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Dimension Theory of Non-conformal Attractors and Overlapping Self-similar Sets

PhD Thesis

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1 Introduction

In my thesis we investigate some properties of self-similar sets and self-affine sets. Especially, we focus on the dimension theory of fractals generated by iterated function systems (IFS).

More precisely, let $\Phi = \{f_1, \dots, f_n\}$ be a set of contracting functions (that is, $\|D_{\underline{x}} f\| < 1$) of $\mathbb{R}^d$ mapping an open bounded set U into itself. Then it is well known (see [H]), that there exists a unique, non-empty, compact subset Λ of $\mathbb{R}^d$ such that

$$\Lambda = \bigcap_{k=1}^{\infty} \bigcup_{i_1,\dots,i_k=1}^{n} f_{i_1} \circ \cdots f_{i_k}(U) \text{ and } \Lambda = \bigcup_{i=1}^{n} f_i(\Lambda).$$

We call the set Λ as the attractor of the iterated function system Φ.

One of the important properties of these sets is the dimension. In this thesis we mainly focus on the so-called *Minkowski dimension* (or box dimension) and *Hausdorff dimension*. We denote the Hausdorff dimension (and respectively the box dimension) of the set Λ by $\dim_H \Lambda$ ($\dim_B \Lambda$). Moreover, let us denote the s-dimensional Hausdorff measure by $\mathcal{H}^s$. For the definition and basic properties of the Hausdorff and box dimension and the Hausdorff measure we refer to [Fa1, Fa2].

2 Self-similar sets

2.1 Introduction for self-similar sets

The simplest case is when the functions of an Iterated Function System are contracting similarities

$$\Phi = \{f_i(x) = \lambda_i x + t_i\}_{i=1}^{n}$$

on the real line. In that case we call the attractor of Φ *self-similar set*. Then the non-trivial upper on the Hausdorff and box dimension of the attractor is the similarity dimension which is defined as the unique solution of

$$\sum_{i=1}^{n} \lambda_i^s = 1.$$

The dimension theory of self-similar sets is quite well understood in the cases when some separation conditions hold. Hutchinson proved that whenever the cylinders $\{f_i(\Lambda)\}_{i=1}^{n}$ are well separated, more precisely, the *open set* *condition* (OSC) holds (there exists an open, bounded subset U of $\mathbb{R}$ such that $f_i(U) \subset U$ for every i and $f_i(U) \cap f_j(U) = \emptyset$ if $i \neq j$) then the similarity dimension is equal to the Hausdorff dimension, see [H]. The box dimension is equal to the Hausdorff dimension independently of separation conditions, see [Fa5].

However, in case of heavy overlaps in between the cylinders we know very little about the structure of attractor Λ. To study such kind of Iterated Function Systems there are two known methods:

- Instead of an individual IFS we consider a one-parameter family of IFS and we use the so-called *transversality condition* introduced by Pollicott Simon [PoSi]. See [PeSo1], [PeSo2] for the most general treatment of this method. In this thesis we use this approach.

- In some very particular cases we can apply the so-called *Weak Separation Condition* [Ze], [LNR], [NW1] or some variants of it. With this method we can handle IFS like $\left\{f_i(x) = \frac{1}{N}x + t_i\right\}_{i=1}^{m}$, where $N, t_i \in \mathbb{Z}$.

In particular, when some of the maps of the IFS have common fixed points then non of the known methods can be applied directly. The simplest situation when two maps share the same fixed point was considered in [B3]. More precisely, in [B3] we considered the IFS $\{\gamma x, \lambda x, \lambda x + 1\}$ and its attractor Λ on the real line, where $\gamma < \lambda$. Let $I = [0, \frac{1}{1-\lambda}]$ be the convex hull of the attractor Λ. See Figure 1 for the image of I by the functions of this IFS. The problem of calculating the dimension was raised by Pablo Shmerkin at the conference in Greifswald in 2008. The novelty of the result obtained in [B3] about the dimension of Λ was to tackle the difficulty which comes from the fact that the first two maps have the same fixed point.

One of the most important novelties of my thesis is to handle the cases of non-distinct fixed points.

2.2 Hausdorff dimension of self-similar sets with heavy overlaps

In this section we study two types of self-similar iterated function systems with non-distinct fixed points. In both of the cases we assume that *the images of the convex hull of the attractor are overlapping only for the functions which share the same fixed point*. In the first case we suppose that there are exactly two different fixed points but a fixed point belongs to arbitrary many functions. For an example see Figure 3.

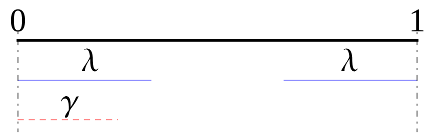

Figure 1: The simplest example of IFS with some of the functions share the same fixed point, considered in [B3].

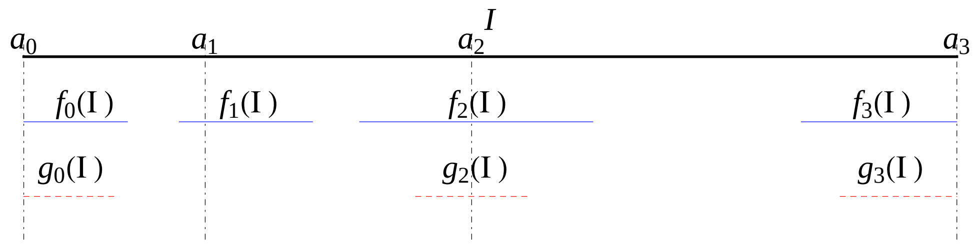

Figure 2: Images of the convex hull of the attractor of IFS $\{f_0, g_0, f_1, f_2, g_2, f_3, g_3\}$, where $a_0 = \mathrm{Fix}(f_0) = \mathrm{Fix}(g_0)$, $a_1 = \mathrm{Fix}(f_1)$, $a_2 = \mathrm{Fix}(f_2) = \mathrm{Fix}(g_2)$ and $a_3 = \mathrm{Fix}(f_3) = \mathrm{Fix}(g_3)$

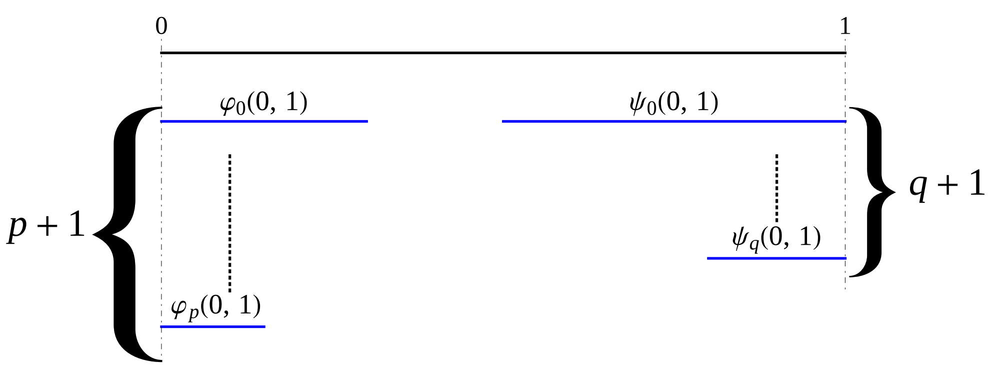

Figure 3: Images of the convex hull of the attractor of IFS $\{\phi_i\}_{i=0}^{p} \cup \{\psi_j\}_{j=0}^{q}$ where $\mathrm{Fix}(\phi_i) = 0$ and $\mathrm{Fix}(\psi_j) = 1$ for every i, j.

Principal Assumptions of Case A:

A1. Let $\mathcal{R}$ be a finite set of linear, real functions such that for every $\varphi \in \mathcal{R}$, $\mathrm{Fix}(\varphi) \in \{0,1\}$ and $\varphi([0,1]) \subseteq [0,1]$.

