Preface

In this thesis we will discuss five loosely related topics about Borel ideals on countable sets.

First of all, we would like to motivate the reader a little bit. How can we measure the “smallness” of a subset of the natural numbers (which will be denoted by $\omega$)? The mathematical approach to this problem gave us the notion of ideal: A family of sets $I \subseteq \mathcal{P}(\omega) = \{X : X \subseteq \omega\}$ is an ideal on $\omega$ if it contains the finite sets, $\omega / \in I$, it is $\subseteq$-descending ($A \subseteq B \in I \Rightarrow A \in I$), and it is closed for the union operation ($A, B \in I \Rightarrow A \cup B \in I$). The following families are well-known classical examples of ideals on $\omega$: the Fréchet ideal of finite subsets of $\omega$ (Fin), the density zero ideal:

$$\mathcal{Z} = \left\{ A \subseteq \omega : \lim_{n \to \infty} \frac{|A \cap n|}{n} = 0 \right\},$$

and the summable ideal:

$$\mathcal{J}_{1/n} = \left\{ A \subseteq \omega : \sum_{n \in A} \frac{1}{n+1} < \infty \right\}.$$

The study of these ideals and their generalizations has become a central topic of infinite combinatorics and forcing theory in the past few years.

An ideal $I$ on $\omega$ is analytic if $I \subseteq \mathcal{P}(\omega) \approx 2^\omega$ is an analytic set in the usual product (Polish space) topology of the Cantor set. $I$ is a P-ideal if for each countable $C \subseteq I$ there is an $A \in I$ such that $C \subseteq^* A$ for each $C \in C$, where $\subseteq^*$ is the almost containing relation, i.e. $A \subseteq^* B$ iff $A \setminus B$ is finite. $I$ is tall if each infinite subset of $\omega$ contains an infinite element of $I$. Fin, $\mathcal{Z}$, and $\mathcal{J}_{1/n}$ are analytic P-ideals, moreover $\mathcal{Z}$ and $\mathcal{J}_{1/n}$ are tall as well, and it is easy to see that $\mathcal{J}_{1/n} \subsetneq \mathcal{Z}$.

Why analytic ideals? Because analyticity is one the most general but still useful notions of definability in descriptive set theory, and in forcing constructions it is much simpler to handle definable objects whose definitions are absolute.

Why P-ideals? Because P-ideals can be seen as an analogue of $\sigma$-ideals (for example the measure zero subsets of a measure space) for countable sets.

We will frequently use Solecki’s characterization theorem: Let $I$ be an ideal on $\omega$. Then $I$ is an analytic P-ideal if and only if

$$I = \text{Exh}(\vartheta) = \left\{ A \subseteq \omega : \lim_{n \to \infty} \vartheta(A \setminus n) = 0 \right\}$$

for some lower semicontinuous submeasure $\vartheta$ on $\omega$.

Let me mention some natural questions concerning tall Borel (or analytic) ideals:

- What is the smallest (infinite or sometimes uncountable) cardinality of a family $A \subseteq \mathcal{P}(\omega) \setminus I$ which is $\subseteq$-maximal respect to the property: $A \cap B \in I$ for each distinct $A, B \in A$? (This property will be called $I$-almost disjointness.)

- What is the smallest cardinality of a family $\mathcal{J} \subseteq I$ such that there is no $A \in \mathcal{J}$ which almost contains all elements of $\mathcal{J}$? (This cardinal will be denoted by $\text{add}^*(I)$.)
• What is the smallest cardinality of a family $C \subseteq I$ such that for all $X \in [\omega]^\omega = \{X \subseteq \omega : |X| = \omega\}$ there is an $A \in C$ such that $|X \cap A| = \omega$? (This cardinal will be denoted by $\text{cov}^*(I)$.)

• Which forcing notions destroy $J$ (i.e. $\Vdash_{P} \exists X \in [\omega]^\omega \forall A \in J \cap V \ |X \cap A| < \omega$)?

• What can we say about the cofinal type of the pre-ordered set $(J, \subseteq^*)$?

• Which ideals can be permuted into $J$?

Outline

In Chapter 1, we give a short summary of basic notions and results of method of forcing, descriptive set theory, absoluteness, and cardinal invariants of the continuum, in particular we present the Cichoń’s diagram and the method of Galois-Tukey connections.

In Chapter 2, we summarize basic notions and notations of the theory of ideals on $\omega$. We recall the useful characterizations of meager ideals, $F_\sigma$ ideals, and analytic $P$-ideals. We define four basic pre-orders on the family of ideals and discuss some important results related to the Katetov-order, such as characterization of forcing indestructibility of ideals. Finally, we give some examples of nice ideals on $\omega$.

In Chapter 3, we investigate $I$-almost-disjoint families and the generalized almost disjointness numbers $a(I)$ and $\bar{a}(I)$. We prove some inequalities between these and classical cardinal invariants, and discuss the method of proving inequalities of the form $\bar{a}(J) \leq a$. We investigate the existence of forcing-indestructible $J$-MAD families under CH. At last, answering a question of L. Soukup, we prove a result about certain forcing-indestructible extensions of $J$-AD families in generic extensions.

In Chapter 4, we investigate general bounding and dominating properties of forcing notions. We show some examples of Borel Galois-Tukey connections between relations associated to analytic $P$-ideals and classical ones. We show their consequences for inequalities between cardinal invariants and for implications between certain properties of forcing notions. At last, we show that the natural generalization of the unbounding and dominating numbers for meager ideals do not give new invariants.

In Chapter 5, we investigate possible generalizations and applications of a measure-theoretic covering theorem due to M. Elekes. We present some counterexamples, and give positive answer in the category case for a big class of ideals containing tall analytic $P$-ideals. Furthermore, we discuss the connection between covering properties and forcing indestructibility of ideals.

In Chapter 6, first we present Hechler’s famous theorem about cofinal subsets of $(\omega^\omega, \leq^*)$ and its known generalizations for the meager and the null ideal. Then we prove Hechler’s theorem for tall analytic $P$-ideals. At last, we show an application of these theorems noticed by L. Soukup.

In Chapter 7, we introduce a new way of associating cardinal invariants to ideals: we introduce the $\sqsubseteq$-intersection number of an ideal where $\sqsubseteq$ is a pre-orders on ideals.
These new invariants are natural generalizations of the pseudo-intersection number $p$. We give some analytic motivations for the considered problems connected to sequential properties of spaces of measures with weak* topology. We show the consistency of some inequalities of these new invariants and classical ones, and other related combinatorial questions. Furthermore, we discuss ideals generated by MAD families and towers, and we analyze related maximality properties of these families.

Acknowledgement

First of all, I am very grateful to my supervisor Lajos Soukup for his support and kind help to find my way in set theory, and for his patience when I obstinately insisted on my favourite topic and I did not work on other ones.

Besides, I would like to thank my colleagues at the department Lajos Rónyai, György Serény, András Simon, and Ferenc Wettl their support and advices.

I also thank my mother, my brother, and my girlfriend their support and their patience when I was quite uptight during the work on this thesis.

Budapest, November 2011

Barnabás Farkas
Contents

Preface ......................................................... iii
Outline ......................................................... iv
Acknowledgement ............................................. v

1 Preliminaries .............................................. 1
  1.1 Method of forcing ........................................ 1
  1.2 Descriptive set theory and absoluteness ................... 3
  1.3 Cardinal invariants of the continuum ....................... 4

2 Ideals on countable sets ................................ 9
  2.1 Basic properties ......................................... 9
  2.2 Ideals associated to submeasures .......................... 10
  2.3 Orders on ideals .......................................... 12
  2.4 Examples ................................................ 14

3 Idealized MADness .................................... 17
  3.1 The almost-disjointness numbers of ideals ................. 17
  3.2 Forcing-indestructible $\mathcal{J}$-MAD families .......... 22
  3.3 Forcing-indestructible extensions of $\mathcal{J}$-AD families 25

4 The extended Cichoń's diagram ...................... 29
  4.1 $\mathcal{R}$-bounding and $\mathcal{R}$-dominating ............... 29
  4.2 Star-invariants and their underlying relations .......... 30
  4.3 Idealized version of $\mathfrak{b}$ and $\mathfrak{d}$ .............. 39

5 Covering properties of ideals ...................... 41
  5.1 The $\mathcal{J}$-covering property .......................... 41
  5.2 Around the category case ................................ 44
  5.3 When the $\mathcal{J}$-covering property “strongly” fails .... 46
  5.4 $\mathcal{J}$-covering properties of ($\mathcal{P}(\omega), \mathcal{I}$) ...... 47

6 Generalizations of Hechler's theorem ........... 51
  6.1 Hechler's original theorem ................................ 51
  6.2 Hechler's theorem for $\mathcal{M}$ and $\mathcal{N}$ ............... 51
  6.3 Hechler's theorem for tall analytic $\mathcal{P}$-ideals ....... 54
  6.4 An application: coding with spectrum ................... 57
7 Generalizations of the pseudo-intersection number

7.1 Intersection numbers of ideals ........................................... 59
7.2 The convex Fréchet-Urysohn property ................................ 61
7.3 Consistency results ............................................................ 64
7.4 Permuting MAD families into ideals ................................. 68
Chapter 1

Preliminaries

1.1 Method of forcing

Forcing is the most useful and powerful method of proving relative consistency results, that are metatheorems of the form “If ZFC is consistent then so is ZFC + Φ” or simply “Con(ZFC) implies Con(ZFC + Φ)” where Φ is a sentence in the language of set theory. In this case we say that Φ is (relative) consistent with ZFC. The first and well-known example of using this method was Paul Cohen’s famous theorem from 1963: The negation of the Continuum Hypothesis (CH) is relative consistent with ZFC.\(^1\)

Theory of forcing has become a central topic of set theory in the past 50 years not only because of its applications in proving relative consistency result but also because it raised new types of problems.

We refer the reader to [35] and [33] for the basic theory of forcing and iterated forcing (see also [5] and [28]). In this section we list our notations and some properties of forcing notions that will be used later in the thesis.

Forcing notions will be denoted by \(P, Q, F, N\) etc. If \(p, q \in P\) then \(p \leq q\) means “\(p\) extends \(q\)” or “\(p\) is stronger than \(q\)” or “\(p\) has more information than \(q\).” We say that \(p\) and \(q\) are compatible (\(p \parallel q\)) if there is an \(r \leq p, q\), and they are incompatible (\(p \perp q\)) if they have no common extensions.

\(V\) always stands for a countable transitive model (c.t.m. in short) of (a large enough finite fragment of) ZFC. If \(P \in V\) is a forcing notion, then the class of \(P\)-names in \(V\) will be denoted by \(V^P\), we will write \(\dot{a}, \dot{x}, \dot{Y}\) etc. for elements of \(V^P\), and the extension of \(V\) by a \((V, P)\)-generic filter \(G \subseteq P\) will be denoted by \(V[G]\).

As it is usual, for simplifying our notations in the forcing language, if \(P \in V\) and \(x \in V\), then we will write \(x\) for its canonical \(P\)-name as well. Furthermore, it is also usual to write \(V^P \models \ldots\) and \(\Vdash_P \ldots\) instead of \(1_P \Vdash_P \ldots\) where \(1_P\) is the largest element of \(P\).

We recall the definition of nice names. If \(P\) is a forcing notion and \(S\) is an arbitrary set, then a nice \(P\)-name for a subset of \(S\) is a \(P\)-name \(\dot{X}\) of the form \(\dot{X} = \bigcup \{s \times A_s : s \in S\}\) where \(A_s \subseteq P\) is an antichain for each \(s \in S\). Clearly, in this case \(\Vdash_P \dot{X} \subseteq S\).

---

\(^1\)Using other methods (namely the constructible universe), in 1940 Gödel proved the relative consistency of CH (with ZFC). This result can also be proved by forcing.
Moreover, we know that if $\check{Y}$ is a $\mathbb{P}$-name, then there is a nice $\mathbb{P}$-name $\check{X}$ for a subset of $S$ such that $\Vdash_{\mathbb{P}} (\check{Y} \subseteq S \rightarrow \check{Y} = \check{X})$.

Let $\mathbb{P}$ be a forcing notion and $H \subseteq \mathbb{P}$. $H$ is $n$-linked if every $\leq n$ of its elements are compatible. $H$ is linked if it is 2-linked. $H$ is centered if it is $n$-linked for each $n$.

$\mathbb{P}$ has the countable chain condition (ccc, in short) if all antichains $A \subseteq \mathbb{P}$ are countable. $\mathbb{P}$ is $\sigma$-$n$-linked (resp. $\sigma$-centered) if it is a countable union of $n$-linked (resp. centered) subsets.

Clearly, if $\mathbb{P}$ is $\sigma$-centered, then it is $\sigma$-$n$-linked for each $n$, and so it has the ccc. We know that if $\mathbb{P}$ has the ccc, then $\mathbb{P}$ does not collapse cardinals (i.e. if $\kappa$ is a cardinal, then $\Vdash_{\mathbb{P}} \kappa$ is a cardinal”).

Let $\mathcal{K}$ be a class of forcing notions. Martin’s Axiom for $\mathcal{K}$, MA($\mathcal{K}$) is the following statement: If $\mathbb{P} \in \mathcal{K}$ and $D$ is a family of dense subsets of $\mathbb{P}$ with cardinality less than $\mathfrak{c} = 2^\omega$, then there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for each $D \in D$.

Assume $\mathbb{P} = (B, \leq_P)$ is a subforcing of $\mathbb{Q} = (Q, \leq_Q)$, i.e. $P \subseteq Q$ and $\leq_P = \leq_Q | P$. Then we say that $\mathbb{P}$ is a complete subforcing of $\mathbb{Q}$ and write $\mathbb{P} \leq \mathbb{Q}$ if maximal antichains of $\mathbb{P}$ are maximal antichains in $\mathbb{Q}$ as well. In this case, if $G$ is a $(V, \mathbb{Q})$-generic filter, then $G \cap \mathbb{P}$ is a $(V, \mathbb{P})$-generic filter.

A forcing notion $\mathbb{P}$ is $\omega^\omega$-bounding if

$$\Vdash_{\mathbb{P}} \forall f \in \omega^\omega \cap V[\check{G}] \exists g \in \omega^\omega \cap V f \leq^* g$$

where $\check{G}$ is the canonical $\mathbb{P}$-name of the generic filter and $f \leq^* g$ iff $\{ n \in \omega : f(n) > g(n) \}$ is finite. $\mathbb{P}$ adds a dominating real if

$$\Vdash_{\mathbb{P}} \exists d \in \omega^\omega \cap V[\check{G}] \forall f \in \omega^\omega \cap V f \leq^* d.$$

Let us denote Slm the set of slaloms, that are functions of the form $S : \omega \rightarrow [\omega]^{<\omega}$ with $|S(n)| \leq n$ for each $n$. A forcing notion $\mathbb{P}$ has the Sacks property if

$$\Vdash_{\mathbb{P}} \forall f \in \omega^\omega \cap V[\check{G}] \exists S \in \text{Slm} \cap V \forall^\infty n \in \omega f(n) \in S(n)$$

where “$\forall^\infty n \in \omega$” stands for “for all but finitely many $n \in \omega$” (and similarly, “$\exists^\infty n \in \omega$” means “there are infinitely many $n \in \omega$”). If $\mathbb{P}$ has the Sacks property, then $\mathbb{P}$ is $\omega^\omega$-bounding because if $f \in \omega^\omega \cap V[G], S \in \text{Slm} \cap V$, and $\forall^\infty n \in \omega f(n) \in S(n)$, then $f \leq^* g$ where $g \in \omega^\omega \cap V, g(n) = \max(S(n))$.

The Cohen forcing. Let $\kappa$ be a cardinal. The Cohen forcing $\mathbb{C}_\kappa$ if the complete Boolean algebra $\text{Borel}(2^\kappa)/\mathcal{M}_\kappa$ (omitting its null element) where $\text{Borel}(2^\kappa)$ is the Boolean algebra of Borel subsets of $2^\kappa$, i.e. the $\kappa$th power of the discrete space $2 = \{0,1\}$, and $\mathcal{M}_\kappa$ is the ideal of (Borel) meager (i.e. countable union of nowhere dense sets) subsets of $2^\kappa$.

There are several forcing equivalent representations of the Cohen forcing: $\mathbb{C}_\kappa$ is forcing equivalent with $\text{Borel}(2^\kappa)/\mathcal{M}_\kappa$ partially ordered by $\subseteq$, with the poset of finite partial functions from $\kappa$ to $2$ (or to $\omega$) partially ordered by $\supseteq$, and with the $\kappa$ stage finite support iteration of $\mathbb{C} = \mathbb{C}_\omega$ as well. $\mathbb{C}_\kappa$ has the ccc, and $\mathbb{C}$ is $\sigma$-centered.

If $V$ is a c.t.m. and $G$ is a $(V, \mathbb{C})$-generic filter, then $\bigcap G$ (computed in the real world with real Borel sets coded in $V$ (see the next section)) contains a single real
The Borel null sets are coded in $\omega_1$. It is well-known that if $V$ is a c.t.m., then a real $c$ is Cohen over $V$ (i.e. there is a $(V, C)$-generic filter $G$ such that $\{c\} = \bigcap G$) iff $c$ is not contained in any Borel meager sets coded in $V$.

Cohen reals in $V^\omega$ show that $C$ is not $\omega^\omega$-bounding so it does not have the Sacks property. Furthermore, we know that $C$ does not add dominating reals.

The random forcing. Let $\kappa$ be a cardinal. The random forcing $B_\kappa$ is the complete Boolean algebra $\text{Borel}(2^\kappa)/\mathcal{N}_\kappa$ where $\mathcal{N}_\kappa$ is the ideal of Lebesgue null subsets of $2^\kappa$. $B_\kappa$ is forcing equivalent with $\text{Borel}(2^\kappa)/\mathcal{N}_\kappa$. $B$ is $\sigma$-n-linked for each $n$ but not $\sigma$-centered.

If $V$ is a c.t.m. and $G$ is a $(V, B)$-generic filter, then $\bigcap G$ contains a single real called random real over $V$. It is well-known that if $V$ is a c.t.m., then a real $r$ is random over $V$ iff $r$ is not contained in any Borel null sets coded in $V$.

$B_\kappa$ is $\omega^\omega$-bounding but it does not have the Sacks property (witnessed by the random real). Note that $B_\kappa$ is not forcing equivalent with the finite support iteration of $B = B_\omega$ because (for example) $B_\kappa$ does not add Cohen reals (and conversely, $C_\kappa$ does not add random reals) but the limit of any nontrivial finite support iterations does.

The Hechler forcing. The Hechler forcing $D$ contains all pairs of the form $p = (s^p, f^p)$ where $s^p \in \omega^{<\omega}$, $f^p \in \omega^\omega$ are strictly increasing, $p \leq q$ iff $s^p \supseteq s^q$, $f^p \supseteq f^q$, and $\forall n \in [s^p] \setminus [s^q]$ there is an $\omega$-centered $\exists f(n) \in s^p(n)$. $D$ is $\sigma$-centered.

If $V$ is a c.t.m. and $G$ is a $(V, D)$-generic filter, then $G = \bigcup \{s^p : p \in G\} \in \omega^\omega \cap V[G]$ is called a Hechler real over $V$. It is easy to see that a Hechler real over $V$ is a dominating real over $V$, i.e. it shows that $D$ adds dominating reals. Furthermore, $D$ adds Cohen reals but does not add random reals.

Let us denote $D_\kappa$ the $\kappa$ stage finite support iteration of $D$.

The localization forcing. Let $T = \bigcup n \in \omega \prod_{k < n} \omega^{\leq k}$ be the tree of initial slaloms. $p = (s^p, f^p) \in \text{LOC}$ iff $s^p \in T$, $f^p \subseteq \omega^\omega$, and $|f^p| \leq |s^p|$. $p \leq q$ iff $s^p \supseteq s^q$, $f^p \supseteq f^q$, and $\forall n \in [s^p] \setminus [s^q]$ there is an $\omega$-centered $f(n) \in s^p(n)$. $\text{LOC}$ is $\sigma$-n-linked for each $n$ but it is not $\sigma$-centered.

If $V$ is a c.t.m. and $G$ is a $(V, \text{LOC})$-generic filter, then $S = \bigcup \{s^p : p \in G\} \in Slm \cap V[G]$ is a slalom over $V$, that is $\forall f \in \omega^\omega \cap V \forall n \in \omega \ f(n) \in S(n)$. In particular, $d(n) = \max(S(n))$ is a dominating real over $V$. Furthermore, $\text{LOC}$ adds both Cohen and random reals.

### 1.2 Descriptive set theory and absoluteness

We refer the reader to [33], [42], and [34] for basics of descriptive set theory and related absoluteness results. In this section, we give a short list of some classical results important to us.

Analytic subsets of $\omega^\omega$ (i.e. elements of $\Sigma_1^1(\omega^\omega)$) can be coded by countable objects, namely trees (or more generally, relations) on $\omega \times \omega$. The analytic set constructed from the tree $T$ will be denoted by $p[T]$. Using this fact, we can talk about the (lightface) $\Sigma_1^1(a)$ sets where $a \in \omega^\omega$ is a parameter: an analytic set $A \subseteq \Sigma_1^1(\omega^\omega)$ is $\Sigma_1^1(a)$ if $A = p[T]$ where $T$ is recursive in $a$, and $A$ is $\Sigma_1^1$ if $T$ is recursive. Similarly, we have $\Pi_1^1(a)$ and $\Pi_1^1$ sets. Note that if $k \in \omega$ then the same procedure works for $(\omega^\omega)^k$ as well.
From now on, transitive model means a transitive model of (a large enough finite fragment of) ZFC. The following theorem is usually called as “analytic absoluteness”.

**Theorem 1.2.1.** (Mostowski’s Absoluteness Theorem) \( \Sigma^1_2(\omega^\omega) \) properties are absolute for transitive models. That is, if \( M \) is a transitive model and \( T \in M \) is recursive in \( a \in \omega^\omega \cap M \), then the formula (with parameter \( a \)) \( x \in p[T] \) is absolute for \( M \).

On the next level of the projective hierarchy we find the \( \Sigma^1_2(\omega^\omega) \) sets. As in the case of analytic sets, we can talk about \( \Sigma^1_2(\omega^\omega) \) sets where \( a \in \omega^\omega \). These sets can be coded by trees on \( \omega \times \omega_1 \) and if \( A = \Sigma^1_2(\omega^\omega) \) then there is a tree \( T \in L[a] \) such that \( A = p[T] \) (where \( L[a] \) is the class of sets constructible form \( a \)).

**Theorem 1.2.2.** (Shoenfield’s Absoluteness Theorem) If \( M \subseteq N \) are transitive models, \( \omega^\omega \subseteq M \), and \( a \in M \), then \( \Sigma^1_2(\omega^\omega) \) properties are absolute between \( M \) and \( N \).

The following theorem helps us to “identify” Borel sets in transitive models with Borel sets from the real world. If \( U \subseteq X \times Y \) is a relation, then let \( U_x = \{ y \in Y : (x, y) \in U \} \).

**Theorem 1.2.3.** (Borel-codes) There are a \( \Pi^1_1 \) set \( BC \subseteq \omega^\omega \) (the set of Borel-codes), a \( \Sigma^1_1 \) set \( U \subseteq \omega^\omega \times \omega^\omega \), and a \( \Pi^1_1 \) set \( V \subseteq \omega^\omega \times \omega^\omega \) (the coding “functions”) such that \( U_x = V_x \) for each \( x \in \mathbb{BC} \) and Borel(\( \omega^\omega \)) = \{ \{ U_x : x \in \mathbb{BC} \} \}.

If \( M \) is a transitive model and \( (B \subseteq \omega^\omega) \) is Borel \( M \), then there is an \( x \in (BC)^M = \mathbb{BC} \cap M \) (by analytic absoluteness) such that \( (B = U_x)^M = (B \cap M^M) = U_x \cap M. \) And conversely, if \( x \in M \) then \( (U_x \cap M) \) is a Borel set \( M \).

