samples.

With ideal downsampling, the spectrum of a high-resolution signal is truncated and transformed back to the spatial domain. The most efficient way of downsampling a spatial signal is cropping its spectrum to the presumed spherical band limit as tightly as possible. Without damaging the spherically bounded spectrum, 30 and 23 percent of the sample points can be discarded by truncating the spectrum to the pass band of the BCC and FCC lattices, respectively. By filling the discarded sample points with the repetition of the primary spectrum, the downsampled BCC/FCC data can be calculated as the inverse Fourier transform of the discrete spectrum. For this purpose, the discrete Fourier transform is derived for non-Cartesian cubic lattices. It can be shown that convolution theorem stands also for BCC and FCC DFTs, thus discrete filtering can be efficiently evaluated as a frequency-domain multiplication.

Besides downsampling, BCC or FCC upsampling of Cartesian data is also advantageous in certain situations. According to the well-known technique, a powerful but expensive resampling filter can be substituted by a simpler yet more efficient one, provided that the discrete data is upsampled in advance. This strategy combines the complementary advantages of frequency-domain and spatial-domain techniques and provides the quality of the more advanced resampling filter at the cost of the increased memory requirements. It is worthwhile to perform the upsampling in the frequency domain which corresponds to a spatial-domain reconstruction with the ideal low-pass filter of the lattice. The most
efficient way of Fourier-domain upsampling is expanding the discrete spectrum as isotropically as possible. This expansion corresponds to the BCC and FCC pass bands, thus we get the upsampled representation on a BCC and an FCC lattice, respectively.

Two equivalent methods are suggested for BCC/FCC upsampling. The first method is zero padding by using the previously derived BCC/FCC DFTs. The second method exploits that a non-Cartesian cubic lattice can be defined as a composition of shifted Cartesian lattices. Consequently, the upsampling can be easily performed as a frequency-domain phase shifting, too.

Volumetric Approximation Filters

In order to be able to efficiently reconstruct the original continuous signal in arbitrary position, spatial-domain resampling filters are required. Due to the shift invariance of the sampling lattice, function reconstruction can be implemented as convolution-based filtering. Nevertheless, it is often not obvious which filter should be used for a specific data or application scenario. Generally, an appropriate filter is chosen by making a compromise between quality and efficiency. The efficiency directly depends on the support of the filter and the complexity of the incorporated numerical operations, whereas the quality can be analyzed from both signal processing and approximation theoretic viewpoints.

The Shannon–Nyquist theory provides an exact system for sampling and interpolating bandlimited signals. The ideal resampling filter can be derived as the inverse Fourier transform of the characteristic function of the pass band. The behavior of the ideal filter can be captured in the frequency domain: it preserves the frequencies in the pass band without any distortion and completely removes the spectrum aliases in the stop band.

In one dimension, the transfer function of the ideal resampling filter is a symmetric window function, and its inverse Fourier transform is the sinc function. The pass band of the hexagonal lattice has a hexagonal shape around the origin, whereas the pass band of the BCC lattice is a rhombic dodecahedron. However, practitioners rarely use these hexagonal and BCC sinc filters because of their infinite support and slow decay. Instead, more compact filters are preferred, particularly polynomial splines, which are more efficient to evaluate in the spatial domain.

Spline filters can be analyzed from both signal processing and approximation theoretic viewpoints. The distortion of the pass band causes the oversmoothing of the reconstructed signal whereas insufficient suppression of spectrum replicas results in postaliasing artifacts. On the other hand, the approximation theoretic approach classifies the filters based on the order of the asymptotic reconstruction error.

For the BCC lattice, multiple efficient spline filters can be defined which realize different compromises between the numerical accuracy of the approximated signal, the directional distribution of the reconstruction errors, and the computational complexity of the resampling. One of such spline families are box splines, that can be constructed as shadows of unit hypercube characteristic functions projected to lower-dimensional subspaces. For example the tent filter is a one-dimensional box spline, the antipodal projection of the cube defines a box spline filter with a hexagonal support, whereas the projection of the hypercube coincides the linear element of the BCC lattice. Box splines offer a mathematically elegant toolbox for constructing multidimensional basis functions. Nevertheless, their implementation on graphics processors (GPUs) is often cumbersome. Thus, an evaluation scheme is designed in order to exploit the efficient trilinear texture fetching of the GPU. In this algorithm, the linear box spline resampling is reformulated as a tetra-
Figure 2: Efficient evaluation of the linear BCC box spline. The subfigures illustrate the six orientation cases of the tetrahedral interpolation.