A2. For arbitrary $\varphi_i, \varphi_j \in \mathcal{R}$ suppose either $\varphi_i([0,1]) \cap \varphi_j([0,1]) = \emptyset$ or $\mathrm{Fix}(\varphi_i) = \mathrm{Fix}(\varphi_j)$.

Theorem 2.1. *Let* $\mathcal{R} = \{\phi_{i,1}(x) = \gamma_{i,1}x\}_{i=0}^{p} \cup \{\phi_{i,2}(x) = \gamma_{i,2}x + (1-\gamma_{i,2})\}_{i=0}^{q}$ *such that* $0 < \gamma_{i,1} < \gamma_{0,1} < 1$ *for* $i = 1, \ldots, p$ *and* $0 < \gamma_{j,2} < \gamma_{0,2} < 1$ *for* $j = 1, \ldots, q$*, then*

$$\dim_B \Lambda = \dim_H \Lambda = \min\{1, s\}, \tag{2.1}$$

where s *is the unique solution of*

$$\prod_{i=0}^{p}(1-\gamma_{i,1}^{s}) + \prod_{i=0}^{q}(1-\gamma_{i,2}^{s}) = 1 \tag{2.2}$$

for Lebesgue almost every $(\underline{\gamma}_1, \underline{\gamma}_2) \in (0,\gamma_{0,1})^p \times (0,\gamma_{0,2})^q$*, where* $\underline{\gamma}_1 = (\gamma_{1,1}, \ldots, \gamma_{p,1})$ *and respectively* $\underline{\gamma}_2 = (\gamma_{1,2}, \ldots, \gamma_{q,2})$.

Moreover $\mathcal{L}(\Lambda) > 0$ *for Lebesgue almost every* $(\underline{\gamma}_1, \underline{\gamma}_2)$ *if* $s > 1$.

Note that whenever $\gamma_{0,1} + \gamma_{0,2} \geq 1$ the attractor of $\mathcal{R}$ is an interval which implies immediately Theorem 2.1. In this way without loss of generality we may assume that $\gamma_{0,1} + \gamma_{0,2} < 1$, which is equivalent to $\varphi_{0,1}([0,1]) \cap \varphi_{0,2}([0,1]) = \emptyset$.

Our assumption in the second case is that every fixed point belongs to at most two functions. For an example of such type of IFS see Figure 2. Precisely,

Principal Assumptions of Case B:

B1. $\mathcal{S} = \mathcal{F} \cup \mathcal{G}$

B2. $\mathcal{F} = \{f_i(x) = \lambda_i x + a_i(1-\lambda_i)\}_{i=0}^{N-1}$ where $0 < \lambda_i < 1$ and the fixed points satisfy: $a_0 < a_1 < \cdots < a_{N-1}$.

B3. Let $I = [a_0, a_{N-1}]$ (the convex hull of the attractor). We require that $f_{i-1}(I) < f_i(I)$ that is

$$f_{i-1}(a_{N-1}) < f_i(a_0) \text{ for every } i = 1, \ldots, N-1. \tag{2.3}$$

B4. $\mathcal{G} = \{g_i(x) = \beta_i x + a_i(1-\beta_i)\}_{i\in\mathcal{J}}$, where $\mathcal{J} \subseteq \{0, \ldots, N-1\}$ and $0 < \beta_i < \lambda_i$ for every $i \in \mathcal{J}$.

Observe that for every $i \in \mathcal{J}$, $\text{Fix}(f_i) = \text{Fix}(g_i) = a_i$.

Denote $\underline{\beta} \in (0,1)^{\sharp\mathcal{J}}$ the vector of contraction ratios of $\mathcal{G}$ and $\underline{\lambda} \in (0,1)^N$ the vector of contraction ratios of $\mathcal{F}$. Moreover, let $\underline{a} \in \mathbb{R}^N$ be the vector of fixed points and denote the attractor of $\mathcal{S}$ by Ω. For the simplicity we write $\mathcal{I} = \{0, \dots, N-1\}$.

Theorem 2.2. *Let $\mathcal{S}$ be as in (B1)-(B4) then the attractor Ω of $\mathcal{S}$ satisfies that*

$$\dim_B \Omega = \dim_H \Omega = \min\{1, s\}, \tag{2.4}$$

where s is the unique solution of

$$\sum_{i=0}^{N-1} \lambda_i^s + \sum_{i\in\mathcal{J}} \beta_i^s - \sum_{i\in\mathcal{J}} \lambda_i^s \beta_i^s = 1, \tag{2.5}$$

for Lebesgue almost every $\underline{\beta}$ in

$$\left\{\underline{\beta} : 0 < \beta_i < \min\left\{\lambda_i, \frac{2}{(1+\sqrt{2})(\alpha_i^2 \lambda_{\max} + 2)}\right\}\right\}, \tag{2.6}$$

where $\lambda_{\max} = \max_i \{\lambda_i\}$ and

$$\alpha_i = \frac{\max\{a_{N-1} - a_i, a_i - a_0\}}{\min\{f_{i+1}(a_0) - a_i, a_i - f_{i-1}(a_{n-1})\}} \text{ for every } i \in \mathcal{I}.$$

Moreover $\mathcal{L}(\Omega) > 0$ for Lebesgue almost every $\underline{\beta}$ such that $\underline{\beta}$ satisfies (2.6) and $s > 1$.

On the other hand, we examine the s-dimensional Hausdorff measure of the attractor Ω. It turns out that for every parameter, the measure is zero. This fact implies that s in (2.5) is always an upper bound for the Hausdorff and Box dimension.

Theorem 2.3. *Assume that $\mathcal{S}$ satisfies (B1)-(B4) and let s be the unique solution of (2.5) then*

$$\mathcal{H}^s(\Omega) = 0.$$

To prove Theorem 2.1 and Theorem 2.2, we have used the so-called transversality method. Note, that our original system does not satisfy the transversality condition, but some well-chosen subsystems of the sufficiently high iterations do so. To verify this we have used two methods of checking the transversality condition. One of them was introduced by Simon, Solomyak and Urbański [SSU1], [SSU2] and the other one is due to [PeSo1], [PeSo2].

The method of the proof of Theorem 2.3 is similar to that of [PSS2, Theorem 1.1] obtained by a modification of the Brandt, Graf method [BG].

The results are based on [B1] and [B2].

3 Non-conformal sets

3.1 Introduction for non-conformal sets

In the last two decades considerable attention has been paid to the dimension theory of non-conformal sets. We call a set Λ *conformal* if it is an attractor of an IFS containing $C^{1+\alpha}$ conformal homeomorphisms, where we call a function conformal if its derivative is a similarity transformation at every point. The dimension theory of conformal attractors is very closely related to the dimension theory of self-similar sets.

The dimension theory of non-conformal Iterated Function Systems is very difficult and there are only very few results. The most important tool of this field is *the sub-additive pressure*, which was defined by K. Falconer [Fa4] and L. Barreira [Barr]. Unfortunately, we know very little about sub-additive pressure itself.

The simplest non-conformal situation is the case of self-affine sets. A set $\Lambda \subset \mathbb{R}^d$ is called self-affine if it is an attractor of an IFS containing contracting affine maps $\{f_i(x) = A_i x + a_i\}_{i=1}^m$, where A_i are $d \times d$ real matrices. The dimension theory of self-affine sets is far from well understood even in the diagonal case. That is, when all A_i are diagonal matrices.