As an immediate consequence of this theorem, \( U_x \subseteq U_y \), \( U_x = U_y \), \( U_x = \emptyset \), \( U_x = U_y \cup U_z \), \( U_x = U_y \cap U_z \), \( U_x = \omega^\omega \setminus U_y \), and \( U_x = \bigcup \{ U_{y_n} : n \in \omega \} \) are \( \Pi^1_1 \) properties of Borel-codes, that is, for instance, \( \{ (x, y) \in \omega^\omega \times \omega^\omega : x, y \in \mathbb{BC} \text{ and } U_x \subseteq U_y \} \subseteq \omega^\omega \times \omega^\omega \) is \( \Pi^1_1 \). In particular, they are absolute for transitive models. Besides, “\( U_x \) is meager” and “\( \lambda(U_x) = r \)” are also absolute for transitive models (where \( \lambda \) is the Lebesgue measure on \( \omega^\omega \) and \( r \in \mathbb{R} \)).

### 1.3 Cardinal invariants of the continuum

Loosely speaking, a cardinal invariant is a first order definable (almost always) uncountable cardinal less or equal than the continuum \( \mathfrak{c} = 2^{\omega} \). First of all, we present a (definitely not complete) list of classical cardinal coefficients of the continuum. For more details and for the proofs see [7].

If \( f, g \in \omega^\omega \) write \( f \leq^* g \) if \( \forall n \in \omega \ f(n) \leq g(n) \), i.e. the set \( \{ n \in \omega : f(n) > g(n) \} \) is finite. This is a pre-order on \( \omega^\omega \). The unbounding and dominating numbers of \( (\omega^\omega, \leq^*) \), denoted by \( b \) and \( \delta \) are defined in the natural way: \( b \) is the minimal size of a \( \leq^* \)-unbounded family, i.e.

\[
b = \min \{ |U| : U \subseteq \omega^\omega \text{ and } \forall f \in \omega^\omega \exists g \in U \ g \not\leq^* f \},
\]

and \( \delta \) is the minimal size of a \( \leq^* \)-dominating family, i.e.

\[
\delta = \min \{ |D| : D \subseteq \omega^\omega \text{ and } \forall f \in \omega^\omega \exists g \in D \ f \not\leq^* g \}.
\]
1.3. CARDINAL INVARIANTS OF THE CONTINUUM

A family \( A \subseteq [\omega]^{\omega} \) is an almost-disjoint (AD) family if \( |A \cap B| < \omega \) for each distinct \( A, B \in A \). An infinite(!) AD family \( A \) is maximal (MAD) if \( \forall X \in [\omega]^{\omega} \exists A \in A \) \( |X \cap A| = \omega \), i.e. \( A \) is \( \subseteq \)-maximal among the AD families. The almost-disjointness number \( \alpha \) is the smallest cardinality of a MAD family.

A family \( X \subseteq [\omega]^{\omega} \) has the strong finite intersection property (SFIP, in short) if \( \bigcap X' = \omega \) for each finite (and nonempty) \( X' \subseteq X \). A set \( Y \in [\omega]^{\omega} \) is a pseudo-intersection of \( X \) if \( Y \subseteq^* X \) for each \( X \in \mathcal{X} \) where \( A \subseteq B \) iff \( A \setminus B \) is finite. The pseudo-intersection number \( p \) is the smallest cardinality of a family with the SFIP but without a pseudo-intersection.

A sequence \( T = (T_\alpha)_{\alpha < \gamma} \) in \( [\omega]^{\omega} \) is a tower if \( \gamma \) is regular, it is \( \subseteq^* \)-descending (i.e. \( T_\beta \subseteq^* T_\alpha \) if \( \alpha < \beta < \gamma \)), and it has no pseudo-intersection. The tower number \( t \) is the smallest cardinality of a tower.

A set \( D \subseteq [\omega]^{\omega} \) is open if it is \( \subseteq^* \)-downward closed. \( D \) is dense if \( \forall X \in [\omega]^{\omega} \exists D \in D \) \( D \subseteq X \). The distributivity number \( \hbar \) is the smallest cardinality of a family of dense and open sets with empty intersection.

We know that

\[
\omega < \text{cf}(p) = p \leq \text{cf}(t) = t \leq \text{cf}(\hbar) = \hbar \leq \text{cf}(b) = b \leq \text{cf}(\delta) = \delta \leq c
\]

and \( b \leq a \) hold in ZFC. In the most cases, there are no more connections between these invariants because for instance, if \( V \models \text{GCH} \) then \( V^{\mathbb{M}_{\omega_2}} \models t < \hbar \) (where \( \mathbb{M}_{\omega_2} \) is the \( \omega_2 \)-stage countable support iteration of the Mathias-forcing \( \mathbb{M} \)), \( V^{\mathbb{P}_{\omega_2}} \models \hbar < b \), \( V^{\mathbb{C}_{\omega_2}} \models b < \text{cf}(\delta) < b \), \( V^{\mathbb{C}_{\omega_2}} \models a < b \); furthermore, \( b < a \) is also consistent with ZFC (see [10]). One of the most mysterious open problems of this topic is the equality of \( p \) and \( t \).

Next, we recall the definition of ideals, their main properties, and their cardinal invariants. Let \( X \) be a nonempty set. An \( I \subseteq \mathcal{P}(X) \) is an ideal on \( X \) if \( \emptyset \in I \), \( I \) is \( \subseteq \)-downward closed, and \( I \) is closed under finite unions. Let \( I \) be an ideal on \( X \).

- \( I \) is principal if there is a \( Y \subseteq X \) such that \( I = \mathcal{P}(Y) \).
- \( I \) is proper if \( X \notin I \).

From now on, if we say that \( I \) is an ideal on \( X \), then we always assume that \( I \) is proper and \( [X]^{<\omega} \subseteq I \), in particular \( I \) is non-principal. We will use only one exception: the ideal \( \{\emptyset\} \) on any \( X \).

- \( I \) is a prime ideal if for every \( A \subseteq X \), either \( A \in I \) or \( X \setminus A \in I \); equivalently, \( I \) is \( \subseteq \)-maximal among ideals on \( X \).
- \( I \) is a \( \sigma \)-ideal if \( \bigcup J \in I \) for each countable \( J \subseteq I \).

Write \( I^+ = \mathcal{P}(X) \setminus I \) for the family of \( I \)-positive sets and \( I^+ = \{X \setminus A : A \in I\} \) for the dual filter of \( I \). Filters are the dual objects of ideals, that is, \( F \subseteq \mathcal{P}(X) \) is a filter if \( X \in F \), \( F \) is \( \subseteq \)-upward closed, and \( F \) is closed for finite intersections. A filter is principal / proper / an ultrafilter if its dual ideal, \( F^+ = \{X \setminus H : H \in F\} \) is principal / proper / a prime ideal.
The following four cardinals are the additivity, cofinality, uniformity, and covering numbers of an ideal $I$ on $X$:

\[
\begin{align*}
\text{add}(I) &= \min \{|J| : J \subseteq I, \cup J \notin I\} \\
\text{cof}(I) &= \min \{|J| : J \subseteq I, \forall A \in I \exists B \in J A \subseteq B\} \\
\text{non}(I) &= \min \{|Y| : Y \subseteq X, Y \notin I\} \\
\text{cov}(I) &= \min \{|J| : J \subseteq I, \cup J = X\}
\end{align*}
\]

It is easy to see that the following inequalities hold for each ideal $I$ where an arrow from $a$ to $b$ is to read as $a \leq b$.

\[
\begin{array}{cccc}
\text{cov}(I) & \quad \text{add}(I) & \quad \text{cof}(I) & \quad \text{non}(I)
\end{array}
\]

The two most important examples of $\sigma$-ideals are the ideal $\mathcal{M}$ of meager subsets of the reals (or $\omega^\omega$ or $2^\omega$ or in general, an uncountable Polish space), that is the ideal of sets which can be covered by countable many nowhere dense sets; and the ideal $\mathcal{N}$ of (Lebesgue) measure zero subsets of the reals (or $\omega^\omega$ or $2^\omega$).

The following diagram called Cichoń's diagram shows the very deep connection between $\mathcal{M}$, $\mathcal{N}$, and the unbounding and dominating numbers.

\[
\begin{array}{cccc}
\text{cov}(\mathcal{N}) & \quad \text{non}(\mathcal{M}) & \quad \text{cof}(\mathcal{M}) & \quad \text{cof}(\mathcal{N})
\end{array}
\]

Of course, $\omega_1 \leq \text{add}(\mathcal{N})$ and $\text{cof}(\mathcal{N}) \leq \mathfrak{c}$. Furthermore, $\text{add}(\mathcal{M}) = \min\{b, \text{cov}(\mathcal{M})\}$ and $\text{cof}(\mathcal{M}) = \max\{\text{non}(\mathcal{M}), 0\}$. In this level, there are no more connections between these invariants: If we cut the diagram into two parts, that is, we partition the invariants from it into to parts $L$ (as “left”) and $R$ (as “right”), such that

1. there are no $b \in L$ and $a \in R$ with $a \leq b$ according to the diagram;
2a. if $\text{add}(\mathcal{M}) \in L$ then $\{b, \text{cov}(\mathcal{M})\} \cap L \neq \emptyset$;
2b. if $\text{cof}(\mathcal{M}) \in R$ then $\{\text{non}(\mathcal{M}), 0\} \cap R \neq \emptyset$;

then it is consistent with ZFC that all $a \in L$ are equal to $\omega_1$ and all $b \in R$ are equal to $\omega_2$ (see [3]).
There is an uniform way of defining cardinal invariants and proving inequalities between them. From now on, we will use the notions of \([45]\) instead of \([7]\). A supported relation is a triple \(\mathcal{R} = (A, R, B)\) where \(R \subseteq A \times B\), \(\text{dom}(R) = A\), \(\text{ran}(R) = B\), and we always assume that for each \(b \in B\) there is an \(a \in A\) such that \((a, b) \notin R\). Let \(R^{-1} = \{(b, a) : (a, b) \in R\}\), and if \(R \subseteq A \times B\) and \(X \subseteq A\), then let \(R[X] = \{b \in B : \exists a \in X \ (a, b) \in R\}\).

The unbounding and dominating numbers of \(\mathcal{R}\) are the following cardinals:

\[
\begin{align*}
\text{b}(\mathcal{R}) &= \min \{|A'| : A' \subseteq A \text{ and } \forall b \in B \ a \not\in R^{-1}[\{b\}]\} \\
\text{d}(\mathcal{R}) &= \min \{|B'| : B' \subseteq B \text{ and } A = R^{-1}[B']\}
\end{align*}
\]

Note that \(\text{b}(\mathcal{R})\) and \(\text{d}(\mathcal{R})\) are defined for each \(\mathcal{R}\), but in general \(\text{b}(\mathcal{R}) \leq \text{d}(\mathcal{R})\) does not hold. Of course, if \(\mathcal{R}\) is a “pre-ordered set”, that is \(\mathcal{R} = (Q, \leq, Q)\) where \((Q, \leq)\) is a pre-ordered set in the usual sense, then \(\text{b}(\mathcal{R}) \leq \text{d}(\mathcal{R})\). Furthermore, if \(\mathcal{R}\) is a pre-ordered set, then it is easy to see that \(\text{b}(\mathcal{R})\) is regular.

Let’s see some examples: \(b = \text{b}(\omega^\omega, \leq^*, \omega^\omega)\) and \(d = \text{d}(\omega^\omega, \leq^*, \omega^\omega)\); if \(I\) is an ideal on \(X\), then add\((I) = \text{b}(I, \subseteq, I), \text{cof}(I) = \text{d}(I, \subseteq, I), \text{non}(I) = \text{b}(X, \in, I), \text{and cov}(I) = \text{d}(X, \in, I)\).

We recall the definition of Galois-Tukey connections between supported relations. Let \(\mathcal{R}_1 = (A_1, R_1, B_1)\) and \(\mathcal{R}_2 = (A_2, R_2, B_2)\) be supported relations. A pair of functions \(\phi : A_1 \rightarrow A_2, \psi : B_2 \rightarrow B_1\) is a Galois-Tukey (GT, in short) connection from \(\mathcal{R}_1\) to \(\mathcal{R}_2\), in notation \((\phi, \psi) : \mathcal{R}_1 \preccurlyeq_{\text{GT}} \mathcal{R}_2\) if \(a_1 R_1 \psi(b_2)\) whenever \(\phi(a_1) R_2 b_2\). In a diagram:

\[
\begin{array}{ccc}
\psi(b_2) \in B_1 & \overset{\psi}{\leftrightarrow} & B_2 \ni b_2 \\
R_1 & \Leftarrow & R_2 \\
\phi \quad a_1 \in A_1 & \phi \rightarrow & A_2 \ni \phi(a_1)
\end{array}
\]

We write \(\mathcal{R}_1 \preccurlyeq_{\text{GT}} \mathcal{R}_2\) if there is a GT-connection from \(\mathcal{R}_1\) to \(\mathcal{R}_2\). If \(\mathcal{R}_1 \preccurlyeq_{\text{GT}} \mathcal{R}_2\) and \(\mathcal{R}_2 \preccurlyeq_{\text{GT}} \mathcal{R}_1\) also hold then we say \(\mathcal{R}_1\) and \(\mathcal{R}_2\) are Galois-Tukey (GT) equivalent, in notation \(\mathcal{R}_1 \equiv_{\text{GT}} \mathcal{R}_2\). It is trivial from the definition that \(\mathcal{R}_1 \preccurlyeq_{\text{GT}} \mathcal{R}_2\) implies \(\text{b}(\mathcal{R}_1) \geq \text{b}(\mathcal{R}_2)\) and \(\text{d}(\mathcal{R}_1) \leq \text{d}(\mathcal{R}_2)\).

Notice that \((A_1, R_1, B_1) \preccurlyeq_{\text{GT}} (A_2, R_2, B_2)\) is equivalent to the existence of a function \(\phi : A_1 \rightarrow A_2\) with that \(\phi[A']\) is \(R_2\)-unbounded (i.e. there is no \(b_2 \in B_2\) such that \(\phi[A'] \subseteq R_2^{-1}[\{b_2\}]\)) whenever \(A'\) is \(R_1\)-unbounded. If \(\mathcal{R}_1\) and \(\mathcal{R}_2\) are pre-ordered sets, then such a map is called Tukey map from \(\mathcal{R}_1\) to \(\mathcal{R}_2\) and we say that \(\mathcal{R}_1\) is Tukey-reducible to \(\mathcal{R}_2\).

If \(\mathcal{R} = (A, R, B)\) is a supported relation then let \(\mathcal{R}^\perp = (B, \sim R^{-1}, A)\) its dual relation where \((b, a) \in \sim R^{-1}\) iff \((a, b) \notin R\). Clearly, \((\mathcal{R}^\perp)^\perp = \mathcal{R}\) and \(\text{b}(\mathcal{R}) = \text{d}(\mathcal{R}^\perp)\). Furthermore, it is easy to see that \((\phi, \psi) : \mathcal{R}_1 \preccurlyeq_{\text{GT}} \mathcal{R}_2\) iff \((\psi, \phi) : \mathcal{R}_2^\perp \preccurlyeq_{\text{GT}} \mathcal{R}_1^\perp\).

In the most cases, if we prove an inequality between bounding / dominating like cardinal invariants, then actually we prove that there is a GT-connection between the underlying relations. For instance, Cichoń’s diagram comes from the following GT-
connections:

\[
(2^\omega, \in, N) \rightarrow (2^\omega, \in, M)^\perp \rightarrow (M, \subseteq, M) \rightarrow (N, \subseteq, N)
\]

\[
(\omega^\omega, \leq^*, \omega^\omega)^\perp \rightarrow (\omega^\omega, \leq^*, \omega^\omega)
\]

\[
(N, \subseteq, N)^\perp \rightarrow (M, \subseteq, M)^\perp \rightarrow (2^\omega, \in, M) \rightarrow (2^\omega, \in, N)^\perp
\]

We will also use the following relations: \((\ell_1^+, \leq^{(s)}, \ell_1^+)\) and \((\omega^\omega, \subseteq^*, \text{Slm})\). \(\ell_1^+\) is the set of positive summable sequences and \(\leq^{(s)}\) is the (almost) everywhere dominating relation. Slm is the set of slaloms, that is, \(\text{Slm} = \prod_{n \in \omega} [\omega]^{\leq n}\) so \(S \in \text{Slm}\) iff \(S : \omega \rightarrow [\omega]^{n}\) and \(|S(n)| \leq n\) for each \(n\), and \(f \subseteq^s S\) iff \(\forall^\infty n \in \omega\) \(f(n) \in S(n)\), i.e. the set \(\{n \in \omega : f(n) \notin S(n)\}\) is finite. The following theorem will be very useful when we will work with \((N, \subseteq, N)\).

**Theorem 1.3.1.** ([26, Corollary 524H])

\[(\ell_1^+, \leq^*, \ell_1^+) \equiv_{\text{cr}} (\omega^\omega, \subseteq^*, \text{Slm}) \equiv_{\text{cr}} (N, \subseteq, N).\]
Chapter 2

Ideals on countable sets

2.1 Basic properties

In this section, we list certain properties of ideals on $\omega$ and related notions that will be needed later. Let us denote $\text{Fin}$ the Fréchet ideal on $\omega$, i.e. $\text{Fin} = [\omega]^{<\omega}$.

If $A \subseteq \wp(\omega)$ then let $\langle A \rangle_{\text{id}} = \{ S \subseteq \omega : \exists A' \in [A]^{<\omega} S \subseteq^* \bigcup A' \}$.

If $\omega \notin \langle A \rangle_{\text{id}}$ then $\langle A \rangle_{\text{id}}$ is the ideal generated by $A$.

The character of an ideal $I$, denoted by $\chi(I)$ is the minimal cardinality of a family generating $I$, and the character of a filter is the character of the dual ideal. In particular, $\chi(\text{Fin}) = 0$ but if $\chi(I) \geq \omega$, then $\chi(I) = \text{cof}(I)$.

If $I$ is an ideal on $\omega$ (or on any set) and $X \in I^+$, then let $I \upharpoonright X$ be the restriction of $I$ to $X$, that is $I \upharpoonright X = \{ A \in I : A \subseteq X \} = \{ X \cap B : B \in I \}$.

If $I$ is an ideal on $\omega$ (or on a countable set), then

- $I$ is analytic (resp. Borel, $\Sigma^0_\alpha$, meager, a null set etc.) if $I \subseteq \wp(\omega) \approx 2^\omega$ is analytic (Borel, $\Sigma^0_\alpha$, meager, a null set etc.) in the usual product topology (and measure) of the Cantor set;

- $I$ is a $P$-ideal if for each countable $C \subseteq I$ there is an $A \in I$ such that $C \subseteq^* A$ for each $C \in C$, i.e. the pre-ordered set $(I, \subseteq^*)$ is $\sigma$-directed;

- $I$ is tall if each infinite subset of $\omega$ contains an infinite element of $I$, i.e. $I^*$ does not have a pseudo-intersection.

One can easily see that an ideal cannot be closed, open, or (using the Baire Category Theorem) $G_\delta$ so the minimal Borel-complexity of an ideal is $F_\sigma$, and for example $I_{1/\alpha}$ is $F_\sigma$.

Informally, the next theorem says that ideals (and filters because of the measure preserving homeomorphism $\wp(\omega) \to \wp(\omega), X \mapsto \omega \setminus X$) are “small” if we can “measure” them.

**Theorem 2.1.1.** (Sierpiński, see [3, Theorem 4.1.1]) Let $I$ be an ideal on $\omega$. 

9
(1) If $I$ has the Baire property (i.e. there is an open set $U \subseteq \mathcal{P}(\omega) \cong 2^\omega$ such that $U \triangle I$ is meager), then $I$ is meager.

(2) If $I$ is measurable, then it is a null set.

It is well-known that analytic sets are Lebesgue measurable and have the Baire property (see e.g. [33, Theorem 11.18]) so we obtain the following:

**Corollary 2.1.2.** Analytic ideals and filters are meager null sets.

There is a nice characterization of meager filters:

**Theorem 2.1.3.** ([44], in English [3, Theorem 4.1.2]) Let $\mathcal{F}$ be a filter on $\omega$. Then the following are equivalent:

(i) $\mathcal{F}$ is meager.

(ii) The set of increasing enumerations of elements of $\mathcal{F}$ is $\leq^*$-bounded.

(iii) There is a partition $\{P_n : n \in \omega\}$ of $\omega$ into finite sets such that

$$\forall F \in \mathcal{F} \forall^\infty n \in \omega F \cap P_n \neq \emptyset.$$ 

Property (iii) is also called feebleness. This property can be reformulated by the following way: A filter $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is feeble iff there is a finite-to-one function $f : \omega \to \omega$ such that $f''[\mathcal{F}]$ is the Fréchet filter $\text{Fin}^*$.

The following theorem is about the characters of meager ideals and filters.

**Theorem 2.1.4.** (R. C. Solomon and P. Simon, [7, Theorem 9.10]) If an ideal (or filter) has character less than $b$ then it is meager but there is a non-meager ideal (resp. filter) generated by $b$ sets.

### 2.2 Ideals associated to submeasures

We present a natural way of defining nice ideals on $\omega$.

**Definition 2.2.1.** A function $\vartheta : \mathcal{P}(\omega) \to [0, \infty]$ is a submeasure on $\omega$ if

1. $\vartheta(\emptyset) = 0$;
2. if $X \subseteq Y \subseteq \omega$ then $\vartheta(X) \leq \vartheta(Y)$;
3. if $X, Y \subseteq \omega$ then $\vartheta(X \cup Y) \leq \vartheta(X) + \vartheta(Y)$;
4. $\vartheta(\{n\}) < \infty$ for $n \in \omega$.

A submeasure $\vartheta$ is lower semicontinuous (lsc in short) if

1. $\vartheta(X) = \lim_{n \to \infty} \vartheta(X \cap n)$ for each $X \subseteq \omega$.

A submeasure $\vartheta$ is finite if $\vartheta(\omega) < \infty$. 

2.2. IDEALS ASSOCIATED TO SUBMEASURES

Notice that if \( \vartheta \) is an lsc submeasure on \( \omega \) then it is \( \sigma \)-subadditive, i.e.

\[
\vartheta \left( \bigcup_{n \in \omega} A_n \right) \leq \sum_{n \in \omega} \vartheta(A_n)
\]

holds for \( A_n \subseteq \omega \).

If \( \vartheta \) is an lsc submeasure on \( \omega \) then for \( X \subseteq \omega \) let

\[
\|X\|_\vartheta = \lim_{n \to \infty} \vartheta(X \setminus n).
\]

It is easy to see that \( \|A \cup B\|_\vartheta \leq \|A\|_\vartheta + \|B\|_\vartheta \) if \( A, B \subseteq \omega \) but \( \|\cdot\|_\vartheta \) is not necessarily \( \sigma \)-subadditive. We assign two ideals to a submeasure \( \vartheta \) as follows

\[
\text{Fin}(\vartheta) = \{X \subseteq \omega : \vartheta(X) < \infty\},
\]

\[
\text{Exh}(\vartheta) = \{X \subseteq \omega : \|X\|_\vartheta = 0\}.
\]

It is easy to see that if \( \text{Fin}(\vartheta) \neq \mathcal{P}(\omega) \), then it is an \( F_\sigma \) ideal; and similarly if \( \text{Exh}(\vartheta) \neq \mathcal{P}(\omega) \), then it is an \( F_\sigma \delta \) \( \mathcal{P} \)-ideal. From now on, if we are dealing with \( \text{Fin}(\vartheta) \) (resp. \( \text{Exh}(\vartheta) \)) then we always assume that \( \text{Fin}(\vartheta) \neq \mathcal{P}(\omega) \), i.e. \( \vartheta(\omega) = \infty \) (resp. \( \vartheta(\omega) \neq \infty \), i.e. \( \|\omega\|_\vartheta > 0 \)).