In addition, a novel filter family is proposed that can be easily defined for BCC and FCC lattices and has properties similar to that of the box splines. The idea behind this spline family is based on the BCC trilinear interpolation. The BCC lattice can be obtained from a dense Cartesian lattice such that only those CC lattice points are kept which have coordinates of identical parity. The BCC trilinear interpolation first performs a linear interpolation of these missing points from their neighbors using a three-way cross filter. Then a conventional trilinear interpolation is evaluated on this dense Cartesian lattice. The filtering is implemented on the GPU by taking three trilinear fetches for each sublattice of the BCC lattice. The BCC trilinear interpolation kernel can be used as a generator element for higher-order filters. Due to the associative property of the convolution operation, all of these filters can be defined as a convolution of a discrete filter and a continuous B-spline filter. Thus, these filters are denominated as discrete-continuous (DC) splines. With an appropriate discrete component, DC-splines can be easily adapted to the FCC lattice, as well.

Volumetric Approximation Schemes

The approximation power of the DC-spline that is the error term of the highest order cannot be exploited by regular sampling. However, this asymptotic error behavior can be significantly improved by a discrete prefiltering. This resampling scheme is called *quasi-interpolation* that enables the linear and higher-order filters representing trilinear and tricubic polynomials, respectively. Discrete prefilters are designed to exploit the approximation power of the DC-splines.

On the other hand, the cubic DC-spline is not interpolating. Nevertheless, it can be applied for *generalized interpolation* if the discrete samples are previously deconvolved by the sampled reconstruction filter.
Figure 3: Gradient errors of different resampling schemes. Angular error of zero degree is mapped to black, whereas angular error of 30 degrees is mapped to white. Tricubic B-spline combined with central differences (a), analytical derivative of the tricubic B-spline (b), tricubic B-spline combined with the optimal discrete derivative filter $d$ (c), and analytical derivative of the tricubic B-spline combined with the discrete prefilter $p$ (d).

Quasi-interpolating and interpolating prefilters should be executed prior to the continuous resampling as in this way the computational cost of the resampling does not increase. For an efficient prefiltering the previously derived BCC DFTs are used.

Prefiltering can improve not just the function value reconstruction but also the accuracy of the gradient reconstruction. Discrete prefilters are derived to improve the accuracy of the tricubic B-spline gradient reconstruction defined for the Cartesian lattice. Interestingly, the prevalent central differences method combined with the tricubic B-spline yields just a second-order derivative filtering. Surprisingly, the error term of the analytical derivative of the cubic B-spline is also of second order. However, if the analytical derivative is combined with the discrete prefilter that optimizes the function reconstruction, the resultant scheme is of third order. The forth-order approximation power of the B-spline can be exploited by a discrete prefilter especially tailored to the gradient reconstruction (see Fig. 3).

In practical applications, a reconstruction filter is hard to be optimized unless the characteristics of the underlying signal are known. This is especially true, when the data contains valuable high-frequency details along with non-negligible noise. In order to support practitioners, a method is proposed that interpolates between different resampling schemes. The naive method would combine schemes after resampling. However, this would significantly increase the cost of the filtering. Instead, combining different discrete prefilters with a common continuous reconstruction filter results in a flexible approximation scheme. This scheme can provide different levels of smoothing and postaliasing. By exploiting the associative property of the operations, the interpolation between the different approximation schemes can be efficiently implemented as a weighted average of the prefiltered coefficient sets. The method is illustrated in Fig. 4 for the non-prefiltered and the interpolating tricubic B-spline schemes. In this case, the trade-off between smoothing and postaliasing can be set on the fly using a single scalar parameter. The user of a volume-rendering system, that implements this flexible resampling scheme, does not have to pay attention to the frequency-domain behavior of different resampling schemes, since the interactive parameter control provides an immediate and intuitive visual feedback.
3D Frequency-Domain Analysis

In order to study the oversmoothing and postaliasing effects of the proposed volumetric resampling schemes, the analysis of their frequency responses is required. For a Cartesian lattice, reconstruction filters are usually designed in 1D and extended to the multivariate setting by a tensor-product extension. Therefore, it is sufficient to analyze the frequency-domain behavior only in 1D. However, this approach is not feasible for the BCC lattice, since it cannot be defined as a separable extension of any lower-dimensional lattice. Instead, the frequency-domain behavior has to be analyzed directly in three dimensions. As a natural tool, direct volume rendering is applied for such a 3D analysis.