To study the dimension of a self-affine attractor we consider the k-th approximation of the attractor with the so called k-th cylinders which are naturally defined by the k fold application of the functions of the IFS. To measure the contribution of such a k cylinder to the covering sum which appears in the definition of the Hausdorff measure for each of these k-th cylinders we consider the singular value function. These are non-negative valued functions defined in a neighborhood of the attractor. The dimension of the attractor is related to the exponential growth rate of the sum of the values of these exponentially many singular value functions in the self affine case. Precisely, the Falconer Theorem (see [Fa6]) states that the Hausdorff- and box dimension of a self-affine attractor coincide for almost every translation parameters and equal to the singularity dimension, whenever the norm of all the affine maps of IFS is smaller than 1/3. This bound was improved to 1/2 by Solomyak in [So1]. To verify this it was essential that the exponential growth rate is the same wherever we evaluate these singular value functions, since the singular value functions are constant in the self-affine case.

Falconer [Fa4] and Barreira [Barr] considered the situation when the IFS is no longer self-affine. They introduced a technical condition named 1-bunched property, which implies that the cylinder sets in each iteration are convex. In this case, it turns out that the exponential growth rate of the sum of the value of the singular value functions does not depend on wherever they are evaluated. We express this phenomenon as the "insensitivity property holds". This is a very important property of the sub-additive pressure and in general we do not know if it holds or not.

Even if the 1-bunched condition is not satisfied, Zhang [Zh] found that the zero of the sub-additive pressure is an upper bound for the Hausdorff dimension.

3.2 Sub-additive pressure of Iterated Function Systems with triangular maps

The main result of this section is to verify the insensitivity property in a special case when the 1-bunched property does not hold but the IFS consists of maps with lower triangular derivative matrices. This result is a generalization of the result of K. Simon and A. Manning [MS2]. They proved the same assertion on the real plane.

Let $M \subset \mathbb{R}^n$ be a non-empty, open and bounded set, and let $F_i : M \mapsto M$ contractive maps for every $i = 1, ..., l$. For an $\mathbf{i} = i_1 i_2 ... i_k$, $i_j \in \{1, ..., l\}$, we write $F_{\mathbf{i}}(\underline{x}) = F_{i_1} \circ F_{i_2} \circ ... \circ F_{i_k}(\underline{x})$. Our principal assumption about the maps F_i, $i = 1, ..., l$ is that

$$F_i(x_1, ..., x_n) = \left(f_i^1(x_1), f_i^2(x_1, x_2), ..., f_i^n(x_1, ..., x_n) \right), \tag{3.1}$$

and $F_i(x_1, ..., x_n) \in C^{1+\varepsilon}(\overline{M})$ for every $i = 1, ..., l$. Moreover we require that $D_{\underline{x}} F_i$ is a regular (non-singular matrix) for every $\underline{x} \in \overline{M}$ and every $i \in \{1, ..., l\}$. Denote the elements of $D_{\underline{x}} F_{\mathbf{i}}$ by $x_{ij}(\mathbf{i}, \underline{x})$.

The singular values of a linear contraction T are the positive square roots of the eigenvalues of TT^*, where T^* is the transpose of T. Let $\alpha_k(D_{\underline{x}} F_{\mathbf{i}})$ be the k-th greatest singular value of the matrix $D_{\underline{x}} F_{\mathbf{i}}$. The singular value function ϕ^s is defined for $0 \leq s \leq n$ as

$$\phi^s(D_{\underline{x}} F_{\mathbf{i}}) := \alpha_1(D_{\underline{x}} F_{\mathbf{i}}) ... \alpha_{k-1}(D_{\underline{x}} F_{\mathbf{i}}) \alpha_k(D_{\underline{x}} F_{\mathbf{i}})^{s-k+1} \tag{3.2}$$

where $k - 1 < s \leq k$ and k is a positive integer. We define the maximum and the minimum of the singular value function as

$$\overline{\phi}^s(\mathbf{i}) := \max_{\underline{x} \in \overline{M}} \phi^s(D_{\underline{x}} F_{\mathbf{i}}) \ , \ \underline{\phi}^s(\mathbf{i}) := \min_{\underline{x} \in \overline{M}} \phi^s(D_{\underline{x}} F_{\mathbf{i}}).$$

We define the sub-additive pressure after K. Falconer [Fa4] and L. Barreira [Barr]:

$$P(s) := \lim_{k \to \infty} \frac{1}{k} \log \sum_{|\mathbf{i}|=k} \overline{\phi}^s(\mathbf{i}) \tag{3.3}$$

and define the lower pressure:

$$\underline{P}(s) := \liminf_{k\to\infty} \frac{1}{k} \log \sum_{|\mathbf{i}|=k} \underline{\phi}^s(\mathbf{i}). \tag{3.4}$$

Theorem 3.1. *For $0 \le s \le n$. If $F_1, ..., F_l$ contractive maps in form (3.1) and $F_i \in C^{1+\varepsilon}$ for every $1 \le i \le l$ then*

$$P(s) = \underline{P}(s) = \lim_{r\to\infty} \frac{1}{r} \log(\max_{\substack{j_1,\dots,j_{m-1}\\ j'_1,\dots,j'_m}} \sum_{|\mathbf{i}|=r} \left(\left|x_{j_1 j_1}(\mathbf{i},\underline{x})\right| \dots \left|x_{j_{m-1}j_{m-1}}(\mathbf{i},\underline{x})\right|\right)^{m-s} \times \\ \times \left(\left|x_{j'_1 j'_1}(\mathbf{i},\underline{x})\right| \dots \left|x_{j'_m j'_m}(\mathbf{i},\underline{x})\right|\right)^{s-m+1}) \tag{3.5}$$

for every $\underline{x} \in M$.

The formula (3.5) shows us that the sub-additive pressure depends only on the diagonal elements of the derivative matrices in the case when the derivative matrices are triangular. This result can be also considered as a generalization of a recent paper by K. Falconer and J. Miao [FM]. They gave a formula to estimate the Hausdorff dimension of self-affine fractals generated by upper-triangular matrices.

The result Theorem 3.1 is based on [B4] which uses the technique of [FM]. The result of the chapter was part of author's Master Thesis.

3.3 Box Dimension of the generalized 4-corner set

We consider a special family of self-affine sets which is called the generalized four corner set $\Lambda(\underline{\alpha},\underline{\beta})$ on the real plane. The generalized 4-corner set is the attractor of the self-affine iterated function system (IFS) of Figure 4.

Precisely, let $\Psi = \{f_0(\underline{x}), f_1(\underline{x}), f_2(\underline{x}), f_3(\underline{x})\}$ be an iterated function system on the real plane and $\Lambda(\underline{\alpha},\underline{\beta})$ its attractor, where

$$\begin{aligned} f_0(\underline{x}) &= \begin{pmatrix} \alpha_0 & 0 \\ 0 & \beta_0 \end{pmatrix} \underline{x}, \\ f_1(\underline{x}) &= \begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix} \underline{x} + \begin{pmatrix} 0 \\ 1-\beta_1 \end{pmatrix}, \\ f_2(\underline{x}) &= \begin{pmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{pmatrix} \underline{x} + \begin{pmatrix} 1-\alpha_2 \\ 0 \end{pmatrix}, \\ f_3(\underline{x}) &= \begin{pmatrix} \alpha_3 & 0 \\ 0 & \beta_3 \end{pmatrix} \underline{x} + \begin{pmatrix} 1-\alpha_3 \\ 1-\beta_3 \end{pmatrix}. \end{aligned} \tag{3.6}$$

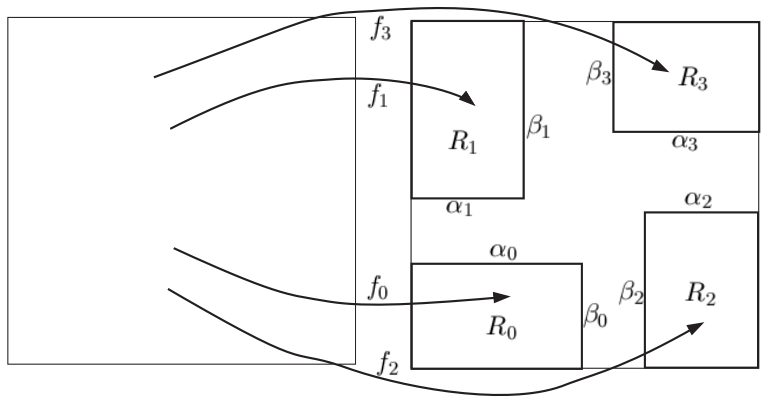

Figure 4: Maps of the generalized 4-corner set.