It is straightforward to see that if \( \vartheta \) is an lsc submeasure on \( \omega \) then

\[
\text{Exh}(\vartheta) \text{ is tall iff } \lim_{n \to \infty} \vartheta(\{n\}) = 0.
\]

Furthermore, we can assume the following without changing \( \text{Exh}(\vartheta) \):

(i) \( \vartheta(\{k\}) > 0 \) for each \( k \in \omega \);

(ii) \( \vartheta \) is finite so \( \|\omega\|_\vartheta < \infty \) too.

Indeed, let \( \vartheta_1(A) = \vartheta(A) + \sum_{k \in \omega} 2^{-k} \vartheta_1(\{k\}) > 0 \) for each \( k \in \omega \), and \( \text{Exh}(\vartheta_1) = \text{Exh}(\vartheta) \). If \( \vartheta_2(A) = \max\{\vartheta_1(A), 1\} \) then \( \vartheta_2 \) is an lsc submeasure, \( \vartheta_2(\omega) \leq 1 \) so it satisfies (i) and (ii), and \( \text{Exh}(\vartheta_2) = \text{Exh}(\vartheta_1) = \text{Exh}(\vartheta) \).

The following characterization theorem gives us the most useful method to work on combinatorics of \( F_\sigma \) ideals and analytic \( \mathcal{P} \)-ideals.

**Theorem 2.2.2.** ([37] and [40]) Let \( \mathcal{I} \) be an ideal on \( \omega \).

- \( \mathcal{I} \) is an \( F_\sigma \) ideal iff \( \mathcal{I} = \text{Fin}(\vartheta) \) for some lsc submeasure \( \vartheta \).

- \( \mathcal{I} \) is an analytic \( \mathcal{P} \)-ideal iff \( \mathcal{I} = \text{Exh}(\vartheta) \) for some (finite) lsc submeasure \( \vartheta \).

- \( \mathcal{I} \) is an \( F_\sigma \delta \mathcal{P} \)-ideal iff \( \mathcal{I} = \text{Fin}(\vartheta) = \text{Exh}(\vartheta) \) for some lsc submeasure \( \vartheta \).

Therefore each analytic \( \mathcal{P} \)-ideal is \( F_\sigma \delta \), so a Borel subset of \( 2^\omega \).

Furthermore, Solecki proved (see [40]) that an ideal \( \mathcal{I} \) on \( \omega \) is an analytic \( \mathcal{P} \)-ideal (or \( \mathcal{I} = \mathcal{P}(\omega) \)) iff it is Polishable, that is, there is Polish group topology on \( \mathcal{I} \) with respect to the symmetric difference as group operation such that the Borel structure of this topology coincides with the Borel structure inherited from \( 2^\omega \).
Using the previous theorem, we do not need analytic absoluteness to see that if $\mathcal{I}$ is an analytic P-ideal coded in the transitive model $M$, then $\mathcal{I}^M = \mathcal{I} \cap M$. Lsc submeasures are determined by their restrictions to $[\omega]^{<\omega}$ and there is an lsc submeasure $\vartheta$ such that $\vartheta \upharpoonright [\omega]^{<\omega} \in M$ and $M \models \mathcal{I} = \text{Exh}(\vartheta \upharpoonright \mathcal{P}_\omega(\omega))$. Furthermore, it is easy to see that $A \in \text{Exh}(\vartheta)$ is absolute for transitive models. In particular, we can talk about the “same” ideal in forcing extensions of $V$ by using $\vartheta$.

### 2.3 Orders on ideals

Recall some classical pre-orders on the family of ideals on $\omega$:

- **Katětov-order:** $\mathcal{I} \preceq_k \mathcal{J}$ iff $\exists f : \omega \to \omega$ (A $\in \mathcal{I} \Longrightarrow f^{-1}[A] \in \mathcal{J}$).
- **Katětov-Blass-order:** $\mathcal{I} \preceq_{KB} \mathcal{J}$ iff $\exists f : \omega \to \omega$ (A $\in \mathcal{I} \Longrightarrow f^{-1}[A] \in \mathcal{J}$).
- **Rudin-Keisler-order:** $\mathcal{I} \preceq_{RK} \mathcal{J}$ iff $\exists f : \omega \to \omega$ (A $\in \mathcal{I} \iff f^{-1}[A] \in \mathcal{J}$).
- **Rudin-Blass-order:** $\mathcal{I} \preceq_{RB} \mathcal{J}$ iff $\exists f : \omega \to \omega$ (A $\in \mathcal{I} \iff f^{-1}[A] \in \mathcal{J}$).

We have the following trivial implications between these pre-orders:

- $\mathcal{I} \preceq_{RB} \mathcal{J} \iff \mathcal{J} \preceq_{KB} \mathcal{I}$
- $\mathcal{I} \preceq_{BK} \mathcal{J} \iff \mathcal{J} \preceq_{RB} \mathcal{I}$
- $\mathcal{I} \preceq_k \mathcal{J} \iff \mathcal{J} \preceq_k \mathcal{I}$

Of course, we can use these pre-orders for filters as well, e.g. $\mathcal{F} \preceq_k \mathcal{G}$ iff $\mathcal{F}^* \preceq_k \mathcal{G}^*$.

We refer the reader to [19], [31], and [38] for algebraic and combinatorial aspects of these pre-orders. In [38] using these pre-orders and some classical ideals (see e.g. the next section), some really nice characterization theorems were proved.

It is easy to see that if $\mathcal{F}$ is feeble (i.e. meager) iff $\text{Fin} \preceq_{RB} \mathcal{F}^*$. According to Corollary 2.1.2 and Theorem 2.1.3, analytic filters are feeble. This fact has an easy proof if we assume additionally that our ideal is a P-ideal:

**Proposition 2.3.1.** $\text{Fin} \preceq_{RB} \mathcal{I}$ for any analytic P-ideal $\mathcal{I}$.

**Proof.** Let $\vartheta = \text{Exh}(\vartheta)$ for some finite lsc submeasure $\vartheta$ on $\omega$. Since $\omega \notin \mathcal{I}$ we have $||\omega||_\vartheta = \lim_{n \to \omega} \vartheta(\omega \setminus n) = \varepsilon > 0$. Hence by the lsc property of $\vartheta$ for each $n > 0$ there is $m > n$ such that $\vartheta([n,m)) > \varepsilon/2$.

So there is a partition $\{P_n : n < \omega\}$ of $\omega$ into finite pieces such that $\vartheta(P_n) > \varepsilon/2$ for each $n \in \omega$. Define the function $f : \omega \to \omega$ by the stipulation $f(P_n) = \{n\}$. Then $f$ witnesses $\text{Fin} \preceq_{RB} \mathcal{I}$. \qed

It is easy to see that if $\mathcal{B}$ is the set of Borel-codes of Borel ideals, then $\mathcal{B}$ is $\Pi_1^1$. Furthermore, if $\mathcal{I}$ is the family of pre-orders defined above, then $\mathcal{I} \preceq \mathcal{J}$ is a $\Sigma_1^1$ property, that is, the set $\{(x,y) \in \mathcal{B} : U_x \leq U_y\}$ is $\Sigma_1^1$ (where $U$ is the coding “function”, see Theorem 1.2.3). In particular, using Shoenfield’s Absoluteness Theorem, this property is absolute between any pair of transitive models $M \subseteq N$ with $\omega_1^N \subseteq M$.

We prove some easy but useful facts about the Katětov-order.
Proposition 2.3.2. ([38, Proposition 1.7.2] and A. Blass (private communication))
The Katětov-order is $\mathfrak{c}^+$-downward and $\mathfrak{c}^+$-upward directed, that is, every family of ideals
on $\omega$ with cardinality at most $\mathfrak{c}$ has both a $\leq_\omega$-lower bound and a $\leq_\omega$-upper bound.

Proof. Assume $\{j_\alpha : \alpha < \mathfrak{c}\}$ is a family of ideals on $\omega$. We will show that it has a $\leq_\omega$
lower bound $J$ and a $\leq_\omega$-upper bound $j$. Let $\{A_\alpha : \alpha < \mathfrak{c}\} \subseteq [\omega]^{\mathfrak{c}}$ be an almost disjoint
family on $\omega$ (i.e. $|A_\alpha \cap A_\beta| < \omega$ for each $\alpha < \beta < \mathfrak{c}$) and fix bijections $e_\alpha : \omega \rightarrow A_\alpha$. The
ideal $J$ generated by $\bigcup \{e_\alpha^{\uparrow}[j_\alpha] : \alpha < \mathfrak{c}\}$ is then a Katětov-lower bound of $\{j_\alpha : \alpha < \mathfrak{c}\}$
because $e_\alpha$ witnesses $J \leq_\omega j_\alpha$ for each $\alpha$.

Now, let $\{f_\alpha : \alpha < \mathfrak{c}\} \subseteq [\omega]^{\omega}$ be an independent family of functions, that is, for
every $\alpha_0, \ldots, \alpha_k < \mathfrak{c}$ and $n_0, \ldots, n_{k-1} \in \omega$ there is some $x \in \omega$ such that $f_\alpha(x) = n_i$
for all $i < k$ (see [18, Theorem 3] for the proof of the existence of such a family). Let $\mathcal{J}$ be the ideal generated
by the family $\bigcup\{|f_\alpha^{-1}[\{j_\alpha\}] : \alpha < \mathfrak{c}\}$. Notice that no finite union of elements from the family covers $\omega$, so $J \nsubseteq \mathcal{J}$. Indeed, if $A_i \in \mathcal{J}_\alpha$ and $n_i \in \omega \setminus A_i$ for $i < k \in \omega$, then there is an $x \in \omega$ such that $f_\alpha(x) = n_i$ for each $i < k$ so
$x \notin \bigcup\{|f_\alpha^{-1}[\{j_\alpha\}] : i < k\}$. Clearly, $J_\alpha \leq_\omega \mathcal{J}$ is witnessed by $f_\alpha$ for each $\alpha$.

Proposition 2.3.3. Meager filters are cofinal in the Katětov-order.

Proof. Let $\mathcal{F}$ be an arbitrary filter on $\omega$. Fix a partition $(P_n)$ of $\omega$ into finite sets such
that $|P_n| = n$, $P_n = \{p_k^n : k < n\}$. Let $\mathcal{G}$ be the filter generated by the sets $\mathcal{F} = \{p_k^n : F \in F \cap n\}$ for $F \in \mathcal{F}$. Clearly, $\mathcal{F} \cap P_n \neq \emptyset$ for each $n > \min(F)$ so $\mathcal{G}$ is meager by Theorem
2.1.3. Now, the function $g : \omega \rightarrow \omega$ defined by $g(p_k^n) = k$ witnesses that $\mathcal{F} \leq_\omega \mathcal{G}$.  

At last in this section, we recall an important characterization theorem due to M. Hrušák and J. Zapletal.

Assume $\mathcal{J}$ is a tall ideal on $\omega$ and $\mathbb{P}$ is a forcing notion. We say that $\mathcal{J}$ is $\mathbb{P}$-
indestructible if $\Vdash_\mathbb{P} \exists A \in \mathcal{J} [X \cap A = \mathbb{R}]$ for each $\mathbb{P}$-name $X$ for an infinite subset
of $\omega$, i.e. in $V^{\mathbb{P}}$ the ideal generated by $\mathcal{J}$ is tall. If $\mathcal{J}$ is not $\mathbb{P}$-indestructible, then we also say: $\mathbb{P}$ destroys $\mathcal{J}$. Forcing-indestructibility of ideals has been widely studied for years.

The G$_\delta$-closure of a set $A \subseteq 2^{<\omega}$ (or $\omega^{<\omega}$) is

$$[A] = \{f \in 2^{\omega}$ (or $\omega^{<\omega}) : \exists^{\infty} n f \upharpoonright n \in A\}.$$

The trace ideal of a $\sigma$-ideal $I$ on $\omega^n$ (or on $\omega^{<\omega}$) is

$$\text{tr}(I) = \{A \subseteq 2^{<\omega}$ (or $\omega^{<\omega}) : [A] \in I\}.$$

If $I$ is a $\sigma$-ideal on a Polish space $X$, then $\mathbb{P}_I$ denotes the forcing notion Borel$(X) \setminus I$
partially ordered by the reverse inclusion. Properties of forcing notions of the form $\mathbb{P}_I$ is a central topic of the theory of forcing methods. For more details and notions, for instance the property continuous readings of names (CRN), see [46] or [32].

The following theorem shows the crucial role of the Katětov-order in forcing indestructibility of ideals:

Theorem 2.3.4. ([32]) Let $I$ be a $\sigma$-ideal on $2^{\omega}$ or on $\omega^{<\omega}$, and let $\mathcal{J}$ be a tall ideal
on $\omega$. Assume furthermore that $\mathbb{P}_I$ is proper and has the CRN. Then $\mathcal{J}$ is $\mathbb{P}_I$-indestructible
if, and only if $\mathcal{J} \nsubseteq \text{tr}(I) \upharpoonright X$ for any $X \in \text{tr}(I)^+$. 

In some classical cases we know a little bit more: we do not need to investigate \( J \notin \text{tr}(I) \) for any \( X \in \text{tr}(I)^+ \) only \( J \notin \text{tr}(I) \) because of homogeneous properties of \( \text{tr}(I) \) (for more details see [38, Section 2.1.1]).

For example:

- \( J \) is Cohen-indestructible iff \( J \notin \text{tr}(M) \).
- \( J \) is random-indestructible iff \( J \notin \text{tr}(N) \).
- \( J \) is Sacks-indestructible iff \( J \notin \text{tr}([2^\omega]^{<\omega}) \).

For other types of general characterization theorems see [14].

### 2.4 Examples

We define some important (classes of) Borel ideals.

#### Summable ideals. Let \( h : \omega \to [0, \infty) \) be a function such that \( \sum_{n \in \omega} h(n) = \infty \). The summable ideal associated to \( h \) is

\[
J_h = \left\{ A \subseteq \omega : \sum_{n \in A} h(n) < \infty \right\}.
\]

It is easy to see that a summable ideal \( J_h \) is tall iff \( \lim_{n \to \infty} h(n) = 0 \), and that summable ideals are \( F_\sigma \) ideals. The function \( \theta_h : \omega \to [0, \infty] \), \( \theta_h(A) = \sum_{n \in A} h(n) \) is an lsc (sub)measure on \( \omega \) and \( J_h = \text{Fin}(\theta_h) = \text{Exh}(\theta_h) \).

The classical summable ideal \( J_{1/n} = J_h \) where \( h(n) = 1/(n+1) \), or \( h(0) = 1 \) and \( h(n) = 1/n \) if \( n > 0 \).

We know that there are tall \( F_\sigma \) ideals which are not summable ideals (see [19, Example 1.11.1] for the details): Consider \( J = \text{Fin}(\theta) \) where \( \theta \) is the following submeasure on \( \omega \):

\[
\theta(A) = \sum_{n < \omega} \min\{n, |A \cap [2^n, 2^{n+1})|\}.
\]

#### Density ideals. Let \( (P_n)_{n \in \omega} \) be a sequence of pairwise disjoint finite sets of \( \omega \) and let \( \bar{\mu} = (\mu_n)_{n \in \omega} \) be a sequences of measures, \( \mu_n \) is concentrated on \( P_n \) such that \( \limsup_{n \to \infty} \mu_n(P_n) > 0 \). The density ideal generated by \( \bar{\mu} \) is

\[
Z_{\bar{\mu}} = \{ A \subseteq \omega : \lim_{n \to \infty} \mu_n(A \cap P_n) = 0 \}.
\]

A density ideal

\[
Z_{\bar{\mu}} \text{ is tall iff } \lim_{n \to \infty} \max_{i \in P_n} \mu_n(\{i\}) = 0
\]

and density ideals are \( F_{\sigma, \delta} \) ideals. The function \( \theta_{\bar{\mu}}(A) = \sup_{n \in \omega} \mu_n(A \cap P_n) \) is an lsc submeasure on \( \omega \) and \( Z_{\bar{\mu}} = \text{Exh}(\theta_{\bar{\mu}}) \).

The density zero ideal \( Z = \{ A \subseteq \omega : \lim_{n \to \infty} |A \cap n|/n = 0 \} \) is a tall density ideal because let \( P_n = [2^n, 2^{n+1}) \) and \( \mu_n(A) = |A|/2^n \) for \( A \subseteq P_n \). It is easy to see that \( J_{1/n} \subseteq Z \).

We will need the following:
2.4. EXAMPLES

Lemma 2.4.1. ([19, Lemma 1.13.4]) If \( \mathcal{I} \) is a density ideal, then there is a sequence of measures \( \mu = (\mu_n)_{n \in \omega} \), \( \mu_n \) is concentrated on \( P_n \) such that \( \mu_n(P_n) \geq 1 \) for each \( n \) and \( \mathcal{I} = \mathcal{I}_g \).

**Erdős-Ulam ideals.** These ideals are special density ideals. Assume \( f : \omega \to (0, \infty) \) and \( \sum_{n \in \omega} f(n) = \infty \). Then the Erdős-Ulam ideal associated to \( f \) is

\[
\mathcal{E} U_f = \left\{ A \subseteq \omega : \lim_{n \to \infty} \frac{\sum_{k \in \Delta_n} f(k)}{\sum_{k \in n} f(k)} = 0 \right\}.
\]

It is easy to see that \( \mathcal{E} U_f \) is a tall \( F_{\sigma\delta} \) ideal. \( \mathcal{Z} \) is an Erdős-Ulam ideal because \( \mathcal{Z} = \mathcal{E} U_f \) where \( f \equiv 1 \).

I. Farah proved (see [19, Theorem 1.13.3]) that each Erdős-Ulam ideal \( \mathcal{E} U_f \) is equal to a density ideal \( \mathcal{Z} = \mathcal{E} U f \) where \( f \equiv 1 \).

Some tall \( F_{\sigma\delta} \) ideals:

1. Tall summable ideals (see above).

2. The eventually different ideals:

\[
\mathcal{E} D = \left\{ A \subseteq \omega \times \omega : \exists \ m \in \omega \ \forall \ n \in \omega \ \|(A)_n\| \leq m \right\} = \left\{ A \subseteq \omega \times \omega : \limsup_{n \to \infty} |(A)_n| < \infty \right\}
\]

where \( (A)_n = \{ k \in \omega : (n,k) \in A \} \), and \( \mathcal{E} D_{\text{fin}} = \mathcal{E} D | \Delta \) where \( \Delta = \{ (n,m) \in \omega \times \omega : m \leq n \} \). \( \mathcal{E} D \) and \( \mathcal{E} D_{\text{fin}} \) are not \( P \)-ideals.

3. The van der Waerden ideal:

\[
\mathcal{W} = \{ A \subseteq \omega : A \text{ does not contain arbitrary long arithmetic progressions} \}.
\]

Because of van der Waerden’s well-known theorem, \( \mathcal{W} \) is a proper ideal. \( \mathcal{W} \) is not a \( P \)-ideal. Szemerédi’s famous theorem says that \( \mathcal{W} \subseteq \mathcal{Z} \) (see [43]). The stronger statement \( \mathcal{W} \subseteq \mathcal{I}_{1/2} \) is a still open Erdős prize problem ($3000).

For several interesting theorems about this ideal see J. Flašková’s papers.

4. The random graph ideal:

\[
\text{Ran} = \langle \text{homogeneous subsets of the random graph} \rangle_{\text{id}}.
\]

Ran is not a \( P \)-ideal. For more details about this ideal see [38].

5. Solecki’s ideal: Let \( \text{CO}(2^\omega) \) be the family of clopen subsets of \( 2^\omega \) (clearly \( |\text{CO}(2^\omega)| = \omega \)), and let \( \Omega = \{ A \in \text{CO}(2^\omega) : \lambda(A) = 1/2 \} \) where \( \lambda \) is the Lebesgue measure on \( 2^\omega \). The ideal \( \mathcal{S} \) is generated by \( \{ I_x : x \in 2^\omega \} \) where \( I_x = \{ A \in \Omega : x \in A \} \).

\( \mathcal{S} \) is not a \( P \)-ideal, for more details and characterizations see [38].

Some tall \( F_{\sigma\delta} \) ideals:

1. Tall density ideals (see above).
(2) The ideal of nowhere dense subsets of \( \mathbb{Q} \): \( \text{Nwd} = \{ A \subseteq \mathbb{Q} : \text{int}(A) = \emptyset \} \) where int stands for the interior operation on subsets of the reals, and \( A \) is the closure of \( A \) in \( \mathbb{R} \). Nwd is not a P-ideal.

Some tall \( F_{\sigma \delta \sigma} \) (non P-)ideals:

1. The ideal \( \text{Conv} \) is generated by the infinite subsets of \( \mathbb{Q} \cap [0, 1] \) that are convergent in \( [0, 1] \), in other words
   \[
   \text{Conv} = \{ A \subseteq \mathbb{Q} \cap [0, 1] : |\text{accumulation points of } A \text{ (in } \mathbb{R})| < \omega \}.
   \]
2. The Fubini product of \( \text{Fin} \) by itself:
   \[
   \text{Fin} \otimes \text{Fin} = \{ A \subseteq \omega \times \omega : \forall \mathbb{N} n \in \omega \ |(A)_n| < \omega \}.
   \]

Some non tall ideals:

1. An important \( F_{\sigma} \) ideal:
   \[
   \text{Fin} \otimes \{\emptyset\} = \{ A \subseteq \omega \times \omega : \forall \mathbb{N} n \in \omega \ (A)_n = \emptyset \},
   \]
2. and its \( F_{\sigma \delta} \) brother:
   \[
   \{\emptyset\} \otimes \text{Fin} = \{ A \subseteq \omega \times \omega : \forall n \in \omega \ |(A)_n| < \omega \}.
   \]

Fubini products of ideals. We mentioned three important examples of Fubini products of ideals. In general, if \( I \) and \( J \) are ideals on \( \omega \) then their Fubini product is the following ideal on \( \omega \times \omega \):

\[
I \otimes J = \{ A \subseteq \omega \times \omega : \{ n : (A)_n \notin J \} \in I \}.
\]

Ideals generated by AD families. Let \( \mathcal{A} \subseteq [\omega]^\omega \) be an infinite AD family. Then \( \langle \mathcal{A} \rangle_{\text{id}} \) is not a P-ideal, and it is tall iff \( \mathcal{A} \) is a MAD family. If \( \mathcal{A} \) is a MAD family, then \( \langle \mathcal{A} \rangle_{\text{id}} \) is non analytic (see [36]) but it is meager by the following fact, by Theorem 2.1.3, and by the characterization of feebleness using \( \leq_{rb} \).

Fact 2.4.2. If \( \mathcal{A} \) is an almost disjoint family, then \( \text{Fin} \leq_{rb} \langle \mathcal{A} \rangle_{\text{id}} \).

Proof. We can assume that there is a partition of \( \omega \) in this AD family: \( \{ A_n : n \in \omega \} \subseteq \mathcal{A} \). Let \( A_n = \{ a_{nk} : k \in \omega \} \). Define \( f : \omega \to \omega \) by \( f(a_{ik}) = i + j \). \( f \) witnesses that \( \text{Fin} \leq_{rb} \langle \mathcal{A} \rangle_{\text{id}} \) because if \( X \in [\omega]^\omega \), then \( |f^{-1}[X] \cap A_n| = \omega \) for each \( n \), so \( f^{-1}[X] \notin \langle \mathcal{A} \rangle_{\text{id}} \). \( \square \)
Chapter 3

Idealized MADness

3.1 The almost-disjointness numbers of ideals

In this section we introduce a natural generalization of (M)AD families and the almost-disjointness number. All results without references in this section are from [23] and [22].