As the frequency responses of the reconstruction filters are analytically known, their characteristic isosurfaces can be rendered separately in the pass band and in the stop band. Moreover, the visualization of the frequency responses conveys information not only on the absolute postaliasing and oversmoothing effects, but also on their direction dependence. The power spectrum of the resampling schemes is visualized inside the rhombic-dodecahedral pass band of the BCC lattice. This reveals the high-frequency components that are suppressed by the filtering. On the other hand, rendering of the stop band identifies the directions in which severe post-aliasing occurs.
New Scientific Results

This thesis work presents efficient representation and approximation techniques for volumetric signals. The proposed methods justify that the benefits of non-Cartesian sampling lattices can be exploited not just in theory, but also in practice. Different methods are recommended for generating and filtering BCC and FCC discrete data, applying them for efficient spatial-domain filtering on the GPU, utilizing the approximation power of the resampling filters, and analyzing the resampling schemes in the Fourier domain.

The thesis points are discussed in four groups. The thesis parts stand for the thesis groups whereas chapters reflect the proposed theses. The first thesis group addresses discrete volumetric data generation on BCC and FCC lattices. The second thesis group discusses the efficient evaluation of resampling filters on the optimal BCC lattice. The third thesis group introduces discrete prefilters for exploiting the approximation power of the resampling filters. In the final thesis group, a comprehensive frequency-domain analysis of the previously proposed schemes is presented and the correspondence between the directional resampling artifacts and the imperfect Fourier-domain behavior of resampling schemes is emphasized.

**Thesis Group 1. Volumetric Representation**

Discrete volumetric data generation algorithms are proposed for non-Cartesian cubic lattices, such as tomographic reconstruction, ideal frequency-domain downsampling, and upsampling.

**Thesis 1.1 Tomographic Reconstruction on Non-Cartesian Cubic Lattices**

Discrete representations of practical volumetric signals are typically generated by tomographic imaging devices. The Kepler conjecture implies that volumetric signals with a spherically bounded spectrum can be more efficiently sampled on BCC and FCC lattices than on the conventional Cartesian lattice. To exploit this fact also in practice, the filtered back-projection algorithm (FBP) is adapted to non-Cartesian cubic lattices [18].

**Thesis 1.2 Downsampling Cartesian Data on Non-Cartesian Cubic Lattices**

Even though tomographic reconstruction can be evaluated on BCC and FCC lattices, the conventional Cartesian representation still retains its hegemony for practical applications. Hence, methods are proposed for both downsampling and upsampling the Cartesian representation on BCC and FCC lattices. Assuming the Shannon-Nyquist framework, a Fourier-domain downsampling algorithm is designed. Using this technique, signals with spherically bounded spectra can be downsampled on the BCC lattice without any distortion by using 30 percent less samples.
Thesis 1.3 Upsampling Cartesian Data on Non-Cartesian Cubic Lattices

In addition, a discrete Fourier-domain algorithm is derived for BCC or FCC upsampling of Cartesian data. Zero padding and phase shifting is applied for upsampling the Cartesian input data on non-Cartesian cubic lattices. On the upsampled data, a simple linear filter can provide the resampling quality of a more advanced resampling filter at the cost of increased memory requirements [8,12].

Thesis Group 2. Volumetric Approximation Filters

The linear box spline is transformed into the trilinear B-spline basis. Using this evaluation scheme the efficient trilinear texture fetching of the GPU can be exploited. In addition, a novel resampling filter family is proposed that can be easily defined for non-Cartesian cubic lattices. Since spline family members can be defined as a convolution of a discrete filter and a continuous B-spline filter, they are referred to as discrete-continuous (DC) splines. The linear members of B-spline, box spline, and DC-spline families are compared in terms of their performance.

Thesis 2.1 Evaluation of the Linear Box Spline from Trilinear Texture Fetches

Among the BCC resampling filters, the box splines have the smallest support. Moreover, when only elementary operations are assumed, the box spline has the lowest computational cost, too. However, this non-separable filter cannot exploit the fast trilinear texture fetching of the graphics processing unit (GPU). Hence, the observed performance of box spline filtering is lower than that of the separable B-spline resampling. In order to reduce the filter evaluation time by exploiting the trilinear texture fetching of the GPU, the box spline filtering is transformed into the trilinear B-spline basis [2].