The parameters $\underline{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ and $\underline{\beta} = (\beta_0, \beta_1, \beta_2, \beta_3)$ are chosen such that the rectangles R_0, R_1, R_2, R_3 on Figure 4 are disjoint. Our goal is to determine the box dimension of this set for Lebesgue typical parameters.

Before we compute the box dimension of the generalized 4-corner set, we state a general theorem on the box dimension of diagonally self-affine sets.

Let

$$f_i(x, y) = (\alpha_i x + t_i, \beta_i y + u_i) \tag{3.7}$$

for $i = 0, \ldots, m$ such that

$$\begin{aligned} & 0 < \alpha_i, \beta_i < 1 \\ & f_i([0,1]^2) \subseteq [0,1]^2 \text{ for } i = 0, \ldots, m \\ & f_i((0,1)^2) \bigcap f_j((0,1)^2) = \emptyset \text{ for } i \neq j. \end{aligned} \tag{3.8}$$

Denote the attractor of $\Psi = \{f_i(x, y)\}_{i=0}^m$ by $\mathbf{\Lambda}$ and define $\text{proj}_x\mathbf{\Lambda}$ (and $\text{proj}_y\mathbf{\Lambda}$) as the projection of $\mathbf{\Lambda}$ onto the x-axis (and y-axis, respectively).

Theorem 3.2. *Let f_i be in form (3.7) for $i = 0, \ldots, m$ and let us suppose that $\Psi = \{f_i(x, y)\}_{i=0}^m$ satisfies (3.8). Then the attractor $\mathbf{\Lambda}$ of Ψ satisfies*

$$\dim_B \mathbf{\Lambda} = \max \{d_\alpha, d_\beta\}$$

where d_α and d_β are the unique solutions of

$$\sum_{i=0}^m \alpha_i^{s_\alpha} \beta_i^{d_\alpha - s_\alpha} = 1 \text{ and } \sum_{i=0}^m \beta_i^{s_\beta} \alpha_i^{d_\beta - s_\beta} = 1,$$

where $s_\alpha = \dim_B \text{proj}_x\mathbf{\Lambda}$ and $s_\beta = \dim_B \text{proj}_y\mathbf{\Lambda}$.

The proof of Theorem 3.2 is based on [B1] which follows the method of Feng, Wang [FW, Theorem 1] and Barański [Bara, Theorem B] with slight modifications.

Using this and [SS, Theorem 2.1] we can compute the box dimension of the attractor at least for almost all translations such that (3.8) holds.

Corollary 3.3. *Let f_i be in form (3.7) for $i = 0, \ldots, m$ and let $\mathcal{T} \subset \mathbb{R}^{2m+2}$ be the set of translation vectors such that $\Psi = \{f_i(x,y)\}_{i=0}^{m}$ satisfies (3.8). Then the attractor $\mathbf{\Lambda}$ of Ψ satisfies*

$$\dim_B \mathbf{\Lambda} = \max\{d_\alpha, d_\beta\} \text{ for almost every translations in } \mathcal{T} \text{ with respect to } 2m+2\text{-dimensional Lebesgue measure}$$

where d_α and d_β are the unique solutions of

$$\sum_{i=0}^{m} \alpha_i^{\min\{1,s_\alpha\}} \beta_i^{d_\alpha - \min\{1,s_\alpha\}} = 1 \text{ and } \sum_{i=0}^{m} \beta_i^{\min\left\{1,s_\beta\right\}} \alpha_i^{d_\beta - \min\left\{1,s_\beta\right\}} = 1,$$

and s_α, s_β are the unique solutions of

$$\sum_{i=0}^{m} \alpha_i^{s_\alpha} = 1 \text{ and } \sum_{i=0}^{m} \beta_i^{s_\beta} = 1.$$

Now, using Theorem 3.2 and the earlier result of Section 2.2 we are able to calculate the box dimension of the generalized 4-corner set for almost every parameters.

Theorem 3.4. *Let $\Lambda(\underline{\alpha}, \underline{\beta})$ be the attractor of the self-affine IFS of Figure 4. Then*

$$\dim_B \Lambda(\underline{\alpha}, \underline{\beta}) = \max\{d_\alpha, d_\beta\}, \text{ for Lebesgue almost every } (\underline{\alpha}, \underline{\beta}) \text{ such that}$$
$$\max\{\alpha_i + \alpha_{i+2}, \beta_i + \beta_{i+2}\} < 1 \text{ and } \min\{\alpha_i + \alpha_{3-i}, \beta_i + \beta_{3-i}\} < 1 \text{ for } i = 0,1 \tag{3.9}$$

where d_α and d_β are defined in two steps. First we define two numbers s_α, s_β as the unique solutions of the equations

$$\alpha_0^{s_\alpha} + \alpha_1^{s_\alpha} + \alpha_2^{s_\alpha} + \alpha_3^{s_\alpha} - \alpha_0^{s_\alpha}\alpha_1^{s_\alpha} - \alpha_2^{s_\alpha}\alpha_3^{s_\alpha} = 1$$
$$\beta_0^{s_\beta} + \beta_1^{s_\beta} + \beta_2^{s_\beta} + \beta_3^{s_\beta} - \beta_0^{s_\beta}\beta_2^{s_\beta} - \beta_1^{s_\beta}\beta_3^{s_\beta} = 1.$$

Then we can define d_α and d_β as the unique real numbers such that

$$\sum_{i=0}^{3} \alpha_i^{\min\{1,s_\alpha\}} \beta_i^{d_\alpha - \min\{1,s_\alpha\}} = 1, \quad \sum_{i=0}^{3} \beta_i^{\min\left\{1,s_\beta\right\}} \alpha_i^{d_\beta - \min\left\{1,s_\beta\right\}} = 1. \tag{3.10}$$

4 Dimension Theory of the intersections of the Sierpiński Gasket and lines with rational slope

Denote by $\Delta \subset \mathbb{R}^2$ the usual Sierpiński gasket, that is, Δ is the unique non-empty compact set satisfying

$$\Delta = S_0(\Delta) \cup S_1(\Delta) \cup S_2(\Delta),$$

where

$$S_0(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y\right),\ S_1(x,y) = \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y\right),\ S_2(x,y) = \left(\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{\sqrt{3}}{4}\right). \tag{4.1}$$

It is well known that $\dim_H \Delta = \dim_B \Delta = \frac{\log 3}{\log 2} = s$.

Let us denote by proj_θ the projection onto the line through the origin making angle θ with the x-axis. For $a \in \mathrm{proj}_\theta(\Delta)$ we let

$$L_{\theta,a} = \{(x,y) \,:\, \mathrm{proj}_\theta(x,y) = a\} = \{(x, a + x\tan\theta) \,:\, x \in \mathbb{R}\}.$$

Our goal is to analyze the dimension theory of the slices $E_{\theta,a} = L_{\theta,a} \cap \Delta$. In particular, we describe the multifractal analysis of the size of the slices which correspond to a countable dense set of angles.

Since Δ is rotation and reflection invariant, without loss of generality we may assume that $\theta \in [0, \frac{\pi}{3})$.