Let $I$ be an ideal on $\omega$. A family $A \subseteq I^{+}$ is $I$-almost-disjoint ($I$-AD in short), if $A \cap B \in I$ for each $\{A, B\} \in [A]^{2}$. An infinite $I$-AD family $A$ is an $I$-MAD family if for each $X \in I^{+}$ there exists an $A \in A$ such that $X \cap A \in I^{+}$, i.e. $A$ is $\subseteq$-maximal among the $I$-AD families.

Let $a(I)$ denote the minimum of cardinalities of $I$-MAD families, $a = a(\text{Fin})$.

Example 3.1.1. We claim that $a(\text{Fin} \otimes \{\emptyset\}) = a$. $a(\text{Fin} \otimes \{\emptyset\}) \leq a$ because if $A$ is a MAD family, then $\{A \times \omega : A \in A\}$ is a $\text{Fin} \otimes \{\emptyset\}$-MAD family. Conversely, if $B$ is a $\text{Fin} \otimes \{\emptyset\}$-MAD family, then there is an $f \in \omega^{\omega}$ such that $\{B \in B : B \cap \Delta_f \not\in \text{Fin} \otimes \{\emptyset\}\}$ is infinite where $\Delta_f = \bigcup_{n \in \omega} \{n\} \times [0, f(n)]$. It is easy to see that $a(\text{Fin} \otimes \{\emptyset\}) > \omega$ (or we can use the next proposition) so in particular $\{B \in B : |B \cap \Delta_f| = \omega\}$ is a MAD family on $\Delta_f$.

Remark 3.1.2. It is easy to see that $a(I \otimes J) \leq a(I)$ for all ideals $I$ and $J$.

Proposition 3.1.3. $a(I) > \omega$ for any $F_\sigma$ ideal $I$.

Proof. Let $I = \text{Fin}(\emptyset)$ for some lsc submeasure $\emptyset$, and assume that $\{A_n : n < \omega\} \subseteq I^{+}$ is an $I$-AD family. For each $n \in \omega$ let $B_n \subseteq A_n \setminus \cup \{A_m : m < n\}$ be finite (by the lsc property of $\emptyset$) such that $\emptyset(B_n) > n$, and put $B = \bigcup \{B_n : n \in \omega\} \in I^{+}$. Then clearly $\emptyset(B \cap A_n) < \infty$ i.e. $B \cap A_n \in I$ for each $n \in \omega$.

Theorem 3.1.4. $a(\mathbb{Z}_\mu) = \omega$ for any tall density ideal $\mathbb{Z}_\mu$.

Proof. Write $\bar{\mu} = (\mu_n)_{n \in \omega}$ where $\mu_n$ is concentrated on $P_n$. Using Lemma 2.4.1, we can assume that $\mu_n(P_n) \geq 1$ for each $n$.

Since $I$ is tall, $\lim_{n \to \infty} \max_{k \in P_n} \emptyset\{k\} = 0$ so $\lim_{n \to \infty} |P_n| = \infty$. It is easy to see that for all but finitely many $n$ there is a $P'_n \subseteq P_n$ such that $1 \leq \mu_n(P'_n) \leq 2$. Assume that if $n \geq N$ then there is such a $P'_n$. If $A = \bigcup \{P_n \setminus P'_n : n \geq N\} \in \mathbb{Z}_\mu^{+}$ then it will be an element of the desired countable $\mathbb{Z}_\mu$-MAD family.
Now for \( n \geq N \) choose a \( k_n \in \omega \) and a partition \( \{ P'_{n,k} : k < k_n \} \) of \( P'_n \) such that

\[
\lim_{n \to \infty} k_n = \infty \quad \text{and} \quad \forall n \geq N \quad \forall k < k_n \quad \mu_n(P'_{n,k}) \geq \frac{\mu_n(P'_n)}{2^{k+1}}.
\]

Put \( A_k = \bigcup\{ P'_{n,k} : n \geq N \land k < k_n \} \). We show that \( \{ A_k : k \in \omega \} \) is a \( \mathbb{Z}_\beta \)-MAD family on \( \omega \setminus A \) (i.e. a \( \mathbb{Z}_\beta \upharpoonright (\omega \setminus A) \)-MAD family) so if \( A \in \mathbb{Z}_\beta^+ \) then \( \{ A_k : k \in \omega \} \cup \{ A \} \) is a \( \mathbb{Z}_\beta \)-MAD family, if not then \( \{ A_k : k \in \omega \} \) is a \( \mathbb{Z}_\beta \)-MAD family.

If \( n \geq N \) and \( k < k_n \) then \( \mu_n(A_k \cap P'_n) = \mu_n(P'_{n,k}) \geq \frac{\mu_n(P'_n)}{2^{k+1}} \). Since for an arbitrary \( k \) for all but finitely many \( n \) we have \( k < k_n \) it follows that

\[
\limsup_{n \to \infty} \mu_n(A_k \cap P'_n) = \limsup_{n \to \infty} \mu_n(P'_{n,k}) \geq \limsup_{n \to \infty} \frac{\mu_n(P'_n)}{2^{k+1}} \geq \frac{1}{2^{k+1}} > 0,
\]

thus \( A_k \in \mathbb{Z}_\beta^+ \).

Assume that \( X \subseteq \omega \setminus A \) and \( X \in \mathbb{Z}_\beta^+ \). There is a \( \delta > 0 \) such that the set \( Y = \{ n \in \omega : \mu_n(X \cap P'_n) > \delta \} \) is infinite. For a large enough \( k \) we have \( 2^{k+1} < \frac{\delta}{2} \) so if \( n \in Y \) and \( k < k_n \) then

\[
\mu_n(P'_n \setminus \bigcup\{ P'_{n,i} : i \leq k \}) \leq \frac{\mu_n(P'_n)}{2^{k+1}} \leq \frac{2}{2^{k+1}} < \frac{\delta}{2}.
\]

So for each large enough \( n \in Y \) there is \( i_n \leq k \) such that \( \mu_n(X \cap P'_{n,i_n}) > \frac{\delta}{2^{k+1}} \).

Then \( i_n = i \) for infinitely many \( n \in Y \) so \( \limsup_{n \to \infty} \mu_n(X \cap A_i) \geq \frac{\delta}{2^{k+1}} \), in particular \( X \cap A_i \in \mathbb{Z}_\beta^+ \).

By the following theorem of Farah we can characterize analytic P-ideals with countable \( \mathcal{J} \)-MAD families.

**Theorem 3.1.5.** (reformulation of [19, Theorem 5.11.1]) Let \( \mathcal{J} \) be an analytic P-ideal. Then \( a(\mathcal{J}) > \omega \) if, and only if \( \mathcal{J} \) is \( F_\sigma \).

**Problem 3.1.6.** Is there any reasonable characterization of Borel (or even analytic) ideals with \( a(\mathcal{J}) = \omega \)?

At this point, it is not clear how large an \( \mathcal{J} \)-AD family can be.

**Observation 3.1.7.** Assume that \( \mathcal{J} \) and \( \mathcal{J}' \) are ideals on \( \omega \), \( \mathcal{J} \leq_{nk} \mathcal{J}' \) witnessed by a function \( f : \omega \to \omega \). If \( \mathcal{A} \) is an \( \mathcal{J} \)-AD family then, \( (f^{-1})'[\mathcal{A}] = \{ f^{-1}[A] : A \in \mathcal{A} \} \) is a \( \mathcal{J}' \)-AD family.

Using that \( \text{Fin} \leq_{nk} \mathcal{J} \) for each meager ideal \( \mathcal{J} \) (see Theorem 2.1.3 and the note before Proposition 2.3.1), we obtain that there are \( \mathcal{J} \)-MAD families of cardinality \( \epsilon \) for each meager ideal \( \mathcal{J} \).

For any meager ideal \( \mathcal{J} \) let \( \tilde{a}(\mathcal{J}) \) denote the minimum of cardinalities of uncountable \( \mathcal{J} \)-MAD families. Clearly \( a(\mathcal{J}) > \omega \) implies \( a(\mathcal{J}) = \tilde{a}(\mathcal{J}) \).
3.1. THE ALMOST-DISJOINTNESS NUMBERS OF IDEALS

Example 3.1.8. \( \alpha(\{\emptyset\} \otimes \text{Fin}) = \omega \) because the columns in \( \omega \times \omega \) form a \( \{\emptyset\} \otimes \text{Fin-MAD} \) family. We claim that \( \bar{\alpha}(\{\emptyset\} \otimes \text{Fin}) = a \). If \( A \) is a MAD-family, then we can consider its copies on all columns and we obtain an uncountable \( \{\emptyset\} \otimes \text{Fin-MAD} \) family. Conversely, if \( B \) is an uncountable \( \{\emptyset\} \otimes \text{Fin-MAD} \) family, then there is a column \( \{n\} \times \omega \) such that \( \{B \in B : |(B)_n| = \omega\} \) is infinite and hence it is a MAD family on this column.

After clarifying the case of countable \( \aleph \)-MAD families at least for analytic \( P \)-ideals, it is natural to ask: what can we say about lower bounds of \( \bar{\alpha} \)?

Theorem 3.1.9. \( b \leq \bar{\alpha}(\aleph) \) provided that \( \aleph \) is an analytic \( P \)-ideal.

Proof. \( \aleph = \text{Exh}(\vartheta) \) for some finite lsc submeasure \( \vartheta \). Let \( A \) be an uncountable \( \aleph \)-AD family of cardinality smaller than \( b \). We show that \( A \) is not maximal.

There exists an \( \varepsilon > 0 \) such that the set \( A_{\varepsilon} = \{A \in A : \|A\|_\vartheta > \varepsilon\} \) is uncountable. Let \( A' = \{A_n : n \in \omega\} \subseteq A_{\varepsilon} \) be a set of pairwise distinct elements of \( A_{\varepsilon} \). We can assume that these sets are pairwise disjoint. For each \( A \in A \setminus A' \) choose a function \( f_A \in \omega^{< \omega} \) such that

\[ (A \cap A_n) \setminus f_A(n) < 2^{-n} \text{ for each } n \in \omega. \]

Using the assumption \( |A| < b \) there exists a strictly increasing function \( f \in \omega^{< \omega} \) such that \( f_A \leq f \) for each \( A \in A \setminus A' \). For each \( n \) pick \( g(n) > f(n) \) such that \( \vartheta(A_n \cap [f(n), g(n)]) > \varepsilon \), and let

\[ X = \bigcup_{n \in \omega} (A_n \cap [f(n), g(n)]). \]

Clearly, \( X \in \mathcal{Z}_{\vartheta}^+ \) because for each \( n < \omega \) there is \( m \) such that \( A_m \cap [f(m), g(m)] \subseteq X \setminus n \) and so \( \vartheta(X \setminus n) \geq \vartheta(A_n \cap [f(m), g(m)]) > \varepsilon \), hence \( \|X\|_\vartheta \geq \varepsilon \).

We have to show that \( X \cap A \in \mathcal{Z}_{\vartheta} \) for each \( A \in A \). If \( A = A_n \) for some \( n \) then \( X \cap A = A_n \cap [f(n), g(n)] \), i.e. the intersection is finite.

Assume now that \( A \in A \setminus A' \). Let \( \delta > 0 \). We show that if \( k \) is large enough then \( \vartheta((A \cap X) \setminus k) < \delta \).

There is \( N \in \omega \) such that \( 2^{-N+1} < \delta \) and \( f_A(n) \leq f(n) \) for each \( n \geq N \). Let \( k \) be so large that \( k \) contains the finite set \( \bigcup_{n < N} [f(n), g(n)] \).

Now we have

\[ (X \cap A) \setminus k = \bigcup_{n \in \omega} (A_n \cap A \cap [f(n), g(n)]) \setminus k \]

and \( (A_n \cap A \cap [f(n), g(n)]) \setminus k = \emptyset \) if \( n < N \) so

\[ (X \cap A) \setminus k = \bigcup_{n \geq N} (A_n \cap A \cap [f(n), g(n)]) \setminus k \subseteq \bigcup_{n \geq N} ((A_n \cap A) \setminus f_A(n)). \]

Thus by \((\ast_A)\) we have

\[ \vartheta((X \cap A) \setminus k) \leq \sum_{n \geq N} \vartheta(A_n \cap A \setminus f_A(n)) \leq \sum_{n \geq N} \frac{1}{2^n} = 2^{-N+1} < \delta. \]
Notice that if $A \subseteq [\omega]^{\omega}$ is an infinite AD family, then there is a tall ideal $\mathcal{I}$ such that $\mathcal{A}$ is $\mathcal{I}$-MAD. So Theorem 3.1.9 does not hold for an arbitrary (tall) ideal on $\omega$. Moreover, according to the following theorem of J. Brendle, Theorem 3.1.9 does not hold even for all tall $\mathcal{F}_\sigma$ ideals:

**Theorem 3.1.10.** (see [12]) Assume GCH. Then $V^{\mathcal{D}_\omega} \models a(\mathcal{D}_{\text{fin}}) < b$.

We present two more lower bounds for $a(\mathcal{I})$ in special cases.

**Proposition 3.1.11.** $b \leq a(\text{Fin} \otimes \text{Fin})$.

**Proof.** Let $A \subseteq (\text{Fin} \otimes \text{Fin})^+$ be a $\text{Fin} \otimes \text{Fin}$-AD family with $|A| < b$. We show that $A$ is not maximal. Fix a countable family $A_0 = \{A_n : n \in \omega\} \subseteq A$. We can assume that these sets are pairwise disjoint. For each $A \in A \setminus A_0$ let $g_A \in \omega^\omega$ be the following function:

$$g_A(n) = \min \{k \in \omega : \forall \ell \geq k \ |(A \cap A_n)_\ell| < \omega\}.$$

Because of our assumption on the cardinality of $A$, there is a strictly increasing $g \in \omega^\omega$ such that $g_A \leq^* g$ for each $A \in A \setminus A_0$. Since $A_n \in (\text{Fin} \otimes \text{Fin})^+$ we can assume that $(A_n)_{g(n)}$ is infinite for each $n \in \omega$.

Now for each $A \in A \setminus A_0$ let $f_A \in \omega^\omega$ be the following function:

$$f_A(n) = \begin{cases} 0 & \text{if } (A \cap A_n)_{g(n)} = \emptyset \text{ or } |(A \cap A_n)_{g(n)}| = \omega, \\ \max |(A \cap A_n)_{g(n)}| & \text{if } 0 < |(A \cap A_n)_{g(n)}| < \omega. \end{cases}$$

As in the case of $g_A$'s, we can choose a strictly increasing $f \in \omega^\omega$ such that $f_A \leq^* f$ for each $A \in A \setminus A_0$. Finally, let

$$B = \bigcup_{n \in \omega} \{g(n)\} \times ((A_n)_{g(n)} \setminus f(n)) \in (\text{Fin} \otimes \text{Fin})^+.$$

It is easy to see that $B \cap A \in \text{Fin} \otimes \emptyset \subseteq \text{Fin} \otimes \text{Fin}$ for each $A \in A$. \qed

Before the next proposition we recall an important characterization of add($\mathcal{M}$):

**Theorem 3.1.12.** ([3, Theorem 2.2.6]) $\text{add}(\mathcal{M}) > \kappa$ if, and only if for every family $\{D_\alpha : \alpha < \kappa\}$ of dense open subsets of $\mathbb{R}$ and for every countable dense $X \subseteq \mathbb{R}$, there exists a dense $Y \subseteq X$ such that $Y \subseteq^* D_\alpha$ for each $\alpha < \kappa$.

**Proposition 3.1.13.** $\alpha(\text{Nwd}) = \omega$ and add($\mathcal{M}$) $\leq \bar{a}(\text{Nwd})$.

**Proof.** $\{[n, n+1) \cap \mathbb{Q} : n \in \mathbb{Z}\}$ is a countable Nwd-MAD family.

Assume $\mathcal{A} \subseteq \text{Nwd}^+$ is an uncountable Nwd-AD family with cardinality less than add($\mathcal{M}$). There is an interval $(r, q)$ with rational endpoints such that $\mathcal{A}' = \{A \in \mathcal{A} : (r, q) \subseteq \bar{A}\}$ is uncountable. As usual, first we fix a countable subfamily $\mathcal{A}_0 = \{A_n : n \in \omega\} \subseteq \mathcal{A}'$, and we assume that the elements of $\mathcal{A}_0$ are pairwise disjoint. For each $A \in \mathcal{A} \setminus \mathcal{A}_0$ and $n \in \omega$ there is a dense open $D_n^A \subseteq (r, q)$ such that $A \cap A_n \cap D_n^A = \emptyset$.

Using the characterization of add($\mathcal{M}$), for every $n$ we can find a $B_n \subseteq A_n$ such that $B_n$ is dense in $(r, q)$ and $B_n \subseteq^* D_n^A$ for each $A \in \mathcal{A} \setminus \mathcal{A}_0$. These $B_n$’s will be the “columns.”
Let $B_n = \{b^n_k : k \in \omega\}$ be enumerations and for each $A \in A \setminus A_0$ let $f_A \in \omega^\omega$ be the following function:

$$f_A(n) = \min\{k : \forall \ell \geq K b^n_\ell \notin A\}.$$

Since $\text{add}(\mathbb{M}) \leq b$, there is an $f \in \omega^\omega$ such that $f_A \leq^* f$ for each $A \in A \setminus A_0$. Fix an enumeration $\{(r_n, q_n) : n \in \omega\}$ of open subintervals of $(r, q)$ with rational endpoints. Finally, for each $n$ choose a $k_n \geq f(n)$ such that $b^n_{k_n} \in (r_n, q_n)$, then $\{b^n_k : n \in \omega\} \in \text{Nwd}^+$ witnesses that $A$ is not maximal.

In the rest of this section we present a natural method of proving inequalities between almost-disjointness numbers of ideals. Assume $\mathcal{I}$ and $\mathcal{J}$ are ideals on $\omega$ and there is a regular embedding $e : \mathcal{P}(\omega)/\mathcal{I} \to \mathcal{P}(\omega)/\mathcal{J}$, that is, $e$ is an embedding in the sense of Boolean algebras and the image of any maximal antichains of $\mathcal{P}(\omega)/\mathcal{I}$ (i.e. $\mathcal{I}$-MAD families) is also maximal in $\mathcal{P}(\omega)/\mathcal{J}$. In other words, $e$ is a complete embedding in the sense of forcing notions but avoiding confusions when we are working with Boolean algebras, we will talk about regular embeddings.

In this case clearly $a(\mathcal{I}) \geq a(\mathcal{J})$ and $\bar{a}(\mathcal{I}) \geq \bar{a}(\mathcal{J})$. How could we find regular embeddings? The simplest way is to find a function $f : \omega \to \omega$ such that

$$f^{-1}[\cdot] : \mathcal{P}(\omega)/\mathcal{I} \to \mathcal{P}(\omega)/\mathcal{J} \text{ is a regular embedding} \quad \text{(reg)}$$

where we do not distinguish equivalence classes and representatives. We know that an $f$ has property (reg) iff the following hold:

(a) $A \in \mathcal{I}$ iff $f^{-1}[A] \in \mathcal{J}$, i.e. $f$ witnesses $\mathcal{I} \leq^* \mathcal{J}$;

(b) $\forall X \in \mathcal{J}^+ \exists Y \in \mathcal{J}^+ \forall Z \in (\mathcal{J} \upharpoonright Y)^+ f^{-1}[Z] \cap X \in \mathcal{J}^+$ (This $Y$ is called the pseudo-projection of $X$.)

If there exists such an $f$, then we write $\mathcal{I} \leq^r \mathcal{J}$.

**Proposition 3.1.14.** $\text{Fin} \leq^r \mathcal{J}$ holds for $\mathcal{J} = Z_{\mu}, \mathcal{E} \mathcal{D}, \mathcal{E} \mathcal{D}_\text{fin}, \mathcal{W}, \text{Fin} \otimes \text{Fin}, \text{Fin} \otimes \{\emptyset\}, \{\emptyset\} \otimes \text{Fin}$. In particular, $\bar{a}(\mathcal{J}) \leq a(\mathcal{J})$ for these ideals.

**Proof.** $\mathcal{J} = Z_{\mu}$: Let $\mu_n$ be concentrated on $P_n$. By Lemma 2.4.1 we can assume that $\mu_n(P_n) \geq 1$ for each $n$. Let $f : \omega \to \omega$ be the function defined by $f^{-1}[\{0\}] = \omega \setminus \bigcup\{P_n : n \geq 1\}$ and $f^{-1}[\{n\}] = P_n$ for $n > 0$. Clearly, $f$ witnesses $\text{Fin} \leq^* Z_{\mu}$.

Let $A$ be a (Fin-)MAD family. We show that $(f^{-1})''[A] = \{f^{-1}[A] : A \in A\}$ is a $Z_{\mu}$-MAD family. To show the maximality, let $X \in Z_{\mu}^+$ be arbitrary, $\limsup_{n \to \infty} \mu_n(X \cap P_n) = \varepsilon > 0$. Thus $J = \{n \in \omega : \mu_n(X \cap P_n) > \varepsilon/2\}$ is infinite. So there is an $A \in A$ such that $A \cap J$ is infinite. Then $f^{-1}[A] \in (f^{-1})''[A]$ and $X \cap f^{-1}[A] \in Z_{\mu}^+$ because there are infinitely many $n$ such that we have $P_n \subseteq f^{-1}[A]$ and $\mu_n(X \cap P_n) > \varepsilon/2$.

$\mathcal{J} = \mathcal{E} \mathcal{D}, \mathcal{E} \mathcal{D}_\text{fin}, \text{Fin} \otimes \text{Fin}, \text{Fin} \otimes \{\emptyset\}$: It is easy to see that the projection onto the first coordinate, i.e. $f(n, m) = n$ generates regular embeddings.

$\mathcal{J} = \{\emptyset\} \otimes \text{Fin}$: In this case, the projection onto the second coordinate works.

$\mathcal{J} = \mathcal{W}$ (Flašková): It is easy to see that the function $f : \omega \to \omega$ defined by $f^{-1}(\{n\}) = [2^n, 2^{n+1})$ generates a regular embedding.
CHAPTER 3. IDEALIZED MADNESS

It is easy to see that if \( X \in \mathcal{I}^+ \), then \( a(J) \leq a(J \upharpoonright X) \) and \( \tilde{a}(J) \leq \tilde{a}(J \upharpoonright X) \) because if \( \mathcal{A} \) is an \( \mathcal{J} \upharpoonright X \)-MAD family, then \( \mathcal{A} \cup \{ \omega \setminus X \} \) is an \( \mathcal{J} \)-MAD. In particular, \( a(\mathcal{E}D) \leq a(\mathcal{E}D_{\text{fin}}) \).

We finish this section with a list of related questions:

**Question 3.1.15.**
- Does \( \tilde{a}(\mathcal{J}) \leq a \) hold for each analytic \( \mathcal{P} \)-ideal \( \mathcal{J} \)?
- Is there any reasonable lower bound of \( a(\mathcal{W}) \)?
- Do \( b \leq \tilde{a}(\text{Nwd}) \) or \( \tilde{a}(\text{Nwd}) \leq a \) hold?
- What can we say about the almost-disjointness numbers of \( \text{Ran}, \text{S}, \) and \( \text{Conv} \)?
- Is there any nontrivial upper bound for \( a(\mathcal{J}) \) when \( \mathcal{J} \) is an \( F_\sigma \) ideal?

### 3.2 Forcing-indestructible \( \mathcal{J} \)-MAD families

If \( \mathbb{P} \) is a forcing notion and \( \mathcal{A} \) is a MAD family, then we say that \( \mathcal{A} \) is **\( \mathbb{P} \)-indestructible** if \( \mathbb{P} \vdash \mathcal{A} \) is a MAD family, in other words, the ideal \( \langle \mathcal{A} \rangle_{\text{id}} \) is \( \mathbb{P} \)-indestructible. Kunen [35, Ch.VIII, Theorem 2.3] constructed a Cohen-indestructible MAD family assuming CH. His method was later extended to other forcing notions and there were proved many similar indestructibility results assuming typically that a cardinal invariant of the continuum is equal to \( c \).