Thesis 2.2 DC-splines: A Novel Filter Design Approach for Non-Cartesian Cubic Lattices

A novel filter family is proposed as a generalization of the BCC trilinear interpolation. First, it is demonstrated that the BCC trilinear interpolation has comparable capabilities than that of the linear box spline but with a simpler evaluation scheme that can be efficiently implemented in the GPU. Taking the BCC trilinear kernel as a generator element, higher-order filters are derived as the convolution of a non-separable discrete filter and a separable B-spline filter. The discrete component as well as the DC-spline can be easily derived for both the FCC lattice and higher-dimensional non-Cartesian lattices.

As a result of thesis 2.1 and 2.2, an evaluation of each linear resampling kernel exists that can exploit the trilinear texture fetching of the GPU. This allows us to compare these filters in terms of their performance [6].

Thesis Group 3. Volumetric Approximation Schemes

Three different approximation schemes are defined for enhancing the capabilities of volumetric resampling filters. First, quasi-interpolating and interpolating discrete prefilters are derived for the DC-splines. Then, discrete prefilters are designed for minimizing the error of the gradient reconstruction. Finally, an efficient technique is proposed that enables real-time interpolation between different resampling schemes.
Thesis 3.1 *Discrete Prefilters for the DC-Splines*

In order to exploit the approximation power of the DC-spline, appropriate analysis filters are required. To improve the asymptotic error behavior, discrete FIR filters are derived that ensure the same approximation orders as that of the B-splines and box splines. In addition, a discrete IIR filter is defined that makes the cubic DC-spline interpolating. For quasi-interpolation and generalized interpolation, the formerly introduced BCC DFT is applied [1].

Thesis 3.2 *Prefiltered Gradient Estimation*

An accurate estimation of gradients has a high importance for volume rendering and volumetric segmentation. In the rendering equation, the estimation of the isosurface normal vectors are required, whereas segmentation relies on accurate localization of rapid changes in a scalar field. Different discrete prefilters are proposed for enhancing the gradient estimation. The asymptotic error behavior of the tricubic B-spline can be significantly improved compared to the omnipresent central differences. Discrete prefilters are designed to minimize the gradient error of the tricubic B-spline and its analytical derivative. These gradient reconstruction schemes are applied for synthetic and noisy data, and evaluated in both the spatial and the frequency domain [4].

Thesis 3.3 *Interpolation Between Different Reconstruction Schemes*

In volume rendering resampling filters is a challenging task. This is especially true, when the data contains valuable high-frequency components along with significant noise. To help practitioners with optimizing their resampling filters, a method is suggested that interpolates between different resampling schemes. The suggested technique can be incorporated into volume rendering applications thus it enables monitoring the results in real time [10].

Thesis Group 4. *3D Frequency-Domain Analysis*

In the final thesis, the previously proposed resampling schemes are evaluated in terms of signal processing. Volume rendering is applied for the Fourier-domain evaluation of non-separable volumetric approximation schemes. This technique allows for an analysis of the oversmoothing and postaliasing effects of different BCC resampling schemes.

Thesis 4.1 *3D Fourier Analysis of Reconstruction Schemes on the Body-Centered Cubic Lattice*

Oversmoothing and postaliasing effects of resampling schemes can be studied by using a frequency-domain analysis. However, one-dimensional results cannot be generalized to the multivariate setting for the BCC lattice, since it cannot be defined as a separable extension of any lower-dimensional lattice. Instead, frequency-domain behavior has to be directly analyzed in three dimensions by using direct volume rendering. The power spectrum of the previously recommended resampling schemes is visualized inside the rhombic dodecahedral pass band of the BCC lattice. This revealed the high-frequency components that are suppressed by the filtering. On the other hand, rendering of the stop band identified the directions with severe post-aliasing (see Fig. 5) [1, 7, 17].
Figure 5: Energy distribution of quasi-interpolating linear resampling schemes in the pass band (a)-(c) and in the stop band (d)-(f) along with the direction dependent artifacts of the reconstruction in diagonal (g), uniform (h), and axial (i) directions.
List of Publications

International Peer-Reviewed Journals


International Peer-Reviewed Conference Proceedings


[9] András Wirth, Áron Cserkaszky, Béla Kári, Dávid Légrády, Sándor Fehér, Szabolcs Csifrus, Balázs Domonkos: Implementation of 3D Monte Carlo PET reconstruction algorithm on GPU. In *Nuclear Science Symposium Conference Record*


Hungarian Peer-Reviewed Conference Proceedings


Other Publications