Denote by ν the natural self-similar measure of Δ. That is, $\nu = \frac{\mathcal{H}^s|_\Delta}{\mathcal{H}^s(\Delta)}$. In this case, ν satisfies that

$$\nu = \sum_{i=0}^{2} \frac{1}{3} \nu \circ S_i^{-1}.$$

Denote by ν_θ the projection of ν by angle θ. That is, $\nu_\theta = \nu \circ \mathrm{proj}_\theta^{-1}$. Similarly, let Δ_θ be the projection of Δ.

Let us define the (upper and lower) local dimension of a measure η at the point x by

$$\underline{d}_\eta(x) = \liminf_{r\to 0} \frac{\log \eta(B_r(x))}{\log r},\ \overline{d}_\eta(x) = \limsup_{r\to 0} \frac{\log \eta(B_r(x))}{\log r}.$$

Our first result, Proposition 4.1 implies that a dimension conservation principle holds, which connecting the local dimension of the projected natural measure and the box dimension of the slices. Manning and Simon proved such dimension conservation phenomena for the Sierpiński carpet, (see [MS1, Proposition 4]).

Proposition 4.1. *For every $\theta \in (0, \frac{\pi}{3})$ and $a \in \Delta_\theta$*

$$\underline{d}_{\nu_\theta}(a) + \overline{\dim}_B E_{\theta,a} = s, \tag{4.2}$$

$$\overline{d}_{\nu_\theta}(a) + \underline{\dim}_B E_{\theta,a} = s. \tag{4.3}$$

Applying Proposition 4.1 and the results of Feng and Hu [FH, Theorem 2.12] and Young [You] one can easily deduce.

Corollary 4.2. *For every $\theta \in (0, \frac{\pi}{3})$ and ν_θ-almost every $a \in \Delta_\theta$ we have*

$$\dim_B E_{\theta,a} = s - \dim_H \nu_\theta \geq s - 1,$$

where $\dim_H \nu_\theta$ denotes the Hausdorff dimension of the measure ν_θ.

Liu, Xi and Zhao showed a formula to estimate the box and Hausdorff dimension of the intersections of the Sierpiński carpet with Lebesgue-typical planar lines of rational slopes and conjectured that this value is strictly less than the dimension of the Sierpiński carpet minus one (for precise details see [LXZ]). Manning and Simon verified the conjecture in [MS1]. Our second statement is that the theorems is valid for the Sierpiński gasket.

Furthermore, Theorem 4.3 implies that whenever $\tan\theta = \frac{\sqrt{3}p}{2q+p}$ for positive integers p, q, the direction θ is exceptional in Marstrand's Theorem (see [Mar1] or [Mat, Theorem 10.11]).

Theorem 4.3. *Let $p, q \in \mathbb{N}$ and let us suppose that $\tan\theta = \frac{\sqrt{3}p}{2q+p}$ and $\theta \in (0, \frac{\pi}{3})$. Then there exist constants $\alpha(\theta), \beta(\theta)$ depending only on θ such that*

1. *for Lebesgue almost all $a \in \Delta_\theta$*

$$\alpha(\theta) := \dim_B E_{\theta,a} = \dim_H E_{\theta,a} < s - 1,$$

2. *for ν_θ-almost all $a \in \Delta_\theta$*

$$\beta(\theta) := \dim_B E_{\theta,a} = \dim_H E_{\theta,a} > s - 1.$$

A simple calculation reveals that the tangent of the set of angles in this theorem is equal to $\mathbb{Q}' = \left\{0 < \sqrt{3}\frac{m}{n} < \sqrt{3} : \text{ if } m \text{ is odd then } n \text{ is odd } \right\}$.

In [Fur], Furstenberg introduced and proved a dimension conservation formula [Fur, Definition 1.1] for homogeneous fractals (for example self-similar sets with IFS containing only homothetic similarities). As a consequence of Theorem 4.3(2) and Corollary 4.2 we state the special case of Furstenberg dimension conservation formula for the Sierpiński gasket and rational slopes. By [Fur, Theorem 6.2], the formula is valid for arbitrary angles.

Corollary 4.4 (Furstenberg)**.** *Let us fix $p, q \in \mathbb{N}$ and let us suppose that $\tan\theta = \frac{\sqrt{3}p}{2q+p}$ and $\theta \in (0, \frac{\pi}{3})$. Then the* proj_θ *satisfies the dimension conservation formula [Fur, Definition 1.1] by $\beta(\theta)$. Precisely,*

$$\beta(\theta) + \dim_H \{a \in \Delta_\theta : \dim_H E_{\theta,a} \geq \beta(\theta)\} = s. \tag{4.4}$$

The result of Corollary 4.4 is also valid with the formula

$$\beta(\theta) + \dim_H \{a \in \Delta_\theta : \dim_H E_{\theta,a} = \beta(\theta)\} = s.$$

Now, we would like to analyze the behavior of the function $\Gamma : \delta \mapsto \dim_H \{a \in \Delta_\theta : \dim_H E_{\theta,a} \geq \delta\}$ in the case when $\tan\theta = \frac{\sqrt{3}p}{2q+p}$, where $p, q \in \mathbb{N}$ and $(p, q) = 1$. For the analysis we use two matrices generated naturally by the projection and the IFS $\{S_0, S_1, S_2\}$. For the simplicity, we illustrate these matrices for the right-angle gasket. More precisely, the so-called right-angle Sierpiński gasket Λ is the attractor of iterated function system

$$\Phi = \left\{ F_0(x,y) = \left(\frac{x}{2}, \frac{y}{2}\right),\ F_1(x,y) = \left(\frac{x}{2} + \frac{1}{2}, \frac{y}{2}\right),\ F_2(x,y) = \left(\frac{x}{2}, \frac{y}{2} + \frac{1}{2}\right) \right\}. \tag{4.5}$$

There is a linear transformation T

$$T = \begin{pmatrix} 1 & -\frac{\sqrt{3}}{3} \\ 0 & \frac{2\sqrt{3}}{3} \end{pmatrix} \tag{4.6}$$

which maps the Sierpiński gasket into the right-angle Sierpińsi gasket. Since an invertible linear transformation does not change the dimension of any set we state our results for the usual Sierpiński gasket and for appropriate slopes.

Denote the angle θ projection of Λ to the y-axis by Λ_θ. Then $\Lambda_\theta = [-\tan\theta, 1]$. Moreover, let us consider the projected IFS of Φ. Namely, let

$$\phi = \left\{ f_0(t) = \frac{t}{2}, f_1(t) = \frac{t}{2} + \frac{1}{2}, f_2(t) = \frac{t}{2} - \frac{p}{2q} \right\}.$$