In general, if \( \mathbb{P} \) is a classical forcing notion (such as the Cohen, the random, the Sacks, the Laver, or the Miller forcing), then the existence of a \( \mathbb{P} \)-indestructible MAD family (in ZFC) is still an open problem.

We can generalize this notion for \( \mathcal{J} \)-MAD families: If \( \mathbb{P} \) is a forcing notion, \( \mathcal{J} \) is a Borel ideal, and \( \mathcal{A} \) is an \( \mathcal{J} \)-MAD family, then \( \mathcal{A} \) is **\( \mathbb{P} \)-indestructible** if \( \mathbb{P} \vdash \langle \mathcal{A} \rangle_{\text{id}} \) is \( \mathcal{J} \)-MAD.

**Observation 3.2.1.** The property “\( \mathcal{J} \) is a Borel ideal, and \( \mathcal{A} \) is a countable \( \mathcal{J} \)-MAD family” is \( \Pi^1_1 \) so it is absolute for transitive models. In particular, countable \( \mathcal{J} \)-MAD families are \( \mathbb{P} \)-indestructible for each forcing notion \( \mathbb{P} \).

Unfortunately, the property \( \mathcal{J} \leq_{\text{rk}} \mathcal{J} \) on Borel ideals seems to be \( \Sigma^1_4 \). We introduce a stronger version of this pre-order: we want to guarantee that the pseudo-projection of any set \( X \) is “computable” form \( X \).

**Definition 3.2.2.** Assume \( \mathcal{I} \) and \( \mathcal{J} \) are Borel ideals. We say that a functions \( f : \omega \to \omega \) generates a Borel-regular embedding if there is a Borel-measurable \( F : \mathcal{P}(\omega) \to \mathcal{P}(\omega) \) such that the following hold

\[
(a') \ f \text{ witnesses } \mathcal{I} \leq_{\text{rk}} \mathcal{J}, \text{ and } \forall X \in \mathcal{J}^+ \ F(X) \in \mathcal{J}^+; \\
(b') \ \forall X \in \mathcal{J}^+ \ \forall Z \in (\mathcal{J} \upharpoonright F(X))^+ \ f^{-1}[Z] \cap X \in \mathcal{J}^+.
\]

If there exists such an \( f \), then we write \( \mathcal{I} \leq_{\text{rk}}^{\text{B-r}} \mathcal{J} \).
3.2. FORCING-INDESTRUCTIBLE $\mathcal{J}$-MAD FAMILIES

It is not hard to see that this pre-order on Borel ideals is $\Sigma^1_2$ so it is absolute between any pair $M \subseteq N$ of transitive models with $\omega_1^M \subseteq M$.

In Proposition 3.1.14 we actually proved that $\text{Fin} \leq_{\text{RK}}^M \mathcal{J}$ holds for all $\mathcal{J}$ in the list. For example, $\text{Fin} \leq_{\text{RK}}^M \text{Fin} \otimes \text{Fin}$ because let $f$ be the projection onto the first coordinate and let $F(X) = \{ n \in \omega : |(X)_n| = \omega \}$ for each $X \subseteq \omega \times \omega$. Similarly, $\text{Fin} \leq_{\text{RK}} \mathcal{E}D$ because let $f$ be the projection as in the previous case and let

$$F(X) = \{ \min \{ n \geq k : |(X)_n| \geq k \} : k \in \omega \}
$$

if $X \in \mathcal{E}D^+$ and let $F(X) = \emptyset$ if $X \in \mathcal{E}D$.

At this moment, I am unable to show that this pre-order is really stronger than $\leq^r_{\text{RK}}$ (maybe it is not):

**Question 3.2.3.** Do there exist Borel ideals $\mathcal{I}$ and $\mathcal{J}$ such that $\mathcal{I} \leq_{\text{RK}}^M \mathcal{J}$ but $\mathcal{J} \not\leq_{\text{RK}}^M \mathcal{I}$.

Using the absoluteness of this pre-order on Borel ideals, we have a method of constructing $\mathcal{P}$-indestructible $\mathcal{J}$-MAD families form $\mathcal{P}$-indestructible MAD families:

**Fact 3.2.4.** Assume $\mathcal{J}$ and $\mathcal{J}$ are Borel ideal, $f$ witnesses $\mathcal{J} \leq_{\text{RK}}^M \mathcal{J}$, $\mathcal{P}$ is a forcing notion, and $\mathcal{A}$ is a $\mathcal{P}$-indestructible $\mathcal{J}$-MAD family. Then $\{ f^{-1}[A] : A \in \mathcal{A} \}$ is a $\mathcal{P}$-indestructible $\mathcal{J}$-MAD family.

In particular, under CH, there are Cohen-indestructible $\mathcal{J}$-MAD families for $\mathcal{J} = \mathcal{Z}_\alpha, \mathcal{E}D, \mathcal{E}D_{\text{fin}}, \mathcal{W}, \text{Fin} \otimes \text{Fin}, \text{Fin} \otimes \{ \emptyset \}, \{ \emptyset \} \otimes \text{Fin}$.

Although, I do not no whether $\text{Fin} \leq_{\text{RK}}^M \mathcal{J}$ holds for all $\mathcal{P}_\sigma$ ideals or for all analytic $\mathcal{P}$-ideals, we can construct Cohen-indestructible $\mathcal{J}$-MAD families for these $\mathcal{J}$’s by modifying Kunen’s proof.

**Theorem 3.2.5.** Assume CH and let $\mathcal{J}$ be an $\mathcal{P}_\sigma$ ideal or an analytic $\mathcal{P}$-ideal. Then there is an uncountable Cohen-indestructible $\mathcal{J}$-MAD family.

**Proof.** The construction for $\mathcal{P}_\sigma$ ideals is similar to the other case but a bit simpler so we prove the theorem for analytic $\mathcal{P}$-ideals only.

We will define the uncountable Cohen-indestructible $\mathcal{J}$-MAD family $\{ A_\alpha : \alpha < \omega_1 \} \subseteq \mathcal{J}^+$ by recursion on $\alpha \in \omega_1$. The family $\{ A_\alpha : \alpha < \omega_1 \}$ will be $\text{Fin}-\text{AD}$ as well. Our main concern is that we do have that $\{ A_\xi : \xi < \alpha \}$ is not maximal for $\alpha < \omega_1$ because it is not automatic (e.g. $\text{a}(\mathcal{Z}) = \omega_1$).

Let $\mathcal{J} = \text{Exh}(\theta)$ be an analytic $\mathcal{P}$-ideal. We can assume that $\|\omega\|_\theta = 1$. Let $\{ (p_\alpha, \dot{X}_\alpha, \delta_\alpha) : \omega \leq \alpha < \omega_1 \}$ be an enumeration of all triples $(p, \dot{X}, \delta)$ such that $p \in \mathcal{C}$, $\dot{X}$ is a nice $\mathcal{C}$-name for a subset of $\omega$, and $\delta$ is a positive rational number.

Partition $\omega$ into infinite sets $\{ A_n : n < \omega \}$ such that $\|A_n\|_\theta = 1$ for each $n < \omega$.

Assume $\alpha \geq \omega$ and we have $A_\xi \in \mathcal{J}^+$ for $\xi < \alpha$ such that $\{ A_\xi : \xi < \alpha \}$ is an AD so especially an $\mathcal{J}$-AD family.

**Claim.** There is $X \in \mathcal{J}^+$ such that $|X \cap A_\xi| < \omega$ for each $\xi < \alpha$.

**Proof of the Claim.** Write $\alpha = \{ \xi_i : i < \omega \}$. Recursion on $m \in \omega$ we can choose $F_m \in \{ A_{\xi_k} \}_{k < m}$ for some $k_m \in \omega$ such that $\theta(F_m) \geq 1/2$ and $F_m \cap (A_{\xi_k} \cup A_{\xi_j} \cup \cdots \cup A_{\xi_0}) = \emptyset$.

Why? Assume that $(F_i)_{i < m}$ are chosen. Pick a $k_m \in \omega \setminus \{ \xi_i : i \leq m \}$ and let $D \in \omega$ such
that $A_{k_n} \cap \{A_{\xi_i} : i \leq m\} \subseteq D$. Since $\vartheta(A_{k_n} \setminus D) \geq 1$ there is $F_m \in [A_{k_n} \setminus D]^{<\omega}$ with $\vartheta(F_m) \geq 1/2$.

Let $X = \bigcup \{F_m : m < \omega\}$. Then $|A_{\xi} \cap X| < \omega$ for $\xi < \alpha$ and $|X|_{\vartheta} \geq 1/2$.  

If $p_a$ does not force (a) and (b) below then let $A_a$ be $X$ from the Claim.

\[(a)_a \ | \hat{X}_a|_{\vartheta} > \delta_a, \quad (b)_a \ \forall \xi < \alpha \ \hat{X}_a \cap A_{\xi} \in \mathcal{J}.\]

Assume $p_a \Vdash (a)_a \land (b)_a$. Let $\{B_k^a : k \in \omega\} = \{A_{\xi} : \xi < \alpha\}$ and $\{p_k^a : k \in \omega\} = \{p' \in \mathbb{C} : p' \leq p_a\}$ be enumerations. Clearly, for each $k \in \omega$ we have

$$p_k^a \Vdash \|\hat{X}_a \setminus \bigcup \{B_l^a : l \leq k\}\|_{\vartheta} > \delta_a,$$

so we can choose a $q_k^a \leq p_k^a$ and a finite $a_k^a \subseteq \omega$ such that $\vartheta(q_k^a) > \delta_a$ and $q_k^a \Vdash a_k^a \subseteq (\hat{X}_a \setminus \bigcup \{B_l^a : l \leq k\}) \setminus k$. Let $A_a = \bigcup \{a_k^a : k \in \omega\}$. Clearly $A_a \in \mathcal{J}^+$ and $\{A_{\xi} : \xi \leq \alpha\}$ is an AD family.

Thus we obtain the AD family $\mathcal{A} = \{A_a : \alpha < \omega_1\} \subseteq \mathcal{J}^+$.

We show that $\mathcal{A}$ is a Cohen-indestructible $\mathcal{J}$-MAD. Assume otherwise that there is an $\alpha$ such that $p_a \Vdash \|\hat{X}_a\|_{\vartheta} > \delta_a \land \forall \xi < \omega_1 \ \hat{X}_a \cap A_{\xi} \in \mathcal{J}$, in particular $p_a \Vdash (a)_a \land (b)_a$. There is a $p_k^a \leq p_a$ and an $N$ such that $p_k^a \Vdash 0^\mathcal{A} \setminus \vartheta((\hat{X}_a \cap A_{\alpha}) \setminus N) < \delta_a$. We can assume that $k \geq N$ so $p_k^a \Vdash \vartheta((\hat{X}_a \cap A_{\alpha}) \setminus k) < \delta_a$. By the choice of $q_k^a$ and $a_k^a$ we have $q_k^a \Vdash a_k^a \subseteq (\hat{X}_a \cap A_{\alpha}) \setminus k$, so $q_k^a \Vdash \vartheta((\hat{X}_a \cap A_{\alpha}) \setminus k) > \delta_a$, contradiction.

We can also generalize the construction of Sacks-indestructible MAD families (see [7, page 476]).

**Theorem 3.2.6.** Assume CH. Let $\mathcal{J}$ be an $F_\sigma$ ideal or an analytic $\mathbb{P}$-ideal and let $\mathbb{P}$ be an $\omega^\omega$-bounding proper forcing notion of cardinality $\omega_1$. Then there is an uncountable $\mathbb{P}$-indestructible $\mathcal{J}$-MAD family.

**Proof.** As in the previous proof we will construct the desired $\mathcal{J}$-MAD family $\{A_a : \alpha < \omega_1\}$ by recursion on $\alpha$ such that this family will be an AD family as well. Assume $\mathcal{J} = \text{Exh}(\vartheta)$ is an analytic $\mathbb{P}$-ideal and $\|\omega\|_{\vartheta} = 1$.

It is well-known that under CH, a proper forcing of size $\mathfrak{c}$ adds at most $\mathfrak{c}$ many new reals. Let $\{(p_a, \hat{X}_a) : \omega \leq \alpha < \omega_1\}$ be an enumeration of all pairs such that $p \in \mathbb{P}$ and $\hat{X}$ is a $\mathbb{P}$-name for a subset of $\omega$ from our list of essentially different names of reals.

Let $\{A_n : n \in \omega\}$ be a partition of $\omega$ such that $|A_n|_{\vartheta} = 1$ for each $n \in \omega$, and assume $\{A_{\xi} : \xi < \alpha\}$ is done.

If $p_a \Vdash \hat{X}_a \in \mathcal{J}$ or $p_a \not\Vdash \forall \xi < \alpha \ \hat{X}_a \cap A_{\xi} \in \mathcal{J}$, then let $A_a = X$ form the Claim of previous proof. Assume that $p_a$ forces $\hat{X}_a \in \mathcal{J}^+$ and that $\hat{X}_a \cap A_{\xi} \in \mathcal{J}$ for each $\xi < \alpha$. Then let $\{B_k^a : k \in \omega\} = \{A_{\xi} : \xi < \alpha\}$ be an enumeration, and let $C_0^a = B_0^a$, $C_k^a = B_k^a \setminus (B_0^a \cup B_1^a \cup \cdots \cup B_{k-1}^a)$ for $k > 0$. Furthermore, fix enumerations $C_k^a = \{c_k^a(m) : m \in \omega\}$ for each $k$.

Since $p_a \Vdash \forall k \in \omega \ \hat{X}_a \cap C_k^a \in \mathcal{J}$, there is a $\mathbb{P}$-name $\hat{f}$ for an element of $\omega^\omega$ such that

$$p_a \Vdash \forall k \in \omega \ \vartheta(\hat{X}_a \cap \{c_k^a(m) : m \geq \hat{f}(k)\}) < 2^{k-2} \|\hat{X}_a\|_{\vartheta}.$$
By the $\omega^\omega$-bounding property of $\mathbb{P}$ there is a $q_a \leq p_a$ and a $g \in \omega^\omega$ (in the ground model) such that $q_a \Vdash f \leq g$. Let $A_a = \{c^a_\xi(m) : k \in \omega, m \leq g(k)\}$. Clearly, $\{A_\xi : \xi \leq \alpha\}$ is an AD family.

Using the $\sigma$-subadditivity of $\|\cdot\|$, we show that $q_a \Vdash \|X_a\|_\varnothing < \|X_a\|_\varnothing/2$:

$$q_a \Vdash \|X_a\|_\varnothing = \varnothing\left(\bigcup_{k \in \omega} (X_a \cap \{c^a_k(m) : m > g(k)\})\right) \leq \sum_{k \leq \omega} \varnothing(X_a \cap \{c^a_k(m) : m > g(k)\}) < \sum_{k \leq \omega} 2^{-k-2} \|X_a\|_\varnothing = \|X_a\|_\varnothing/2.$$  

It implies that $q_a \Vdash \|X_a\|_\varnothing < \|X_a\|_\varnothing/2$ so by the subadditivity of $\|\cdot\|$, we obtain that $q_a \Vdash \|X_a \cap A_a\|_\varnothing > \|X_a\|_\varnothing/2 > 0$, in particular $A_a \in \mathcal{I}^+$. We are done by an easy argument similar to the end of the previous proof.

The proof for $F_\sigma$ ideals is a natural modification of this proof. $\square$

**Question 3.2.7.** Assume that $\mathcal{I}$ is as above. Can we construct an uncountable Miller-indestructible $\mathcal{I}$-MAD family under CH?

### 3.3 Forcing-indestructible extensions of $\mathcal{I}$-AD families

The motivation of this section is based on the following theorem and the related question after that.

**Theorem 3.3.1.** [27, Theorem 11] In $V^{C_{\omega_1}}$ there are AD families $\mathcal{A}$ and $\mathcal{B}$ such that, in any generic extension of $V^{C_{\omega_1}}$ by a ccc forcing notion $\mathbb{P}$ such that $\mathbb{P} \in V$, $\mathcal{A}$ cannot be extended to a Cohen-indestructible MAD family and $\mathcal{B}$ cannot be extended to a random-indestructible MAD family.

L. Soukup asked if any AD family could be extended to a Cohen-indestructible MAD family in a ccc forcing extension. Using Kunen’s idea we give a general positive answer for this question. (In a special case see [20].)

We will need the following easy lemma:

**Lemma 3.3.2.** The formula which says that $\mathbb{P}$ is a forcing notion, $p \in \mathbb{P}$, $X$ is a nice $\mathbb{P}$-name for a subset of $\omega$, $\varnothing$ is an lsc submeasure on $\omega$, $e \in \mathbb{P}$, and $p \Vdash \varnothing(X) > e$ (or $p \Vdash \|X\|_\varnothing > e$) is absolute for transitive models.

**Proof.** (Sketch:) We show that this formula can be expressed with bounded quantifiers only, i.e. it is equivalent to a $\Delta_0$ formula in ZFC. Let $X = \bigcup\{k \times A_k : k \in \omega\}$ where $A_k \subseteq \mathbb{P}$ is an antichain for each $k$. The set $Y_k = \{Y \in [\omega]^{<\omega} : \varnothing(Y) > e\}$ is definable form $\varnothing$ and $e$ by a $\Delta_0$ formula. We know that

$$(p \Vdash \varnothing(X) > e) \iff (p \Vdash \exists Y \in Y_k Y \subseteq X).$$

The negation of the second formula is equivalent to $\exists q \leq p \forall Y \in Y_k q \Vdash Y \not\subseteq X$ so it is enough to show that $q \Vdash Y \not\subseteq X$ is equivalent to a $\Delta_0$ formula. It is easy to see that

$$(q \Vdash Y \not\subseteq X) \iff \forall r \in \mathbb{P} \ (\forall y \in Y \exists a \in A_y r \leq a) \to r \perp q).$$

A similar argument shows that $p \Vdash \|X\|_\varnothing > e$ is also absolute for transitive models. $\square$
Theorem 3.3.3. Assume $\mathbb{F}$ is a forcing notion, $\mathcal{I}$ is an $F_\sigma$ ideal or an analytic $P$-ideal, and $\mathcal{A}$ is an infinite $\mathcal{I}$-AD family. Then in a ccc forcing extension $\mathcal{A}$ can be extended to an $\mathbb{F}$-indestructible $\mathcal{I}$-MAD family.

Proof. We prove the theorem for analytic $P$-ideals only because for $F_\sigma$ ideals the proof is very similar. Let $\mathcal{I} = \text{Exh}(\theta)$ for some lsc submeasure $\theta$, and let $\kappa = |\mathbb{F}|$.

By recursion on $\kappa^+$ we will define a finite support iteration of ccc forcing notions $(\mathbb{P}_\alpha, \dot{\mathcal{A}}_\alpha : \alpha \leq \kappa^+, \beta < \kappa^+)$ and a sequence $(\dot{\mathcal{A}}_\alpha)_{\alpha \leq \kappa^+}$ such that $\dot{\mathcal{A}}_\alpha$ is a $\mathbb{P}_{\alpha+1}$-name, $\forces_{\alpha} \mathcal{A} \cup \{\dot{\mathcal{A}}_\beta : \beta < \alpha\}$ is an $\mathcal{I}$-AD family, and $|\mathbb{P}_{\kappa^+}| \leq 2\kappa$. In $V^{\mathbb{P}_{\kappa^+}}$ the family $\mathcal{A} \cup \{\dot{\mathcal{A}}_\beta : \beta < \kappa^+\}$ will be an $\mathbb{F}$-indestructible $\mathcal{I}$-MAD family.

At stage $\alpha$ we will work with a condition $p$ from $\mathbb{F}$ such that each $p \in \mathbb{F}$ will be worked at cofinally many stages in $\kappa^+$.

Assume $\mathbb{P}_\alpha$ and $\{\dot{\mathcal{A}}_\beta : \beta < \alpha\}$ are done and we have a $p \in \mathbb{F}$. From now on we are working in $V^{\mathbb{P}_\alpha}$. Let $\dot{\mathcal{A}}_\alpha = \mathcal{A} \cup \{\dot{\mathcal{A}}_\beta : \beta < \alpha\}$ and let $\dot{X}_\alpha$ be the set of all $\mathbb{P}$-names $\dot{X}$ such that $\dot{X}$ is a nice $\mathbb{F}$-names for a subsets of $\omega$, there is a real $e(\dot{X}) > 0$ with $p \forces_{\mathbb{F}} \|\dot{X}\|_p > e(\dot{X})$, and $p \forces_{\mathbb{F}} \mathcal{A} \cup \{\dot{X}\}$ is an $\mathcal{I}$-AD family. Let $\mathbb{Q}_\alpha$ be the following forcing notion:

$$(n,s,F,B,Y) \in \mathbb{Q}_\alpha \text{ iff }$$

(1) $n \in \omega$ and $s \subseteq n$;

(2) $F$ is a finite subset of $\{q \in \mathbb{P} : q \leq_{\mathbb{P}} p\} \times \omega$;

(3) $B$ is a finite subset of $\mathcal{A}_\alpha$;

(4) $Y$ is a finite subset of $\dot{X}_\alpha$.

(a) $n_1 \geq n_0$ and $s_1 \cap n_0 = s_0$;

(b) $F_1 \supseteq F_0$, $B_1 \supseteq B_0$, and $Y_1 \supseteq Y_0$;

(c) $(s_1 \setminus s_0) \cap \bigcup B_0 = \emptyset$;

(d) $\forall (q,k) \in F_0 \forall \dot{X} \in Y_0 \exists r \leq_{\mathbb{P}} q \ r \forces_{\mathbb{F}} \theta((s_1 \setminus k) \cap \dot{X}) > e(\dot{X})$.

Notation: $c = (n^c, s^c, F^c, B^c, Y^c) \in \mathbb{Q}_\alpha$.

Claim. $\mathbb{Q}_\alpha$ is $\sigma$-centered, $|\mathbb{Q}_\alpha| \leq 2^\kappa$, and the following sets are dense in $\mathbb{Q}_\alpha$:

(i) $\{c \in \mathbb{Q}_\alpha : (q,k) \in F^c\}$ for each $(q,k) \in \{q \in \mathbb{P} : q \leq p\} \times \omega$;

(ii) $\{c \in \mathbb{Q}_\alpha : B \in B^c\}$ for each $B \in \mathcal{A}_\alpha$;

(iii) $\{c \in \mathbb{Q}_\alpha : \dot{X} \in Y^c\}$ for each $\dot{X} \in \dot{X}_\alpha$.

Proof. $\sigma$-centeredness: We show that conditions with the same first and second coordinates are compatible. Let $(n,s,F_0,B_0,Y_0), (n,s,F_1,B_1,Y_1) \in \mathbb{Q}_\alpha$ and let

$$(F_0 \times Y_0) \cup (F_1 \times Y_1) = \{(q_\ell, k_\ell) : \ell < L\}$$

and

$$(F_0 \times Y_0) \cup (F_1 \times Y_1) = \{(q_\ell, k_\ell) : \ell < L\}$$

and

$$(F_0 \times Y_0) \cup (F_1 \times Y_1) = \{(q_\ell, k_\ell) : \ell < L\}.$$
be an enumeration. For each \( \ell < L \) we know that \( q_\ell \Vdash \exists X_\ell \cap (B_0 \cup B_1) \in \mathcal{I} \) so \( q_\ell \Vdash ||X_\ell \cap (B_0 \cup B_1)||_\theta > \varepsilon(X_\ell) \). Therefore, we can choose an \( r_\ell \leq \mathcal{P} q_\ell \) and a finite \( K_\ell \subseteq \omega \setminus \max \{ n, k_\ell \} \) such that \( \mathcal{U}(K_\ell) > \varepsilon(X_\ell) \) and \( r_\ell \Vdash K_\ell \subseteq X_\ell \setminus (B_0 \cup B_1) \). Let \( s' = s \cup \bigcup \{ K_\ell : \ell < L \} \), \( n' = \max(s') + 1 \), \( F' = F_0 \cup F_1 \), \( B' = B_0 \cup B_1 \), and \( y' = y_0 \cup y_1 \). Then \((s', n', F', B', y')\) is a common extension of our two conditions.

\[ |q_{\ell}| \leq 2^\kappa \] is trivial. (i), (ii), and (iii) are implied by the following observation: If \((n,s,F,B,y)\in \mathcal{Q}_a\), then this condition is compatible with \((n,s,F \cup \{ (q,k) \}, B \cup \{ y \}, y \cup \{ X \})\) where \((q,k)\in \{ q \in \mathcal{P} : q \leq p \} \times \omega, B \in \mathcal{A}_a \), and \( X \in \mathcal{X}_a \) are arbitrary by the proof of \( \sigma \)-centeredness.

Let \( \mathcal{Q}_a \) be a \( \mathcal{P}_a \)-name for \( \mathcal{Q}_a \), and \( \mathcal{A}_a \) be a \( \mathcal{P}_{a+1} \)-name for the set \( \bigcup \{ s^c : c \in \mathcal{H}_a \} \) where \( \mathcal{H}_a \) is a \( \mathcal{P}_{a+1} \)-name for the \( \mathcal{Q}_a \)-generic filter. Then

- \( V^{\mathcal{P}_{a+1}} \models p \Vdash \mathcal{A}_a \cap X \in \mathcal{X}_a \) holds for each \( X \in \mathcal{X}_a \) by (i), (iii), (d), and by Lemma 3.3.2 (in particular \( V^{\mathcal{P}_{a+1}} \models p \Vdash \mathcal{A}_a \supseteq \mathcal{A}_a \cap X \in \mathcal{I}^+ \));

- \( V^{\mathcal{P}_{a+1}} \models \mathcal{A} \cup \dot{\mathcal{A}}_{\beta} : \beta < \alpha \) is an \( \mathcal{I} \)-AD family" because of the previous point, (ii), and (c).

At last we prove that \( \dot{\mathcal{A}}_\kappa^+ = \mathcal{A} \cup \{ \dot{\mathcal{A}}_\alpha : \alpha < \kappa^+ \} \) is an \( \mathcal{I} \)-indestructible \( \mathcal{I} \)-MAD family in \( V^{\mathcal{P}_{\kappa^+}} \).

Assume on the contrary that there is a \( \mathcal{P}_{\kappa^+} \)-generic \( G_{\kappa^+} \), a \( \mathcal{P} \)-name \( X \in V[G_{\kappa^+}] \) for an infinite subset of \( \omega \), and a real \( \varepsilon > 0 \) such that

\[ V[G_{\kappa^+}] \models p \Vdash \varepsilon(X) > \varepsilon \wedge p \Vdash \mathcal{A}_\kappa^+ \cup \{ X \} \) is an \( \mathcal{I} \)-AD family".

Then there is an \( \alpha < \kappa^+ \) such that at the stage \( \alpha \) we worked with \( p \) and \( X \in \mathcal{X}_a \) (because \( |X| \leq \kappa \)). Of course, \( \varepsilon(X) \) is not necessarily equal to \( \varepsilon \) but it does not matter. Then in particular \( V[G_{\kappa^+} \cap \mathcal{P}_{a+1}] \models p \Vdash \dot{\mathcal{A}}_\alpha \cap X \in \mathcal{I}^+ \) so by Lemma 3.3.2 this also holds in \( V[G_{\kappa^+}], \) a contradiction.

**Corollary 3.3.4.** Assume \( \mathcal{I} \) is an \( F_\sigma \) ideal or an analytic \( P \)-ideal, and \( \mathcal{A} \) is an infinite \( \mathcal{I} \)-AD family. Then in a ccc forcing extension \( \mathcal{A} \) can be extended to a Cohen-indestructible \( \mathcal{I} \)-MAD family.

Unfortunately, our theorem does not say anything about those forcing notions which are not absolute for transitive models (e.g. the random forcing) so the following question is still open.

**Problem 3.3.5.** Assume that \( \mathcal{I} \) is as above and \( \mathcal{A} \) is an infinite \( \mathcal{I} \)-AD family. Does there exist a ccc forcing extension in which \( \mathcal{A} \) can be extended to a random-indestructible \( \mathcal{I} \)-MAD family? What can we say about other classical forcing notions, such as the Sacks or the Miller forcing?

**Remark 3.3.6.** Using Theorem 2.3.4 we can prove that that any AD family can be extended to a random-indestructible MAD family in a ccc forcing extension (see [22]).
Chapter 4

The extended Cichoń’s diagram

4.1 \(\mathcal{R}\)-bounding and \(\mathcal{R}\)-dominating

In this section we generalize some classical bounding and dominating properties of forcing notions. Then we discuss special cases related to analytic \(P\)-ideals. The results without references in this section are mainly reformulated and improved versions of theorems from [23]. We have to note that there are other known ways of proving these results but the ideas are the same and we prefer the language of (Borel) GT-connections because this is an uniform method for proving all theorems in this section.

A supported relation \(\mathcal{R} = (A, R, B)\) is called Borel relation iff \(A, B \subseteq \omega^\omega\) and \(R \subseteq \omega^\omega \times \omega^\omega\) are Borel sets. Similarly, a GT-connection \((\phi, \psi) : \mathcal{R}_1 \preceq_{gt} \mathcal{R}_2\) between Borel relations is called Borel GT-connection iff \(\phi\) and \(\psi\) are Borel functions (or equivalently, \(\phi, \psi \subseteq \omega^\omega \times \omega^\omega\) are Borel sets). In this case, we write \((\phi, \psi) : \mathcal{R}_1 \preceq_{gt}^b \mathcal{R}_2\), and we write \(\mathcal{R}_1 \preceq_{gt}^b \mathcal{R}_2\) if there is a Borel GT-connection from \(\mathcal{R}_1\) to \(\mathcal{R}_2\).

For instance, all GT-connections mentioned in Section 1.3 are Borel. Notice that \(M\) and \(N\) can be seen as Borel sets because for example, in all proofs it is enough to work with \(G_\delta\) elements of \(N\) (they are cofinal in \((N, \subseteq)\)), and we can consider the (Borel) set of Borel-codes of \(G_\delta\) null sets.

A pair of transitive models \((V, W)\) is an extension if \(V \subseteq W\).

Definition 4.1.1. Let \(\mathcal{R} = (A, R, B)\) be a Borel supported relation. An extension \((V, W)\) is

\(\mathcal{R}\)-bounding if \[
\forall a \in A \cap W \exists b \in B \cap V \ (a, b) \in R;
\]

\(\mathcal{R}\)-dominating if \[
\exists b \in B \cap W \forall a \in A \cap V \ (a, b) \in R.
\]

A forcing notion \(\mathbb{P}\) is \(\mathcal{R}\)-bounding (resp. \(\mathcal{R}\)-dominating) if

\(\models_{\mathbb{P}} \ldots\) (resp. \(\mathcal{R}\)-bounding (\(\mathcal{R}\)-dominating))."

The following is trivial from the definitions.

Fact 4.1.2. Let \((V, W)\) be an extension and assume \(V \models \mathcal{R}_1 \preceq_{gt} \mathcal{R}_2\). Then if \((V, W)\) is \(\mathcal{R}_2\)-bounding / dominating then \((V, W)\) is \(\mathcal{R}_1\)-bounding / dominating as well.
Remark 4.1.3. We should be careful when saying “\( P \) is not \( \mathcal{R} \)-bounding” (resp. “\( P \) is not \( \mathcal{R} \)-dominating”) because in general, it does not imply the stronger property \( \Vdash_{P}(V, V[G]) \) is not \( \mathcal{R} \)-bounding” which is equivalent that \( P \) is \( \mathcal{R}_{\omega} \)-dominating. Although, in all cases if we prove that a forcing notion is not \( \mathcal{R} \)-bounding it means that this stronger property holds.

Several properties of forcing notions can be defined as \( \mathcal{R} \)-bounding / -dominating with a suitable \( \mathcal{R} \). For instance,

- \( P \) is \( \omega^\omega \)-bounding iff \( P \) is \( (\omega^\omega, \leq^*, \omega^\omega) \)-bounding;
- \( P \) adds a dominating real iff \( P \) is \( (\omega^\omega, \leq^*, \omega^\omega) \)-dominating;
- \( P \) has the Sacks property iff \( P \) is \( (\omega^\omega, \subseteq^*, \text{Slm}) \)-bounding;
- \( P \) adds a slalom over the ground model iff \( P \) is \( (\omega^\alpha, \subseteq^*, \text{Slm}) \)-dominating;
- \( P \) adds a Cohen real iff \( P \) is \( (2^\omega, \in, \mathcal{M})^{\perp} \)-dominating;
- \( \Vdash_{\mathcal{P}} 2^\omega \cap V \in \mathcal{M} \) iff \( P \) is \( (2^\omega, \in, \mathcal{M}) \)-dominating.

4.2 Star-invariants and their underlying relations

First of all, the \( * \)-invariants of a tall ideal \( J \) on \( \omega \) are the following cardinals:

\[
\begin{align*}
\text{add}^*(J) &= \min \{ |\mathcal{A}| : \mathcal{A} \subseteq J, \forall \mathcal{A} \in \mathcal{J} \exists B \in \mathcal{J} \mathcal{B} \notin^* \mathcal{A} \}; \\
\text{cof}^*(J) &= \min \{ |\mathcal{C}| : \mathcal{C} \subseteq J, \forall \mathcal{A} \in \mathcal{J} \exists \mathcal{C} \in \mathcal{C} \mathcal{A} \leq^* \mathcal{C} \}; \\
\text{non}^*(J) &= \min \{ |\mathcal{X}| : \mathcal{X} \subseteq [\omega]^\omega, \forall \mathcal{A} \in \mathcal{J} \exists \mathcal{X} \subseteq \mathcal{X} |\mathcal{A} \cap \mathcal{X}| < \omega \}; \\
\text{cov}^*(J) &= \min \{ |\mathcal{J}| : \mathcal{J} \subseteq J, \forall \mathcal{X} \subseteq [\omega]^\omega \exists \mathcal{B} \in \mathcal{J} |\mathcal{X} \cap \mathcal{B}| = \omega \}.
\end{align*}
\]

Clearly, \( J \) is a \( P \)-ideal iff \( \text{add}^*(J) > \omega \).

Using the unbounding and dominating numbers of supported relations we have that \( \text{add}^*(J) = b(J, \subseteq^*, J) \), \( \text{cof}^*(J) = \text{d}(J, \subseteq^*, J) \), \( \text{non}^*(J) = k([\omega]^\omega, R_\infty, J) \), and \( \text{cov}^*(J) = \text{d}(\omega]^{\omega}, R_\infty, J) \) where \( A, B \in R_\infty \) iff \( |\mathcal{A} \cap \mathcal{B}| = \omega \).

Example 4.2.1. (see [38])

- \( \text{add}^*(E\mathcal{D}) = \text{non}^*(E\mathcal{D}) = \omega \), \( \text{cof}^*(E\mathcal{D}) = \text{non}(M) \), and \( \text{cof}^*(E\mathcal{D}) = \mathfrak{c} \);
- \( \text{add}^*(E\mathcal{D}_{\text{fin}}) = \omega \) and \( \text{cof}^*(E\mathcal{D}_{\text{fin}}) = \mathfrak{c} \);
- \( \text{non}(M) = \max\{\text{cof}^*(E\mathcal{D}_{\text{fin}}), b\} \) and \( \text{cov}(M) = \min\{\text{d}, \text{non}^*(E\mathcal{D}_{\text{fin}})\} \);
- \( \text{add}^*(\text{Ran}) = \text{non}^*(\text{Ran}) = \omega \) and \( \text{cof}^*(\text{Ran}) = \text{cof}^*(\text{Ran}) = \mathfrak{c} \);
- \( \text{add}^*(\mathcal{S}) = \text{non}^*(\mathcal{S}) = \omega \), \( \text{cof}^*(\mathcal{S}) = \text{non}(N) \), and \( \text{cof}^*(\mathcal{S}) = \mathfrak{c} \);
- \( \text{add}^*(\text{Nwd}) = \text{non}^*(\text{Nwd}) = \omega \), \( \text{cof}^*(\text{Nwd}) = \text{cov}(M) \), and \( \text{cof}^*(\text{Nwd}) = \text{cof}(M) \);
- \( \text{add}^*(\text{Conv}) = \text{non}^*(\text{Conv}) = \omega \) and \( \text{cof}^*(\text{Conv}) = \text{cof}^*(\text{Conv}) = \mathfrak{c} \);
4.2. STAR-INFRINGEMENTS AND THEIR UNDERLYING RELATIONS

\[ \text{add}^*(\text{Fin} \otimes \text{Fin}) = \text{non}^*(\text{Fin} \otimes \text{Fin}) = \omega, \text{cov}^*(\text{Fin} \otimes \text{Fin}) = b, \text{and } \text{cof}^*(\text{Fin} \otimes \text{Fin}) = \omega. \]

In this section, our goal is to understand deeper these cardinal invariants and related bounding / dominating properties of forcing notions if \( I \) is a tall analytic \( P \)-ideal.

If \( I \) is a tall ideal on \( \omega \), then the associated \( \sigma \)-ideal on \( [\omega^\omega] \) is the following ideal:

\[ I^o = \{ X \in [\omega^\omega] : |X \cap A| = \omega \} : A \in I^{id}. \]

One can easily prove that \( \text{add}^*(I) = \text{add}(I^o), \text{cof}^*(I) = \text{cof}(I^o), \text{non}^*(I) = \text{non}(I^o), \) and \( \text{cov}^*(I) = \text{cov}(I^o). \) In particular, we have:

\[
\begin{array}{ccc}
\text{cov}^*(I) & \rightarrow & \text{non}^*(I) \\
\text{add}^*(I) & \rightarrow & \text{cof}^*(I)
\end{array}
\]

Of course, we do not need \( I^o \) to see these inequalities between the \( * \)-invariants, they come from trivial GT-connections between their defining relations:

\[
\begin{array}{ccc}
([\omega]^{\omega}, R_\infty, I) & \rightarrow & (J, \subseteq^*, I) \\
(J, \subseteq^*, I^o) & \rightarrow & ([\omega]^{\omega}, R_\infty, I^o)
\end{array}
\]

We say that a forcing notion (or extension) is \( J \)-bounding / dominating if it is \( (J, \subseteq^*, I) \)-bounding / dominating. In other words, a forcing notion \( P \) is \( J \)-bounding iff all new elements of \( J \) in \( V^P \) can be covered by an old element of \( J \) from the ground model, i.e. \( J \cap V \) is cofinal in \((J \cap V^P, \subseteq^*)\); and \( P \) is \( J \)-dominating iff it adds a new element of \( J \) which almost contains all the old elements of \( J \), i.e. \( J \cap V \) is bounded in \((J \cap V^P, \subseteq^*)\).

Forcing-indestructibility of Borel ideals can also be seen as a bounding property:

\[
\text{J is } P\text{-indestructible } \iff P \text{ is } ([\omega]^{\omega}, R_\infty, J)\text{-bounding.}
\]

Similarly, the corresponding dominating property means the following:

\[
P \text{ is } ([\omega]^{\omega}, R_\infty, J)\text{-dominating } \iff \forces_P [\omega]^{\omega} \cap V \in J^o.
\]

One can easily show the following fact.

**Fact 4.2.2.** The following implications hold for tall (in (1) and (3)) Borel (in (2) and (4)) ideals:

- \( \text{add}^*(\text{Fin} \otimes \text{Fin}) = \omega, \text{cov}^*(\text{Fin} \otimes \text{Fin}) = b, \text{and } \text{cof}^*(\text{Fin} \otimes \text{Fin}) = \omega. \)
Let $\text{Thm 4.2.4.}$

Moreover, if $\phi$ is

Proof.

If $a \leq k$ for each $n$, then let $a = (a_n) \in \omega^\omega$ is strictly increasing, then

$$(\omega^\omega, \subseteq, \text{Slm}(a)) \cong (\omega^\omega, \subseteq, \text{Slm}).$$

Proof. $(\omega^\omega, \subseteq, \text{Slm}(a)) \cong (\omega^\omega, \subseteq, \text{Slm})$ is trivial because of the identities.

Conversely, we show that $(\omega^\omega, \subseteq, \text{Slm}) \cong (\omega^\omega, \subseteq, \text{Slm}(a))$. Fix a bijection $j : \omega \to \omega^{<\omega}$. If $f \in \omega^\omega$ then let $\phi(f)(n) = j^{-1}(f \restriction a_{n+1})$. If $S \in \text{Slm}(a)$ and $n \in [a_k, a_{k+1})$, then let

$$\psi(S)(n) = \{ j(\ell)(n) : \ell \in S(k) and |j(\ell)| = a_{k+1} \}$$

and if $n \in [0, a_0)$ then let $S(n) = 0$. If $n \in [a_k, a_{k+1})$ then $|S(k)| \leq a_k$ so $|\psi(S)(n)| \leq a_k \leq a_n$ for each $n$ so $\psi(S) \in \text{Slm}$.

We show that $(\phi, \psi) : (\omega^\omega, \subseteq, \text{Slm}) \cong (\omega^\omega, \subseteq, \text{Slm}(a))$. Assume $f(\phi(j))(k) \in S(k)$ if $k \geq K$. Then $j^{-1}(f \restriction a_{k+1}) \in S(k)$ for each $k \geq K$ so if $n \in [a_k, a_{k+1})$ for some $k \geq K$ then $f(n) = (f \restriction a_{k+1})(n) \in \psi(S)(n)$. In other words, $f \equiv \psi(S)$, and we are done.

Theorem 4.2.4. Let $\mathcal{I}$ be an analytic $P$-ideal. Then

$$(\mathcal{I}, \subseteq, \mathcal{I}) \cong (\omega^\omega, \subseteq, \text{Slm}).$$

Moreover, if $\mathcal{I}_n$ is a tall summable ideal, then

$$(\mathcal{I}, \subseteq, \mathcal{I}) \equiv (\omega^\omega, \subseteq, \text{Slm}).$$

Proof. Let $\mathcal{I} = \text{Exh}(\theta)$ and fix a bijection $e : \omega \to [\omega]^{<\omega}$.

If $A \in \mathcal{I}$ and $n \in \omega$, then let $d_A \in \omega^\omega$, $d_A(n) = \min\{k \in \omega : \theta(A \setminus k) < 2^{-n}\}$, and let $\phi(A) \in \omega^\omega$ be defined by

$$\phi(A)(n) = e^{-1}[A \cap [d_A(n), d_A(n+1))].$$
4.2. STAR-INVARIANTS AND THEIR UNDERLYING RELATIONS

If \( S \in \text{Slm} \) is a slalom, then let

\[
\psi(S) = \bigcup_{n \in \omega} \bigcup \{ e(k) : k \in S(n) \text{ and } \vartheta(e(k)) < 2^{-n} \}.
\]

For each \( n \) the set \( \bigcup \{ e(k) : k \in S(n) \land \vartheta(e(k)) < 2^{-n} \} \) is finite and has submeasure less then \( n/2^n \) so \( \psi(S) \in \mathcal{J} \).

We show that \( (\varphi, \psi) : (\mathcal{J}, \subseteq, \mathcal{J}) \preceq_{\text{st}}^{\text{b}} (\omega^\omega, \subseteq^*, \text{Slm}) \). Assume \( A \in \mathcal{J}, S \in \text{Slm}, \) and \( \phi(A) \subseteq^* S, \phi(A)(n) \in S(n) \) for \( n \geq N \). Clearly, \( \vartheta(e(\phi(A)(n))) < 2^{-n} \) so if \( n \geq N \) then \( e(\phi(A)(n)) \subseteq \psi(S) \). Therefore \( A \setminus d\bar{A}(N) \subseteq \psi(S) \) so \( A \subseteq^* \psi(S) \).

Let \( \mathcal{J}_h \) be a tall summable ideal. Using Lemma 4.2.3, it is enough to show that \( (\omega^\omega, \subseteq^*, \text{Slm}(a)) \preceq_{\text{st}}^{\text{b}} (\mathcal{J}_h, \subseteq^*, \mathcal{J}_h) \) where \( a = (2^n) \).

Since \( \mathcal{J}_h \) is tall, we can fix a family \( \{ A^n_k : n, k \in \omega \} \) of pairwise disjoint subsets of \( \omega \) such that \( \sum_{k \in \omega} h(k) \leq 2^{|\omega|} \) for every \( n, k \in \omega \).

If \( f \in \omega^\omega \) then let \( \phi(f) = \bigcup \{ A^n_k(n) : n \in \omega \} \in \mathcal{J}_h \). If \( A \in \mathcal{J}_h \) then let \( \psi_0(A)(n) = \{ k \in \omega : A^n_k \subseteq A \} \). If there would be infinitely many \( n \) such that \( |\psi_0(A)(n)| \geq 2^n \), then \( A \) could not be in \( \mathcal{J}_h \) so we can modify \( \psi_0(A) \) at finitely many coordinates and we obtain \( \psi(A) \) such that \( \psi(A) \in \text{Slm}(a) \).

We claim that \( (\varphi, \psi) : (\omega^\omega, \subseteq^*, \text{Slm}(a)) \preceq_{\text{st}}^{\text{b}} (\mathcal{J}_h, \subseteq^*, \mathcal{J}_h) \). Assume \( \phi(f) \subseteq^* A, \phi(f) \cap N \subseteq A \). Fix an \( M \) such that if \( m \geq M \) then \( N \cap A^m_k = \emptyset \) for each \( k \in \omega \), and \( \psi_0(A)(m) = \psi(A)(m) \). If \( m \geq M \) then \( A^m_k \subseteq \phi(f) \cap N \subseteq A \) so \( f(m) \in \psi_0(A)(m) = \psi(A)(m) \). It means that \( f \subseteq^* \psi(A) \), we are done.

Using Theorem 1.3.1 (that is \( (\mathcal{N}, \subseteq, \mathcal{N}) \equiv_{\text{st}}^{\text{b}} (\omega^\omega, \subseteq^*, \text{Slm}) \)), we have the following:

**Corollary 4.2.5.** Let \( \mathcal{J} \) be a tall analytic \( P \)-ideal. Then the following hold:

1. \( \text{add}(\mathcal{N}) \leq \text{add}^* (\mathcal{J}) \) and \( \text{cof}^*(\mathcal{J}) \leq \text{cof}(\mathcal{N}) \).
2. If \( \mathbb{P} \) has the Sacks property (i.e. \( \mathbb{P} \) is \( (\mathcal{N}, \subseteq, \mathcal{N}) \)-bounding), then \( \mathbb{P} \) is \( \mathcal{J} \)-bounding.

(2a) If \( \mathbb{P} \) adds a slalom over the ground model (i.e. \( \mathbb{P} \) is \( (\mathcal{N}, \subseteq, \mathcal{N}) \)-dominating, in other words, \( \text{lt} \mathbb{P} \bigcup (\mathcal{N} \cap V) \in \mathcal{N} \)), then \( \mathbb{P} \) is \( \mathcal{J} \)-dominating.

More precisely in (2b), if \( e : \omega \to [\omega]^{<\omega} \) is a bijection in \( V \) and \( S \in \text{Slm} \cap V^\mathbb{P} \) is a slalom over \( V \), then

\[
\bigcup_{n \in \omega} \bigcup \{ e(k) : k \in S(n) \land \vartheta(e(k)) < 2^{-n} \} \in \mathcal{J} \cap V^\mathbb{P}
\]

almost contains all elements of \( \mathcal{J} \cap V \).