Let us divide Λ_θ into $p+q$ equal intervals such that $I_k = \left[1 - \frac{k}{q}, 1 - \frac{k-1}{q}\right]$ for $k = 1, \ldots, p+q$. Moreover, let us divide I_k for every k into two equal parts. Namely, let $I_k^0 = \left[1 - \frac{k}{q}, 1 - \frac{2k-1}{2q}\right]$ and $I_k^1 = \left[1 - \frac{2k-1}{2q}, 1 - \frac{k-1}{q}\right]$. Let us define the $(p+q) \times (p+q)$ matrices A_0, A_1 in the following way:

$$(A_n)_{i,j} = \sharp \{k \in \{0,1,2\} : f_k(I_j) = I_i^n\}. \tag{4.7}$$

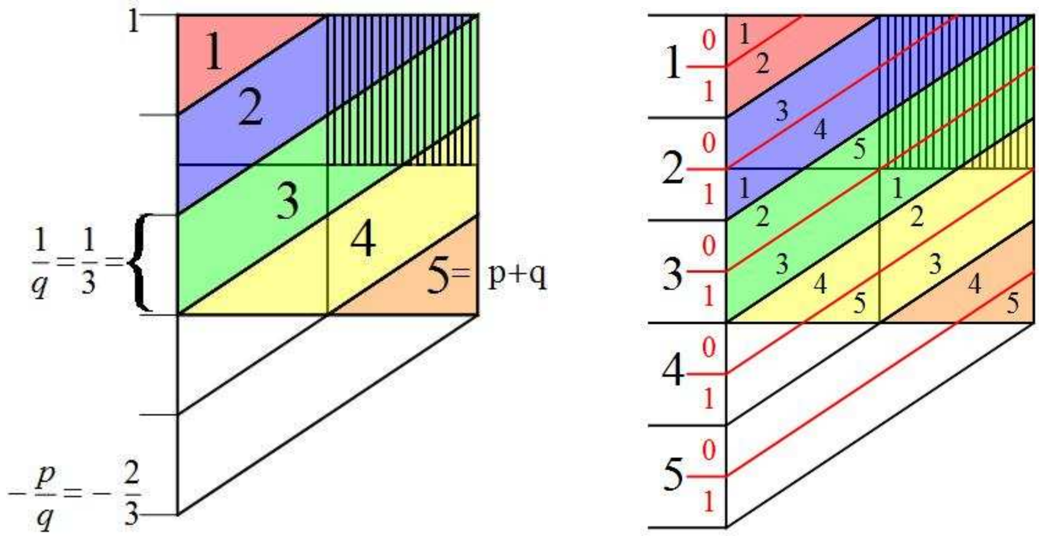

Figure 5: Graph of the projection and construction of matrices A_0, A_1 in the case $\frac{p}{q} = \frac{2}{3}$.

For example, see the case $\frac{p}{q} = \frac{2}{3}$ of the construction in Figure 5 and the matrices are

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proposition 4.5. *Let $p, q \in \mathbb{N}$ and let us suppose that $\tan\theta = \frac{\sqrt{3}p}{2q+p}$ and $\theta \in (0, \frac{\pi}{3})$. Moreover, let $\alpha(\theta)$ and $\beta(\theta)$ be as in Theorem 4.3. Then*

$$\alpha(\theta) = \frac{1}{\log 2} \lim_{n\to\infty} \frac{1}{n} \sum_{\xi_1,\ldots,\xi_n=0}^{1} \frac{1}{2^n} \log \underline{e} A_{\xi_1} \cdots A_{\xi_n} \underline{e},$$

$$\beta(\theta) = \frac{1}{\log 2} \lim_{n\to\infty} \frac{1}{n} \sum_{\xi_1,\ldots,\xi_n=0}^{1} \frac{1}{3^n} \underline{e} A_{\xi_1} \cdots A_{\xi_n} \underline{p} \log \left(\underline{e} A_{\xi_1} \cdots A_{\xi_n} \underline{p} \right),$$

where $\underline{e} = (1, \cdots, 1)$ and $\underline{p}$ is the unique probability vector such that $(A_0 + A_1)\,\underline{p} = 3\,\underline{p}$.

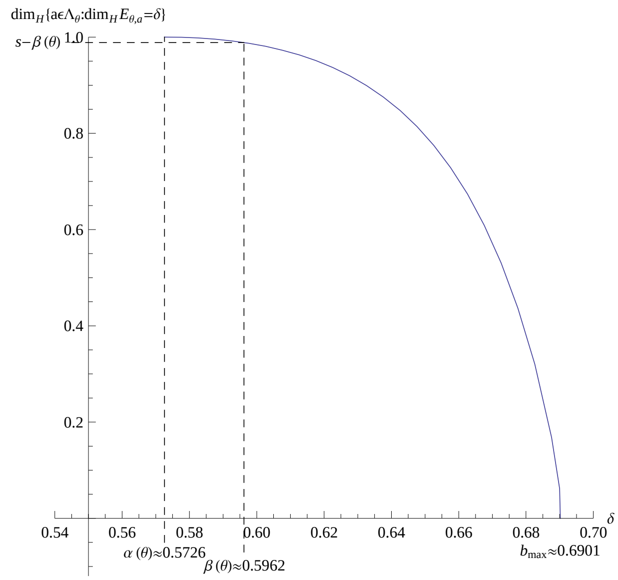

Figure 6: The graph of the function $\delta \mapsto \dim_H \{a \in \Delta_\theta : \dim_H E_{\theta,a} = \delta\}$ of the case $\frac{p}{q} = 1$.

In order to obtain further information on the nature of the function $\Gamma: \ \delta \mapsto \dim_H \{a \in \Delta_\theta : \dim_H E_{\theta,a} \geq \delta\}$ we will employ the theory of multifractal analysis for products of non-negative matrices [Fe1, Fe2, FL2]. Let $P(t)$ denote the pressure function which is defined as

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\xi_1,\ldots,\xi_n=0}^{1} (\underline{e} A_{\xi_1} \cdots A_{\xi_n} \underline{e})^t \tag{4.8}$$

and let us define

$$b_{\max} = \lim_{t\to\infty} \frac{P(t)}{t}.$$

Theorem 4.6. *Let $p, q \in \mathbb{N}$ and let us suppose that $\tan\theta = \frac{\sqrt{3}p}{2q+p}$ and $\theta \in (0, \frac{\pi}{3})$. Then*

1. $\Gamma(\delta) = \dim_H \{a \in \Delta_\theta : \dim_H E_{\theta,a} \geq \delta\} = \inf_{t>0} \left\{-\delta t + \frac{P(t)}{\log 2}\right\}$ *if* $b_{\max} \geq \delta > \alpha(\theta)$ *and* $\Gamma(\delta) = 1$ *if* $\delta \leq \alpha(\theta)$*. The function* Γ *is decreasing and continuous.*

2. $\chi(\delta) = \dim_H \{a \in \Delta_\theta : \dim_H E_{\theta,a} = \delta\} = \inf_{t>0} \left\{-\delta t + \frac{P(t)}{\log 2}\right\}$ *for every* $b_{\max} \geq \delta \geq \alpha(\theta)$*. The function* χ *is decreasing and continuous.*

For an example of the function $\delta \mapsto \dim_H \{a \in \Delta_\theta : \dim_H E_{\theta,a} = \delta\}$ with $\tan\theta = \frac{\sqrt{3}}{3}$ in the usual Sierpiński gasket case, see Figure 6.

The results are based on [BFS] which is a joint work with Andrew Ferguson and Károly Simon.

5 The absolute continuity of the invariant measure of Random Iterated Function Systems

Finally, we investigate some properties of the invariant measure of iterated function systems with random perturbations. Precisely, we are interested in studying absolute continuity with L^2 density.

Let $\{f_1, \dots, f_l\}$ be an iterated function system (IFS) on the real line. Suppose that for each $i \in \{1, \dots, l\}$, f_i maps $[-1,1)$ into itself, such that $f_i([-1,1))$ is bounded away from -1 and 1, $f_i \in C^{1+\alpha}([-1,1))$ and

$$0 < \lambda_{i,\min} \le |f_i'(x)| \le \lambda_{i,\max} < 1 \tag{5.1}$$

for every $x \in [-1,1)$. Moreover let us assume that for every i the fixed point of f_i is $a_i \in (-1,1)$, and

$$i \neq j \ \Rightarrow\ a_i \neq a_j. \tag{5.2}$$

For the IFS $\{f_i\}_{i=1}^n$ the natural coding of the elements of its attractor Λ by the elements of $\Sigma = \{1, \dots, n\}^{\mathbb{N}}$ is called the natural projection π and then $\pi : \Sigma \mapsto \Lambda$. Let $\mu = (p_1, \dots, p_n)^{\mathbb{N}}$ be a Bernoulli measure on the space Σ. Let $h = -\sum_{i=1}^n p_i \log p_i$ be the entropy of the left-shift operator with respect to the Bernoulli measure μ. Denote by ν the push-down measure of μ, that is $\nu = \mu \circ \pi^{-1}$. Then ν satisfies

$$\nu = \sum_{i=1}^{l} p_i \nu \circ f_i^{-1}. \tag{5.3}$$

It was proved in [BNS], for non-linear, contracting on average, iterated function systems (and later extended in [FST]) that

$$\dim_H(\nu) \le \frac{h}{|\chi|},$$

where $\dim_{\mathrm{H}}(\nu)$ is the Hausdorff dimension of the measure ν and χ is the Lyapunov exponent of the IFS associated to the Bernoulli measure μ.