Moreover, if \( \mathcal{J}_h \) is a tall summable ideal, then

3. \( \text{add}(\mathcal{N}) = \text{add}^* (\mathcal{J}_h) \) and \( \text{cof}(\mathcal{N}) = \text{cof}^*(\mathcal{J}_h) \);
4. If \( \mathbb{P} \) has the Sacks property iff \( \mathbb{P} \) is \( \mathcal{J}_h \)-bounding;
5. If \( \mathbb{P} \) adds a slalom over the ground model iff \( \mathbb{P} \) is \( \mathcal{J}_h \)-dominating.
**Remark 4.2.6.** In [30] using Fremlin’s and Farah’s results, the authors proved that \(\text{add}(N) = \text{add}^d(\mathcal{Z}_G)\) and \(\text{cof}(N) = \text{cof}^d(\mathcal{Z}_G)\) also hold for each tall density ideal \(\mathcal{Z}_G\). Unfortunately, at this moment we are unable to show a (Borel) GT-connection from \((N, \subseteq, N)\) to \((\mathcal{Z}, \subseteq^*, \mathcal{Z})\). Although, using a result of Fremlin, we will prove that the \(\mathcal{Z}\)-bounding property is equivalent with the Sacks property.

One of the most important open problems of this topic is the following:

**Problem 4.2.7.** Is \((\mathcal{I}, \subseteq^*, \mathcal{I}) \equiv_{\text{GT}} (N, \subseteq, N)\) for each tall analytic \(P\)-ideal \(\mathcal{I}\)? Is at least \(\text{add}^*(\mathcal{I}) = \text{add}(N)\) (and \(\text{cof}^*\)(\(\mathcal{I}\)) = \(\text{cof}(N)\)) for each tall analytic \(P\)-ideal \(\mathcal{I}\)?

**Theorem 4.2.8.** Let \(\mathcal{I} = \text{Exh}(\theta)\) be a tall analytic \(P\)-ideal. Then

\[
(\omega^\omega, \leq^*, \omega^\omega) \not\equiv_{\text{GT}} (\mathcal{I}, \subseteq^*, \mathcal{I}).
\]

**Proof.** We can assume that \(||\omega||_\theta = 2\).

For each \(f \in \omega^\omega\) we can choose a sequence \((A^f_n)_{n<\omega}\) of pairwise disjoint finite sets such that \(A^f_n \subseteq \omega \setminus f(n)\) and \(2^{-n} \leq \theta(A^f_n) < 2^{-n+1}\) for each \(n\). Why? Since \(\mathcal{I}\) is tall, the set

\[
X = \left\{ k \in \omega \setminus \left( f(n) \cup \bigcup \{ A^f_k : k < n \} \right) : \theta(\{k\}) < 2^{-n} \right\},
\]

is co-finite, and so \(2 = ||\omega||_\theta \leq \theta(X)\). Thus there is a finite \(A^f_n \subseteq X\) with \(2^{-n} \leq \theta(A^f_n) < 2^{-n+1}\). Let \(\phi(f) = \bigcup \{ A^f_n : n \in \omega \} \in \mathcal{I}\). Clearly, \(\phi\) could be chosen as Borel.

For a \(B \in \mathcal{I}\) let \(\psi(B) \in \omega^\omega\) be defined by

\[
\psi(B)(n) = \min \left\{ k \in \omega : \theta(B \setminus k) < 2^{-n} \right\}.
\]

We show that \((\phi, \psi) : (\omega^\omega, \leq^*, \omega^\omega) \not\equiv_{\text{GT}} (\mathcal{I}, \subseteq^*, \mathcal{I})\).

Assume \(\phi(f) \subseteq^* B \in \mathcal{I}\). Then there is an \(N\) such that \(\bigcup \{ A^f_n : n \geq N \} \subseteq B\). If \(n \geq N\) then \(\theta(B \setminus f(n)) \geq \theta(A^f_n) \geq 2^{-n}\) so \(\psi(B)(n) \geq f(n)\). Therefore \(f \leq^* \psi(B)\). \(\square\)

**Corollary 4.2.9.** Let \(\mathcal{I}\) be a tall analytic \(P\)-ideal. Then the following hold:

1. \(\text{add}^*(\mathcal{I}) \leq b\) and \(d \leq \text{cof}^*\)(\(\mathcal{I}\)).
2a. If \(P\) is \(\mathcal{I}\)-bounding, then \(P\) is \(\omega^\omega\)-bounding.
2b. If \(P\) is \(\mathcal{I}\)-dominating, then \(P\) adds dominating reals.

We show that \((\omega^\omega, \leq^*, \omega^\omega)\) and \((\mathcal{I}, \subseteq^*, \mathcal{I})\) are not GT-equivalent for any tall analytic \(P\)-ideals.

Notice that for tall summable and tall density ideals it follows from the cardinal characteristics: from Corollary 4.2.5 (3), Remark 4.2.6, and from the consistent cuts of Cichoń’s diagram. Or simply consider the random forcing: it is \(\omega^\omega\)-bounding but it does not have the Sacks property, i.e. it is not \((\mathcal{I}_h, \subseteq^*, \mathcal{I}_h)\)-bounding by Corollary 4.2.5 (4a) (in the strong sense: it is \((\mathcal{I}_h, \subseteq^*, \mathcal{I}_h)\)-dominating) for all tall summable ideals.

We show that the converse of the implication of Corollary 4.2.9 (2b) does not hold for any tall analytic \(P\)-ideals. The Hechler forcing is a counterexample according to the following theorem.
4.2. STAR-INVARIANTS AND THEIR UNDERLYING RELATIONS

Theorem 4.2.10. If $\mathbb{P}$ is $\alpha$-centered then $\mathbb{P}$ is not $\mathcal{J}$-dominating (in the strong sense: $\mathbb{P}$ is $(\mathcal{J}, \subseteq^*, \mathcal{J}^\perp)$-bounding) for any tall analytic $\mathbb{P}$-ideal $\mathcal{J}$.

Proof. Assume that $\mathcal{J} = \text{Exh}(\vartheta)$ for some finite lsc submeasure $\vartheta$. W.l.o.g. we can assume that $\|\vartheta\|_\vartheta = 1$.

Let $\mathbb{P} = \bigcup\{C_n : n \in \omega\}$ where $C_n$ is centered for each $n$. Assume on the contrary that $\Vdash_{\mathbb{P}} X \in \mathcal{J}$ and $\Vdash_{\mathbb{P}} \forall A \in \mathcal{J} \cap V A \subseteq^* X$ for some $\mathbb{P}$-name $\dot{X}$.

For each $A \in \mathcal{J}$ choose a $p_A \in \mathbb{P}$ and a $k_A \in \omega$ such that

$$p_A \Vdash A \setminus k_A \subseteq \check{X} \land \check{\vartheta}(\check{x} \setminus k_A) < 1/2.$$  

(\sigma)

For each $n, k \in \omega$ let $C_{n,k} = \{A \in \mathcal{J} : p_A \in C_n \land k_A = k\}$, and let $B_{n,k} = \bigcup C_{n,k}$. We show that for each $n$ and $k$

$$\vartheta(B_{n,k} \setminus k) \leq 1/2.$$

Assume on the contrary that $\vartheta(B_{n,k} \setminus k) > 1/2$ for some $n$ and $k$. There is a $k'$ such that $\vartheta(B_{n,k} \cap [k, k')) > 1/2$ and there is a finite $\mathcal{D} \subseteq C_{n,k}$ such that $B_{n,k} \cap [k, k') = (\bigcup \mathcal{D}) \cap [k, k')$. Choose a common extension $q$ of $\{p_A : A \in \mathcal{D}\}$. Now we have $q \Vdash \bigcup\{A[k] : A \in \mathcal{D}\} \subseteq \check{X}$ and so

$$q \Vdash 1/2 < \vartheta(B_{n,k} \cap [k, k')) = \vartheta\left(\left(\bigcup \mathcal{D}\right) \cap [k, k')\right) \leq \vartheta(\check{X} \cap [k, k')) \leq \vartheta(\check{X} \setminus k),$$

which contradicts (\sigma).

So for each $n$ and $k$ the set $\omega \setminus B_{n,k}$ is infinite, so $\omega \setminus B_{n,k}$ contains an infinite $D_{n,k} \in \mathcal{J}$. Let $D \in \mathcal{J}$ such that $D_{n,k} \subseteq^* D$ for each $n, k \in \omega$. Then there is no $n, k$ such that $D \in C_{n,k}$, contradiction.

We will use the following deep result of Fremlin.

Theorem 4.2.11. ([26, Theorem 526F]) There is a family $\{P_f : f \in \omega^\omega\}$ of Borel subsets of $\ell_1^+$ such that the following hold:

(i) $\ell_1^+ = \bigcup\{P_f : f \in \omega^\omega\}$;

(ii) if $f \leq g$ then $P_f \subseteq P_g$;

(iii) $(P_f, \leq, \ell_1^+)^{\omega_1^\omega} (\subseteq, \subseteq, \subseteq)$ for each $f$.

Corollary 4.2.12. $\mathbb{P}$ is $\mathcal{Z}$-bounding iff $\mathbb{P}$ has the Sacks property.

Proof. Let $\{P_f : f \in \omega^\omega\}$ be a family satisfying (i), (ii), and (iii) in Theorem 4.2.11, and fix Borel GT-connections $\langle \phi_f, \psi_f : (P_f, \leq, \ell_1^+) \cong^\mathbb{P} (\mathcal{Z}, \subseteq, \subseteq)\rangle$ for each $f \in \omega^\omega$.

Assume $\mathbb{P}$ is $\mathcal{Z}$-bounding and $\Vdash_{\mathbb{P}} \check{x} \in \ell_1^+$. $\mathbb{P}$ is $\omega^\omega$-bounding by Corollary 4.2.9 so using (ii) we have $\Vdash_{\mathbb{P}} \ell_1^+ = \bigcup\{P_f : f \in \omega^\omega \cap V\}$. We can choose a $\mathbb{P}$-name $\check{f}$ for an element of $\omega^\omega \cap V$ such that $\Vdash_{\mathbb{P}} \check{x} \in P_{\check{f}}$. By the $\mathcal{Z}$-bounding property of $\mathbb{P}$ there is a $\mathbb{P}$-name $\check{A}$ for an element of $\mathcal{Z} \cap V$ such that $\Vdash_{\mathbb{P}} \check{\phi}_{\check{f}}(\check{x}) \subseteq \check{A}$, so $\Vdash_{\mathbb{P}} \check{x} \neq \psi_{\check{f}}(\check{A}) \in \ell_1^+ \cap V$. It means that $\mathbb{P}$ is $(\ell_1^+, \leq^{(\omega)}, \ell_1^+)$-bounding so by Theorem 1.3.1 we obtain that $\mathbb{P}$ has the Sacks property.

The converse implication was proved in Corollary 4.2.5 (2a).
Problem 4.2.13. Does the $\mathcal{I}$-bounding property imply the Sacks property for each tall analytic $\mathcal{P}$-ideal $\mathcal{I}$? Does $\mathcal{Z}$-dominating (or $\mathcal{J}$-dominating) imply adding slaloms over the ground model?

By Theorem 4.2.10 the Cohen forcing is not $\mathcal{I}$-dominating, and it will be followed from the following theorem that $\mathcal{C}$ is not $\mathcal{J}$-bounding for any tall analytic $\mathcal{P}$-ideal $\mathcal{J}$.

Theorem 4.2.14. Let $\mathcal{J} = \text{Exh}(\emptyset)$ be a tall analytic $\mathcal{P}$-ideal. Then

$$
([\omega]^\omega, R_\infty, \mathcal{J}) \not\leq_{\text{gt}}^B (\omega^\omega, \in, \mathcal{M})^+.
$$

Proof. Since $\mathcal{J}$ is tall, we can fix a partition $\{P_n : n \in \omega\}$ of $\omega$ into finite intervals such that $\emptyset(\{x\}) < 2^{-n}$ for $x \in P_{n+1}$ (we cannot say anything about $\emptyset(\{x\})$ for $x \in P_0$). Then $A \in \mathcal{J}$ whenever $|A \cap P_n| \leq 1$ for each $n$. Let $\{i^n_k : k < n\}$ be the increasing enumeration of $P_n$.

In this proof, first we will define $\psi : \omega^\omega \rightarrow \mathcal{J}$. If $f \in \omega^\omega$ then let

$$
\psi(f) = \{i^n_k : f(n) \equiv k \mod k_n\}.
$$

Then $\psi(f) \in \mathcal{J}$ because $|\psi(f) \cap P_n| = 1$ for each $n$.

If $X \in [\omega]^\omega$ then let $\phi(X) = \{g \in \omega^\omega : |\psi(g) \cap X| < \omega\}$. It is enough to show that $\phi(X) \in \mathcal{M}$ because then clearly $\phi, \psi : ([\omega]^\omega, R_\infty, \mathcal{J}) \not\leq_{\text{gt}}^B (\omega^\omega, \in, \mathcal{M})^+$. We have that

$$
\phi(X) = \bigcup_{N \in \omega} \bigcap_{n \in \omega} \{g \in \omega^\omega : g(n) \equiv k (k_n) \Rightarrow i^n_k \notin X\}
$$

and it is easy to see that the countable intersection after the union is nowhere dense so $\phi(X) \in \mathcal{M}$. 

Corollary 4.2.15. Let $\mathcal{J}$ be a tall analytic $\mathcal{P}$-ideal. Then the following hold:

1. $\text{non}^*(\mathcal{J}) \geq \text{cov}(\mathcal{M})$ and $\text{cov}^*(\mathcal{J}) \leq \text{non}(\mathcal{M})$.
2. If $\mathbb{P}$ adds Cohen reals, then $\mathbb{P} \upharpoonright [\omega]^\omega \cap V \in \mathcal{J}^\mathbb{P}$.
3. If $\mathbb{P}$ destroys $\mathcal{J}$, then $\mathbb{P} \upharpoonright \omega^\omega \cap V \in \mathcal{M}$.

This corollary has an interesting application. Let $\mathcal{J}$ be an ideal on $\omega$. We say that a tower $(T_\alpha)_{\alpha < \gamma}$ is a tower in $\mathcal{J}^\mathbb{P}$ if $T_\alpha \in \mathcal{J}^\mathbb{P}$ for $\alpha < \gamma$. Under CH it is straightforward to construct towers in $\mathcal{J}^\mathbb{P}$ for each tall analytic $\mathcal{P}$-ideal $\mathcal{J}$. The existence of such towers is consistent with arbitrary large continuum as well by the following theorem.

Theorem 4.2.16. In $V^{|\omega_1}$ there exist towers in $\mathcal{J}^\mathbb{P}$ for each tall analytic $\mathcal{P}$-ideal $\mathcal{J}$.

Proof. Let $G$ be a $(V, C_{\omega_1})$-generic filter and $\mathcal{J} = \text{Exh}(\emptyset)$ be a tall analytic $\mathcal{P}$-ideal in $V[G]$ with some lsc submeasure $\emptyset$ on $\omega$. There is a $\delta < \omega_1$ such that $\emptyset \upharpoonright [\omega]^{<\omega} \in V[G_\delta]$ where $G_\delta = G \cap C_\delta$, so we can assume that $\emptyset \upharpoonright [\omega]^{<\omega} \in V$.

Working in $V[G]$ by recursion on $\omega_1$ we construct the tower $\mathcal{T} = (T_\alpha)_{\alpha < \omega_1}$ in $\mathcal{J}^\mathbb{P}$ such that $\mathcal{T} \upharpoonright \alpha \in V[G_\alpha]$. 

Because $J$ contains infinite elements we can construct in $V$ a sequence $(T_n)_{n \in \omega}$ in $J^*$ which is strictly $\leq^*$-descending, i.e. $|T_n \setminus T_{n+1}| = \omega$ for $n \in \omega$. Assume $(T_\xi)_{\xi < \alpha}$ are done.

Since $J$ is a $P$-ideal there is $T'_\alpha \in J^* \cap V[G_\alpha]$ with $T'_\alpha \subseteq^* T_\beta$ for $\beta < \alpha$.

By Corollary 4.2.15 (2a) there is a set $A_\alpha \in V[G_{\alpha+1}] \cap J^*$ such that $|A_\alpha \cap X| = \omega$ for each $X \in [\omega]^\omega \cap V[G_\alpha]$. Let $T_\alpha = T'_\alpha \setminus A_\alpha \in V[G_{\alpha+1}] \cap J^*$ so $X \not\subseteq^* T_\alpha$ for any $X \in V[G_\alpha] \cap [\omega]^\omega$. Hence $V[G] \models \forall \alpha < \omega_1 (T_\alpha)_{\alpha < \omega_1}$ is a tower in $J^*$.

\[ \square \]

**Remark 4.2.17.** Recently, J. Brendle proved (unpublished) that consistently there are no towers in the dual filters of any tall analytic $P$-ideals. He used an $\omega_2$ stage finite support iteration of filter based Mathias forcings.

Finally, we show one more property of tall summable ideals.

**Theorem 4.2.18.** If $J_h$ is a tall summable ideal, then

\[ (2^\omega, e, N) \preceq^a \omega^\omega, R_\infty, J_h. \]

**Proof.** Let $\lambda$ be the Lebesgue-measure on $2^\omega$. We can choose pairwise disjoint sets $H(n) \in [\omega]^\omega$ such that $\sum_{\ell \in H(n)} h(\ell) = 1$ and $\sup\{h(\ell) : \ell \in H(n)\} < 2^{-n}$ for each $n \in \omega$. Let $H(n) = \{\ell^n_k : k \in \omega\}$ be an enumeration. For each $n$ fix a partition $\{B^n_k : k \in \omega\}$ of $2^\omega$ into Borel-sets such that $\lambda(B^n_k) = h(\ell^n_k)$ for each $k \in \omega$.

If $f \in 2^\omega$ then let $\phi(f) = \{\ell^n_k : f \in B^n_k\} \in [\omega]^\omega$.

If $A \in J_h$ then let

\[ \psi(A) = \bigcap_{N \in \omega \cap n \geq N} \bigcup_{\ell^n \in A} B^n_{\ell^n}. \]

$\psi(A) \in N$ because for each fixed $N \leq n \in \omega$ we have

\[ \lambda \left( \bigcup_{\ell^n \in A} B^n_{\ell^n} \right) = \sum_{\ell^n \in A} \lambda(B^n_{\ell^n}) = \sum_{\ell \in A \setminus H(n)} h(\ell) \]

and for each $D \in \omega$ if $N$ is large enough, then $D \cap \bigcup_{n \geq N} H(n) = \emptyset$ so

\[ \lambda \left( \bigcup_{n \geq N} \bigcup_{\ell^n \in A} B^n_{\ell^n} \right) \leq \sum_{n \geq N} \sum_{\ell \in A \setminus D} h(\ell) \]

which tends to $0$ if $D \to \infty$ (because $A \in J_h$). This implies that $\psi(A) \in N$.

Finally, we show that $(\phi, \psi) : (2^\omega, e, N) \preceq^a \omega^\omega, R_\infty, J$. Assume $|\phi(f) \cap A| = \omega$ for $f \in 2^\omega$ and $A \in J_h$. Using that $|\phi(f) \cap H(n)| = 1$ for each $n$, we obtain that for all $N \in \omega$ there are $n \geq N$ and $k \in \omega$ such that $\ell^n_k \in \phi(f) \cap A$ so $f \in \bigcup_{n \geq N} \bigcup_{\ell^n \in A} B^n_{\ell^n}$ and therefore $f \in \psi(A)$.

\[ \square \]

**Corollary 4.2.19.** Let $J_h$ be a tall summable ideal. Then the following hold:

1. $\text{non}(N) \geq \text{non}^*(J_h)$ and $\text{cov}(N) \leq \text{cov}^*(J_h)$.

2a. If $\models^p [\omega]^\omega \cap V \in J_{h'}$ then $\models^p 2^\omega \cap V \in N$. 


(2b) If \( P \) adds random reals, then \( P \) destroys \( I_h \).

Remark 4.2.20. Corollary 4.2.19 (2b) does not hold even for tall density ideals because it is known that the random forcing does not destroy \( \mathbb{Z} \). We refer the reader to [17] for a nice proof of this result, and to [14] and [32] for the theory of forcing indestructibility of ideals.

We summarize the results of this section in the following diagrams where \( I \) is a tall analytic \( P \)-ideal and the arrows mean Borel GT-connections:

![Diagram 1](image1)

An arrow with (t.s.) means that this Borel GT-connection holds for tall summable ideals, and with (*) means that this Borel GT-connection holds for each tall analytic \( P \)-ideal and it is a Borel GT-equivalence for tall summable ideals.

![Diagram 2](image2)
4.3. **Idealized version of \( b \) and \( d \)**

Arrows with (t.s.) mean that the inequalities hold for tall summable ideals, and with (\( \downarrow \)) hold for all tall analytic P-ideals and they are equalities for tall summable and tall density ideals.

We give some known supplements for the second diagram (see [30]).

Cardinal characteristics in ZFC:

- For every tall density and tall summable ideal \( I \), \( \text{cov}^*(Z) \leq \text{cov}^*(I) \) and \( \text{non}^*(Z) \geq \text{non}^*(I) \);
- \( \min\{\text{cov}(N), b\} \leq \text{cov}^*(Z) \leq \max\{b, \text{non}(N)\} \);
- \( \min\{d, \text{cov}(N)\} \leq \text{non}^*(Z) \leq \max\{d, \text{non}(N)\} \).

**Problem 4.2.21.** Is \( \text{cov}^*(Z) \) (resp. \( \text{non}^*(Z) \)) minimal (maximal) among covering* (uniformity*) numbers of tall analytic P-ideals?

The following are consistent with ZFC:

- \( \text{cov}(N) < \text{cov}^*(I) \) for all tall analytic P-ideals (so there is no GT-connection from \( [\omega]^{\omega}, R_\omega, \omega \) to \( (2^\omega, \in, N) \));
- \( \text{cov}^*(I) < \text{add}(M) \) for all tall analytic P-ideals (so there is no GT-connection from \( [\omega]^{\omega}, R_\omega, \omega \) to \( (M, \subseteq, M) \));
- \( \text{cov}^*(Z) < \text{cov}(N) \) (so there is no GT-connection from \( (2^\omega, \in, N) \) to \( [\omega]^{\omega}, R_\omega, Z \));
- \( d < \text{cov}^*(I) \) for all totally bounded tall analytic P-ideals (that is, if \( J = \text{Exh}(\theta) \), then \( \theta(\omega) < \infty \)).

The following problem is still open and seems to be very difficult.

**Problem 4.2.22.** Is it consistent with ZFC that \( b \) or even \( d \) is smaller than \( \text{cov}^*(Z) \)?

### 4.3 Idealized version of \( b \) and \( d \)

One can naturally generalize the classical unbounding and dominating number for ideals: Let \( I \) be an ideal on \( \omega \). If \( f, g \in \omega^\omega \) then we write \( f \leq_I g \) if \( \{n \in \omega : f(n) > g(n)\} \in I \), in particular \( \leq^* = \leq_{\text{Fin}} \). Let \( b(I) \) be the minimal size of a \( \leq_I \)-unbounded family in the the pre-ordered set \( (\omega^\omega, \leq_I) \), i.e.

\[
b(I) = b(\omega^\omega, \leq_I, \omega^\omega) = \min\{|U| : U \subseteq \omega^\omega \text{ and } \forall f \in \omega^\omega \exists g \in U \; g \nleq_I f\};
\]

and let \( d(I) \) be the minimal size of a \( \leq_I \)-dominating family in \( (\omega^\omega, \leq_I) \), i.e.

\[
d(I) = d(\omega^\omega, \leq_I, \omega^\omega) = \min\{|D| : D \subseteq \omega^\omega \text{ and } \forall f \in \omega^\omega \exists g \in D \; f \nleq_I g\}.
\]

**Theorem 4.3.1.** If \( I \leq_{\text{fin}} J \) then \( (\omega^\omega, \leq_J, \omega^\omega) \equiv_{\alpha_I} (\omega^\omega, \leq_I, \omega^\omega) \).
Proof. Fix a finite-to-one function \( f : \omega \to \omega \) witnessing \( J \leq_{\text{ns}} J \).