One can expect that, at least "typically", the measure ν is absolutely continuous when $h/|\chi| > 1$. Essentially the only known approach to this is transversality. For example, in the linear case with uniform contraction ratios, see [PeSc] and [PeSo2]. In the linear case for non-uniform contraction ratios, see [N] and [NW2]. In the non-linear case, see for example [SSU2]. We note that there is another direction in the study of iterated function systems with overlaps, which is concerned with concrete, but not-typical systems, often of arithmetic nature, for which there is a dimension drop, see for example [LNR].

In this section, we consider a random perturbation of the functions. The linear case was studied by Peres, Simon and Solomyak in [PSS1]. They proved absolute continuity for random linear IFS, with non-uniform contraction ratios and also L^2 and continuous density in the uniform case. We would like to extend this result by proving L^2 density with non-uniform contraction ratios and in non-linear case.

Let Y_ε be uniformly distributed on $[1-\varepsilon, 1+\varepsilon]$. Denote the probability measure of Y_ε by η_ε. Let

$$f_{i,Y_\varepsilon}(x) = Y_\varepsilon f_i(x) + a_i(1 - Y_\varepsilon) \tag{5.4}$$

for every $i \in \{1, \ldots, l\}$. Then $f_{i,Y_\varepsilon}(x)$ is in $[-1, 1)$ for all values of $x \in [-1, 1)$ and Y_ε, provided ε is sufficiently small. The iterated maps are applied randomly according to the stationary measure μ, with the sequence of independent and identically distributed errors $y_1, y_2, \ldots$, distributed as Y_ε, independent of the choice of the function. The Lyapunov exponent of the IFS is defined by

$$\chi(\mu, \eta_\varepsilon) = \mathbb{E}(\log(Y_\varepsilon f'))$$

and it is easy to see that

$$\chi(\mu, \eta_\varepsilon) < \sum_{i=1}^{l} p_i \log((1+\varepsilon)\lambda_{i,\max}) < 0,$$

for sufficiently small $\varepsilon > 0$. Let Z_ε be the following random variable

$$Z_\varepsilon := \lim_{n\to\infty} f_{i_1,y_{1,\varepsilon}} \circ f_{i_2,y_{2,\varepsilon}} \circ \cdots \circ f_{i_n,y_{n,\varepsilon}}(0), \tag{5.5}$$

where the numbers i_k are i.i.d., with the distribution μ on $\{1, \ldots, l\}$, and $y_{k,\varepsilon}$ are pairwise independent with distribution of Y_ε and also independent of the choice of i_k. Let ν_ε be the distribution of Z_ε.

One can easily prove the following theorem.

Theorem 5.1. *The measure ν_ε converges weakly to the measure ν as $\varepsilon \to 0$, see (5.3).*

Theorem 5.2. *Let ν_ε be the distribution of the limit (5.5). We assume that (5.1) and (5.2) hold, and*

$$\sum_{i=1}^{l} p_i^2 \frac{\lambda_{i,\max}}{\lambda_{i,\min}^2} < 1. \tag{5.6}$$

Then for every sufficiently small $\varepsilon > 0$, we have that ν_ε is absolutely continuous with respect to the Lebesgue measure, with density in L^2, and there exists a constant C such that the density of ν_ε satisfies

$$\|\nu_\varepsilon\|_2 \leq \frac{C}{\sqrt{\varepsilon}}.$$

We can state an easy corollary of the theorem.

Corollary 5.3. *Let $\{\lambda_i Y_\varepsilon x + a_i(1-\lambda_i Y_\varepsilon)\}_{i=1}^{l}$ be a random iterated function system. We assume that (5.2) holds, and*

$$\sum_{i=1}^{l} \frac{p_i^2}{\lambda_i} < 1. \tag{5.7}$$

Then for every sufficiently small $\varepsilon > 0$, we have that ν_ε is absolutely continuous with respect to the Lebesgue measure with density in L^2, and there exists a constant C such that

$$\|\nu_\varepsilon\|_2 \leq \frac{C}{\sqrt{\varepsilon}}.$$

We study another case of random perturbation, namely let $\widetilde{\lambda}_{i,\varepsilon}$ be uniformly distributed on $[\lambda_i - \varepsilon, \lambda_i + \varepsilon]$. Let $\left\{\widetilde{\lambda}_{i,\varepsilon} x + a_i(1-\widetilde{\lambda}_{i,\varepsilon})\right\}_{i=1}^{l}$ be our random iterated function system, where $a_i \neq a_j$ for every $i \neq j$. Let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_l)$, and $X_{\underline{\lambda},\varepsilon}$ be the following random variable

$$X_{\underline{\lambda},\varepsilon} = \sum_{k=1}^{\infty} (a_{i_k}(1-\widetilde{\lambda}_{i_k,\varepsilon})) \prod_{j=1}^{k-1} \widetilde{\lambda}_{i_j,\varepsilon} \tag{5.8}$$

where the numbers i_k are i.i.d., with the distribution μ on $\{1, \ldots, l\}$, and $\widetilde{\lambda}_{i_k,\varepsilon}$ are pairwise independent. Let $\nu_{\underline{\lambda},\varepsilon}$ denote the distribution of the random variable $X_{\underline{\lambda},\varepsilon}$. Moreover let $\nu_{\underline{\lambda}}$ be the invariant measure of the iterated function system $\{\lambda_i x + a_i(1-\lambda_i)\}_{i=1}^{l}$ according to μ.

Theorem 5.4. *The measure $\nu_{\underline{\lambda},\varepsilon}$ converges weakly to the measure $\nu_{\underline{\lambda}}$ as $\varepsilon \to 0$.*

To have a similar statement as in Theorem 5.2 we need a technical assumption, namely

$$\min_{i\neq j}\left|\frac{a_j\lambda_i - a_i\lambda_j}{\lambda_i-\lambda_j}\right| > 1. \tag{5.9}$$

Theorem 5.5. *Let us suppose that (5.9) and (5.2) hold, and moreover that*

$$\sum_{i=1}^{l}\frac{p_i^2}{\lambda_i} < 1. \tag{5.10}$$

Then for every sufficiently small $\varepsilon > 0$, the measure $\nu_{\underline{\lambda},\varepsilon}$ is absolutely continuous with respect to the Lebesgue measure, with density in L^2, and there exists a constant C such that

$$\|\nu_{\underline{\lambda},\varepsilon}\|_2 \leq \frac{C}{\sqrt{\varepsilon}}.$$

The main difference between Theorem 5.5 and Corollary 5.3 is the random perturbation. Namely, in Theorem 5.5 we choose the contraction ratio uniformly in the ε neighborhood of λ_i, but in Corollary 5.3 we choose the contraction ratio uniformly in the $\lambda_i\varepsilon$ neighborhood of λ_i.

The results are based on [BP] which is a joint work with Tomas Persson.

References

[Bara] K. Barański: Hausdorff dimension of the limit sets of some planar geometric constructions, *Advances in Mathematics* **210**, (2007), 215-245.

[B1] B. Bárány: Dimension of the generalized 4-corner set and its projections, *Erg. Th. & Dyn. Sys.*, (2011).

[B2] B. Bárány: Iterated Function Systems with Non-Distinct Fixed Points, *J. Math. Anal. Appl.* **383** No. 1 (2011), 244-258.

[B3] B. Bárány: On the Hausdorff Dimension of a Family of Self-Similar Sets with Complicated Overlaps, *Fund. Math.* **206** (2009), 49-59.

[B4] B. Bárány: Sub-Additive Pressure for IFS with Triangular Maps, *Bulletin of the Pol. Ac. of Sci. Math.* **57** No. 3 (2009), 263-278.

[BFS] B. Bárány, A. Ferguson, K. Simon: Slicing the Sierpiński gasket, preprint, (2011).

[BP] B. Bárány, T. Persson: The Absolute Continuity of the Invariant Measure of Random Iterated Function Systems with Overlaps, *Fund. Math.* **210** No. 1 (2010), 47-62.

[Barr] L. Barreira: A non-additive thermodinamic formalism and applications of dimension theory of hyperbolic dynamical systems. *Erg. Th. & Dyn. Sys.* **16**, (1996), 871-927.

[BG] C. Bandt, S. Graf: Self-Similar Sets 7. A Characterisation of Self-Similar Fractals with Positive Hausdorff Measure, *Proc. Amer. Math. Soc.* **114** No. 4, (1992), 995-1001.

[BNS] D. Broomhead, M. Nicol, N. Sidorov: On the fine structure of stationary measures in systems which contract on average, *J. Theoret. Probab.* **15** No. 3, (2002), 715-730.

[Fa1] K. J. Falconer: *Fractal Geometry: Mathematical Foundations and Applications*. Wiley, 1990.

[Fa2] K. J. Falconer: *Techniques in Fractal Geometry*, Wiley, 1997.

[Fa4] K. Falconer: Bounded distortion and dimension for non-conformal repellers, *Proc. Camb. Phil. Soc.*, (1994), 315-334.

[Fa5] K. J. Falconer: Dimensions and measures of quasi self-similar sets, *Proc. Amer. Math. Soc.* **106** No. 2, (1989), 543-554.

[Fa6] K. Falconer: The Hausdorff dimension of self-affine fractals, *Math. Proc. Camb. Phil. Soc.* **103** No. 2, (1988), 339-350.

[FM] K. Falconer, J. Miao: Dimensions of self-affine fractals and multifractals generated by upper-triangular matrices, *Fractals* **15** No. 3, (2007), 289-299.

[FST] A. H. Fan, K. Simon, H. Tóth: Contracting on average random IFS with repelling fixed point, *J. Statist. Phys.* **122**, (2006), 169-193.

[Fe1] D.J. Feng: Lyapunov exponents for products of matrices and multifractal analysis. Part I: Positive matrices, *Israel Journal of Mathematics* **138**, (2003), 353-376.

[Fe2] D.J. Feng: Lyapunov exponents for products of matrices and multifractal analysis. Part II: General matrices, *Israel Journal of Mathematics* **170**, (2009), 355-394.

[FH] D.J. Feng, H. Hu: Dimension theory of iterated function systems, *Comm. Pure Appl. Math.* **62** Issue 11, (2009), 1435-1500.

[FL2] D.J. Feng, K.S. Lau: The pressure function for products of non-negative matrices, *Mathematical Research Letters* **9**, (2002), 363-378.

[FW] D.-J. Feng, Y. Wang: A Class of Self-Affine Sets and Self-Affine Measures, *Journal of Fourier Analysis and Applications* **11** No. 1, (2005), 107-124.

[Fur] H. Furstenberg: Ergodic fractal measures and dimension conservation, *Erg. Th. & Dyn. Sys.* **28** No. 2, (2008), 405-422.

[H] J. E. Hutchinson: Fractals and Self-Similarity, *Indiana Univ. Math. Journal* **30** No. 5, (1981), 713-747.

[LNR] K.-S. Lau, S.-M. Ngai and H. Rao, Iterated function systems with overlaps and self-similar measures, *J. London Math. Soc.* **63** No. 2, (2001), no. 1, 99-116.

[LXZ] Q.H. Liu, L.F. Xi and Y.F. Zhao: Dimension of intersections of the Sierpiński carpet with lines of rational slopes, *Proceedings of the Edinburgh Mathematical Society* **50**, (2007), 411-428.

[MS1] A. Manning, K. Simon: Dimension of slices through the Sierpiński carpet, preprint, (2011).

[MS2] A. Manning, K. Simon: Subadditive pressure for triangular maps, *Nonlinearity* **20** No. 1, (2007), 133-149.

[Mar1] J.M. Marstrand: Some fundamental geometrical properties of plane sets of fractional dimension, *Proc. London Mathematical Society* **4**, (1954), 257-302.

[Mat] P. Mattila: *Geometry of sets and measures in Euclidean spaces*, Cambridge, 1995.

[N] J. Neunhäuserer: Properties of some overlapping self-similar and some self-affine measures, *Acta Math. Hungar.* **92** (2001), 143-161.

[NW1] S.-M. Ngai and Y. Wang: Hausdorff dimension of self-similar sets with overlaps, *J. London Math. Soc.* **63** No. 2, (2001), 655-672.

[NW2] S.-M. Ngai and Y. Wang: Self-similar measures associated with IFS with non-uniform contraction ratios, *Asian J. Math.* **9** No. 2, (2005), 227-244.

[PeSc] Y. Peres and W. Schlag: Smoothness of projections, Bernoulli convolutions, and the dimension of exceptions, *Duke Math. J.* **102** No. 2, (2000), 193-251.

[PSS1] Y. Peres, K. Simon and B. Solomyak: Absolute continuity for random iterated function systems with overlaps, *J. London Math. Soc.* **74** No. 2, (2006), 739-756.

[PSS2] Y. Peres, K. Simon and B Solomyak: Self-similar sets of zero Hausdorff and positive Packing measure, *Israel Journal of Mathematics* **117**, (2000), 353-379.

[PeSo1] Y. Peres and B. Solomyak: Absolute continuity of Bernoulli convolutions, a simple proof, *Math. Research Letters* **3** No. 2, (1996), 231-239.

[PeSo2] Y. Peres and B. Solomyak: Self-similar measures and intersections of Cantor sets, *Trans. Amer. Math. Soc.* **350** No. 10, (1998), 4065-4087.

[PoSi] M. Pollicott and K. Simon: The Hausdorff dimension of λ-expansions with deleted digits, *Trans. Amer. Math. Soc.* **347** No. 3, (1995), 967-983.

[SS] K. Simon, B. Solomyak: On the dimension of self-similar sets, *Fractals* **10** No. 1, (2002), 59-65.

[SSU1] K. Simon, B. Solomyak and M. Urbański: Hausdorff dimension of limit sets for parabolic IFS with overlaps, *Pacific J. Math.* **201** No. 2, (2001), 441-478.

[SSU2] K. Simon, B. Solomyak, and M. Urbański: Invariant measures for parabolic IFS with overlaps and random continued fractions, *Trans. Amer. Math. Soc.* **353**, (2001), 5145-5164.

[So1] B. Solomyak: Measure and dimension for some fractal families, *Math. Proc. Camb. Phil. Soc.* **124** No. 3, (1998), 531-546.

[You] L.S. Young: Dimension entropy and Lyapunov exponents, *Erg. Th. & Dyn. Sys.* **2** No. 1, (1982), 109-124.

[Ze] M. W. Zerner: Weak separation properties for self-similar sets, *Proc. Amer. Math. Soc.* **124**, No. 11, (1996), 3529-3539.

[Zh] Y. Zhang: Dynamical upper bounds for Hausdorff dimension of invariant sets, *Erg. Th. & Dyn. Sys.* **17**, (1997), 739-756.