Define \( \phi, \psi : \omega^\omega \to \omega^\omega \) as follows:

\[
\phi(x)(i) = \max \left( x \left[ f^{-1}[\{i\}] \right] \right),
\]

\[
\psi(y)(j) = y(f(j)).
\]

We prove two claims.

Claim. \((\phi, \psi) : (\omega^\omega, \leq_\beta, \omega^\omega) \leq_{\text{gr}} (\omega^\omega, \leq_\beta, \omega^\omega)\).

Proof of the claim. We show that if \( \phi(x) \leq_\beta y \) then \( x \leq_\beta \psi(y) \). Indeed, \( A = \{i : \phi(x)(i) > y(i)\} \in J \). Assume that \( f(j) = i \notin A \). Then \( \phi(x)(i) = \max \left( x \left[ f^{-1}[\{i\}] \right] \right) \leq y(i) \). Since \( y(i) = \psi(y)(j) \), so

\[
x(j) \leq \max \left( x \left[ f^{-1}[\{f(j)\}] \right] \right) \leq y(f(j)) = \psi(y)(j)
\]

Since \( f^{-1}[A] \in J \) this yields \( x \leq_\beta \psi(y) \).

Claim. \((\psi, \phi) : (\omega^\omega, \leq_\beta, \omega^\omega) \leq_{\text{gr}} (\omega^\omega, \leq_\beta, \omega^\omega)\).

Proof of the claim. We show that if \( \psi(y) \leq_\beta x \) then \( y \leq_\beta \phi(x) \). Assume on the contrary that \( y \not\leq_\beta \phi(x) \). Then \( X = \{i \in \omega : y(i) > \phi(x)(i)\} \in J^+ \). By definition of \( \phi \), we have \( X = \{i : y(i) > \max \left( x \left[ f^{-1}[\{i\}] \right] \right) \} \).

Let \( Y = f^{-1}[X] \in J^+ \). For \( j \in Y \) we have \( f(j) \in X \) and so

\[
\psi(y)(j) = y(f(j)) > \phi(x)(f(j)) = \max \left( x \left[ f^{-1}[\{f(j)\}] \right] \right) \geq x(j).
\]

Hence \( \psi(y) \not\leq_\beta x \), contradiction.

These claims prove the statement of the theorem, so we are done.

Corollary 4.3.2. If \( J \leq_{\text{ns}} J \) holds then \( b(J) = b(J) \) and \( d(J) = d(J) \).

By Theorem 2.1.3 we know that \( \text{Fin} \leq_{\text{ns}} J \) holds iff \( J \) is meager (see the note before Proposition 2.3.1) so we obtain the following:

Corollary 4.3.3. If \( J \) is a meager ideal, then \( b(J) = b \) and \( d(J) = d \).
Chapter 5

Covering properties of ideals

5.1 The $\mathcal{J}$-covering property

In this chapter we will discuss the generalizations of the following result due to M. Elekes (see [17]). All result of this chapter can be found in [1].

**Theorem 5.1.1.** Let $(X, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space and let $(A_n)_{n\in\omega}$ be a sequence of sets from $\mathcal{A}$ that covers $\mu$-almost every $x \in X$ infinitely many times. Then there exists a set $M \subseteq \omega$ of asymptotic density zero such that $(A_n)_{n\in M}$ also covers $\mu$-almost every $x \in X$ infinitely many times.

Using this result Elekes gave a nice new proof for the fact that the density zero ideal is random-indestructible. He asked about other variants of this theorem.

We can consider the following abstract setting.

**Definition 5.1.2.** Let $X$ be an arbitrary set and $I \subseteq \mathcal{P}(X)$ be an ideal of subsets of $X$. We say that a sequence $(A_n)_{n\in\omega}$ of subsets of $X$ is an $I$-a.e. infinite-fold cover of $X$ if

$$\{x \in X : \{n \in \omega : x \in A_n\} \text{ is finite}\} \in I,$$

i.e.

$$\limsup_{n\in\omega} A_n \in I^*.$$

Of course, the sequence $(A_n)$ above can be indexed by any countable infinite set. Assume furthermore that given a $\sigma$-algebra $\mathcal{A}$ of subsets of $X$ and an ideal $\mathcal{J}$ on $\omega$. We say that the pair $(\mathcal{A}, I)$ has the $\mathcal{J}$-covering property if for every $I$-a.e. infinite-fold cover $(A_n)_{n\in\omega}$ of $X$ by sets from $\mathcal{A}$, there is a set $S \in \mathcal{J}$ such that $(A_n)_{n\in S}$ is also an $I$-a.e. infinite-fold cover of $X$.

In this context, Elekes' theorem says that if $(X, \mathcal{A}, \mu)$ is a $\sigma$-finite measure space, then $(\mathcal{A}, \mathcal{N}_\mu)$ has the $\mathcal{Z}$-covering property where $\mathcal{N}_\mu = \{H \in \mathcal{A} : \mu(H) = 0\}$ (more precisely, $\mathcal{N}_\mu$ is the ideal generated by null sets because we assume that $I$ is an ideal on the underlying set).

Clearly, in the previous definition it is enough to check infinite-fold covers instead of $I$-a.e. infinite-fold covers.

Observe that, if $I_1 \subseteq I_2$ and $(\mathcal{A}, I_1)$ possesses the $\mathcal{J}$-covering property, then also $(\mathcal{A}, I_2)$ possesses it.
Notice that if \((A, I)\) has the \(\mathcal{J}\)-covering property, then \((A[I], I)\) also has this property where \(A[I]\) is the “\(I\)-completion of \(A\)”, that is
\[
A[I] = \{ B \subseteq X : \exists A \in A A \Delta B \in I \}.
\]

For instance, if \((\text{Borel}(\mathbb{R}), \mathcal{N})\) has the \(\mathcal{J}\)-covering property, then \((\text{LM}(\mathbb{R}), \mathcal{N})\) also has this property where \(\text{LM}(\mathbb{R})\) is the \(\sigma\)-algebra of Lebesgue measurable subsets of \(\mathbb{R}\). Similarly in the category case, it is enough to prove that \((\text{Borel}(\mathbb{R}), \mathcal{M})\) has the \(\mathcal{J}\)-covering property, then it also holds for \((\text{BP}(\mathbb{R}), \mathcal{M})\) where \(\text{BP}(\mathbb{R})\) is the \(\sigma\)-algebra of sets with the Baire property.

Furthermore, if \((A, I)\) has the \(\mathcal{J}\)-covering property then for all \(Y \in A \setminus I\) the pair \((A \upharpoonright Y, I \upharpoonright Y)\) also has this property where of course, \(A \upharpoonright Y = \{ Y \cap A : A \in A \} \) is the restricted \(\sigma\)-algebra.

Clearly, if \(\mathcal{J}\) is not tall, then there is no \((A, I)\) with the \(\mathcal{J}\)-covering property.

We give another motivation to this notion. First we reformulate the \(\mathcal{J}\)-covering property:

**Fact 5.1.3**. \((A, I)\) has the \(\mathcal{J}\)-covering property \((X = \bigcup A)\) if, and only if for every \((A, \text{Borel}(\omega^\omega))\)-measurable function \(F : X \to [\omega]^{\omega}\), there is an \(S \in \mathcal{J}\) such that \(\{ x \in X : |F(x) \cap S| < \omega \} \in I\).

Because of this fact and the observation: \((\mathcal{P}(X), \{ \emptyset \})\) has the \(\mathcal{J}\)-covering property iff \(|X| < \text{non}^*(\mathcal{J})\), the \(\mathcal{J}\)-covering property can be seen as “analytic uniformity.”

We present some easy negative results. First we show that, in Elekes’ theorem applied to the Lebesgue measure, one cannot use \(\mathcal{J}_{1/n}\) instead of \(\mathcal{Z}\).

**Example 5.1.4**. We will show that \((\text{Borel}(\mathbb{R}), \mathcal{N})\) does not have the \(\mathcal{J}_{1/n}\)-covering property in a strong sense. First consider the interval \((0, 1)\) and a fixed infinite-fold cover \((A_n)_{n \in \omega}\) of \((0, 1)\) of the form \(A_n = (a_n, b_n), \ b_n - a_n = \frac{1}{n+1}\). Then for each \(S \in \mathcal{J}_{1/n}\) we have \(\sum_{n \in S} \lambda(A_n) < \infty\) where \(\lambda\) stands for Lebesgue measure on \(\mathbb{R}\). Hence by the Borel-Cantelli lemma, \(\lambda(\limsup_{n \in \omega} A_n) = 0\). Fix a homeomorphism \(h\) from \((0, 1)\) onto \(\mathbb{R}\) of class \(C^1\). Then \((h[A_n])_{n \in \omega}\) is an infinite-fold open cover of \(\mathbb{R}\). Since \(h\) is absolutely continuous, we have \(\lambda(\limsup_{n \in \omega} h[A_n]) = 0\), which gives the desired claim.

This example motivates the following question which will be discussed in Section 5.3 (see Example 5.3.1 and Theorem 5.3.2 below).

**Question 5.1.5**. Assume \(X\) is a Polish space and \((\text{Borel}(X), I)\) does not have the \(\mathcal{J}\)-covering property. Does there exist an infinite-fold Borel cover \((A_n)_{n \in \omega}\) of \(X\) such that \(\limsup_{n \in \omega} A_n \in I\) for all \(S \in \mathcal{J}\)?

Let us denote \(\mathcal{K}_\sigma\) be the \(\sigma\)-ideal on \(\omega^\omega\) generated by compact sets. We will use the fact that an \(H \subseteq \omega^\omega\) is in \(\mathcal{K}_\sigma\) iff there is an \(h \in \omega^\omega\) such that \(H \subseteq \{ x \in \omega^\omega : x \leq^* h \}\).

**Example 5.1.6**. Consider the following infinite-fold cover \((A_n)_{n \in \omega}\) of \(\omega^\omega\) by \(F_\sigma\) sets:

Let \(B = \{ x \in \omega^\omega : \forall n \ x(n) = 0 \}\) and \(A_n = \{ y \in \omega^\omega : y(n) \neq 0 \} \cup B\) for \(n \in \omega\). It is easy to see that if \(X \in [\omega]^{\omega}\) with \(\omega \setminus X \in [\omega]^{\omega}\), then \(\omega^\omega \setminus \limsup_{n \in \omega} A_n\) is dense, uncountable, and does not belong to \(\mathcal{K}_\sigma\).
In particular, there is no ideal $\mathcal{J}$ on $\omega$ such that (Borel($\omega^\omega$), $I$) has the $\mathcal{J}$-covering property if $I = [\omega^\omega]^{<\omega}$, NWD (the ideal of nowhere dense sets), or $\mathcal{K}_\sigma$. Consequently (by the Borel isomorphism theorem), given an uncountable Polish space $X$, (Borel($X$), $[X]^{<\omega}$) does not have the $\mathcal{J}$-covering property for any ideal $\mathcal{J}$ on $\omega$.

How could we conclude a $\mathcal{J}_1$-covering property from a $\mathcal{J}_0$-covering property? In special cases we can do it by the following easy observation:

**Fact 5.1.7.** Assume $\mathcal{J}_0 \leq_{\mathcal{J}_0} \mathcal{J}_1$ and that $(A, I)$ has the $\mathcal{J}_0$-covering property. Then $(A, I)$ has the $\mathcal{J}_1$-covering property as well.

The next observation shows a connection between trace ideals and covering properties.

**Proposition 5.1.8.** Assume $I$ is a $\sigma$-ideal on $2^\omega$ (or on $\omega^\omega$) and (Borel($2^\omega$), $I$) has the $\mathcal{J}$-covering property. Then $\mathcal{J} \not\leq_{\mathcal{J}} \text{tr}(I) \restriction X$ for any $X \in \text{tr}(I)^+$.

**Proof.** Assume on the contrary that $f \in \omega^\omega$ shows that $\mathcal{J} \leq_{\mathcal{J}} \text{tr}(I) \restriction X$ for some $X \in \text{tr}(I)^+$. Let $(A_n)_{n \in \omega}$ be the following infinite-fold cover of $[X] \in I^+$:

$$A_n = \{x \in [X] : \exists k \in \omega \ (x \restriction k \in X \text{ and } f(x \restriction k) = n)\}.$$

If $S \in \mathcal{J}$ then $\limsup_{n \in S} A_n = [f^{-1}[S]] \in I$, a contradiction. \qed

Using Theorem 2.3.4 we obtain the following:

**Corollary 5.1.9.** Let $I$ is a $\sigma$-ideal on $2^\omega$ (or on $\omega^\omega$), and $\mathcal{J}$ is a tall ideal on $\omega$. Assume furthermore that $\mathbb{P}_I$ is proper and has the CRN. If (Borel($2^\omega$), $I$) has the $\mathcal{J}$-covering property, then $\mathcal{J}$ is $\mathbb{P}_I$-indestructible.

We do not need to use Theorem 2.3.4, the trace ideal or the CRN property in this result. The following theorem is a natural generalization of Elekes’ result about random-indestructibility of $\mathbb{Z}$.

**Theorem 5.1.10.** Let $X$ be a Polish space, $I$ be a $\sigma$-ideal on $X$, and assume that $\mathbb{P}_I$ is proper. If (Borel($X$), $I$) has the $\mathcal{J}$-covering property, then $\mathcal{J}$ is $\mathbb{P}_I$-indestructible.

**Proof.** Assume on the contrary that $\check{Y}$ is a $\mathbb{P}_I$-name for an infinite subset of $\omega$, i.e. $\forces_{\mathbb{P}_I} \check{Y} \in [\omega]^\omega$ and $B \forces_{\mathbb{P}_I} \forall A \in \mathcal{J} \ | \check{Y} \cap A| < \omega$ for some $B \in \mathbb{P}_I$. Then there are a $C \in \mathbb{P}_I$, $C \subseteq B$, and a Borel function $f : C \rightarrow [\omega]^\omega$ (coded in the ground model) such that $C \forces_{\mathbb{P}_I} f(\check{r}_{\text{gen}}) = \check{Y}$ where $\check{r}_{\text{gen}}$ is a name for the generic real (see [46, Prop. 2.3.1]). For each $n \in \omega$ let

$$Y_n = f^{-1}([S \in [\omega]^\omega : n \in S]) \in \text{Borel}(X).$$

Then $(Y_n)_{n \in \omega}$ is an infinite-fold cover of $C$ (by Borel sets) because $x \in Y_n \iff n \in f(x)$ and $|f(x)| = \omega$. Using the $\mathcal{J}$-covering property of (Borel($X$) $\upharpoonright C$, $I \upharpoonright C$) we can choose an $A \in \mathcal{J}$ such that $(Y_n)_{n \in A}$ is an $I$-a.e. infinite-fold cover of $C$, that is $|f(x) \cap A| = \omega$ for $I$-a.e. $x \in C$, i.e. $\{x \in C : |f(x) \cap A| < \omega\} \in I$, so $C \forces_{\mathbb{P}_I} |f(\check{r}_{\text{gen}}) \cap A| = \omega$, and consequently, $C \forces_{\mathbb{P}_I} |\check{Y} \cap A| = \omega$, a contradiction. \qed
5.2 Around the category case

If $X$ is a Polish space, then let $\mathcal{M}(X)$ be the $\sigma$-ideal of meager subsets of $X$.

**Theorem 5.2.1.** The pair $(\text{Borel}(X), \mathcal{M}(X))$ has the $\mathcal{E}_\text{fin}$-covering property for each Polish space $X$.

**Proof.** Let $(A_{(n,m)}, (n,m) \in \Delta)$ be an infinite-fold cover of $X$ by Borel sets. Without loss of generality, we can assume that all $A_{(n,m)}$'s are open and nonempty.

Enumerate $\{U_k : k \in \omega\}$ a base of $X$. We will define by recursion a sequence $(n_k, m_k)_{k \in \omega}$ of elements of $\Delta$. First, pick $(n_0, m_0) \in \Delta$ such that $A_{(n_0, m_0)} \cap U_0 \neq \emptyset$. Assume $(n_i, m_i)$ are done for $i < k$. Then we can choose an $(n_k, m_k) \in \Delta$ such that $n_k \neq n_i$ for $i < k$ and $A_{(n_k, m_k)} \cap U_k \neq \emptyset$. We obtain the desired set $S = \{(n_k, m_k) : k \in \omega\} \in \mathcal{E}_\text{fin}$. For every $k \in \omega$, the set $\bigcup_{i \geq k} A_{(n_i, m_i)}$ is dense and open. Consequently,

$$\limsup_{(n,m) \in S} A_{(n,m)} = \bigcap_{k \in \omega} \bigcup_{i \geq k} A_{(n_i, m_i)}$$

is a dense $G_\delta$ set, hence it is residual. $\square$

Using the Fact 5.1.7 and Theorem 5.1.10 we obtain the following:

**Corollary 5.2.2.** If $\mathcal{E}_\text{fin} \leq \text{sb} \mathcal{J}$, then $(\text{Borel}(X), \mathcal{M}(X))$ has the $\mathcal{J}$-covering property for each Polish space $X$, and hence $\mathcal{J}$ is Cohen-indestructible.

Note that $\mathcal{E}_\text{fin} \leq \text{sb} \mathcal{J}$ holds for a quite big class of interesting ideals:

**Proposition 5.2.3.** $\mathcal{E}_\text{fin} \leq \text{sb} \mathcal{J}$ holds for each tall analytic $\mathcal{P}$-ideal $\mathcal{J}$.

**Proof.** Let $\mathcal{J} = \text{Exh}(\varphi)$ for some lsc submeasure $\varphi$. For $k \in \omega$ let

$$d(k) = \min \{ \ell_0 \in \omega : \forall \ell \geq \ell_0 \varphi(\{\ell\}) < 2^{-k}\}.$$

We can choose a strictly increasing sequence $(n_k)_{k \in \omega} \in \omega^\omega$ such that $d(k + 1) - d(k) \leq n_k$. Let $f : \omega \to \omega$ be any one-to-one function with the property $f(\{d(k + 1) \setminus d(k)\}) \subseteq \{(n_k, m) : m \leq n_k\}$. Then $f$ shows that $\mathcal{E}_\text{fin} \leq \text{sb} \mathcal{J}$. $\square$

Now, by Corollary 5.2.2 and Proposition 5.2.3 we obtain

**Corollary 5.2.4.** $(\text{Borel}(X), \mathcal{M}(X))$ has the $\mathcal{J}$-covering property for each Polish space $X$ and for each tall analytic $\mathcal{P}$-ideal $\mathcal{J}$.

One can ask if the implications in Corollary 5.2.2 could be equivalences. The answer is no by the following examples and results after them.

**Example 5.2.5.** $\text{Fin} \otimes \text{Fin}$ is Cohen-indestructible but $(\text{Borel}(\omega^\omega), \mathcal{M})$ does not have the $\text{Fin} \otimes \text{Fin}$-covering property.

Cohen-indestructibility of $\text{Fin} \otimes \text{Fin}$: Is easy to see that a forcing notion $\mathbb{P}$ destroys $\text{Fin} \otimes \text{Fin}$ iff $\mathbb{P}$ adds dominating reals.

$(\text{Borel}(\omega^\omega), \mathcal{M})$ does not have the $\text{Fin} \otimes \text{Fin}$-covering property: Enumerate $\{s_m^n : m \in \omega\}$ the elements of $\omega ^{<\omega}$ with first element $n$. Consider the following infinite-fold cover of $\omega^\omega$: $A_{(n,m)} = \{x \in \omega^\omega : s_m^n \leq x\}$ for $(n, m) \in \omega \times \omega$. It is trivial to see that there is no $S \in \text{Fin} \otimes \text{Fin}$ such that $(A_{(n,m)}, (n,m))_{(n,m) \in S}$ is an $\mathcal{M}$-a.e. infinite-fold cover of $\omega^\omega$. 
Example 5.2.6. \( \mathcal{E} \mathcal{D} \) is also Cohen-indestructible: Let \( \mathbb{C} = (2^{<\omega}, \supseteq) \) be the Cohen forcing and assume on the contrary that \( \dot{X} \) is a \( \mathbb{C} \)-name for an infinite subset of \( \omega \times \omega \) such that \( \mathbb{C} \Vdash \forall A \in \mathcal{E} \mathcal{D} \cap V[X \cap A] < \omega \). Enumerate \( \mathbb{C} = \{ p_n : n \in \omega \} \) and let \( f \in \omega^{\omega} \cap \mathcal{V} \) be the following function: if \( p_n \mathbb{C} (X)_n = \emptyset \) then let \( f(n) = 0 \), if not then let
\[
f(n) = \min \{ k \in \omega : \exists q \leq p_n q \mathbb{C} k = \min ((X)_n) \}.
\]
Clearly \( f \in \mathcal{E} \mathcal{D} \) and it is easy to see that \( \mathbb{C} | \dot{X} \cap f| = \omega \).
Of course, \((\text{Borel}(\omega^\omega), \mathcal{M})\) does not have the \( \mathcal{E} \mathcal{D} \)-covering property because \( \mathcal{E} \mathcal{D} \subseteq \text{Fin} \odot \text{Fin} \) (and we can use Example 5.2.5).

It would be nice to know a characterization of forcing notions which destroy \( \mathcal{E} \mathcal{D} \) (similar to the characterization in the case of \( \text{Fin} \odot \text{Fin} \)).

Question 5.2.7. Is it true that a forcing notion \( \mathbb{P} \) destroys \( \mathcal{E} \mathcal{D} \) iff \( \mathbb{P} \) adds an eventually different real, i.e. a real \( r \) such that \( \mathbb{P} \subseteq \text{Fin} \) (and we can use Example 5.2.5).

In the case of the first implication of Corollary 5.2.2 we have only consistent counterexamples.

Theorem 5.2.8. Assume \( t = c \) and \( |A| \leq c \), then there is no \( \leq_{\text{min}} \)-smallest element of
\[
\{ j : (A, I) \text{ has the } j \text{-covering property} \}.
\]

Proof. If \( \{ j : (A, I) \text{ has the } j \text{-covering property} \} = \emptyset \) then we are done. If \( (A, I) \) has the \( j \)-covering property then we will construct a \( j \) such that \( j_0 \not\leq_{\text{min}} j \) but \( (A, I) \) has the \( j \)-covering property.

Enumerate \( (f_\alpha)_{\alpha < \omega} \) all finite-to-one functions from \( \omega \) to \( \omega \), and enumerate \( (\mathcal{A}_n)_{n \in \omega} : \alpha < \omega \) the infinite-fold covers of \( X = \bigcup A \) by sets from \( A \). By recursion on \( \alpha \) we will define a \( <^* \)-increasing sequence \( (S_\xi)_{\xi < \alpha} \) of infinite and co-infinite subsets of \( \omega \) and the ideal \( j \) generated by this sequence will be as required.

Assume \( (S_\xi)_{\xi < \alpha} \) is done for some \( \alpha < \omega \). Because of our assumption on \( t \) we can choose an infinite and co-infinite \( S'_\alpha \) such that \( S_\xi \subseteq^* S'_\alpha \) for each \( \xi < \alpha \). The set \( f_\alpha[\omega \setminus S'_\alpha] \) contains an infinite element \( E \) of \( j_0 \). We want to guarantee that \( f_\alpha^{-1}[E] \not= j \) because then \( f_\alpha \) can not witness \( j_0 \leq_{\text{min}} j \). Let \( H = f_\alpha^{-1}[E \setminus S'_\alpha] \) in \( [\omega]^{\omega} \).

Consider the \( n \)th cover \( (\mathcal{A}_n)_{n \in \omega} \). If \( (A_n)_{n \in \omega} \cap H \) is an \( I \)-a.e. infinite-fold cover of \( X \), then let \( S_\alpha = S'_\alpha \cup (\omega \setminus H) \).

If not, then
\[
C = \{ x \in X : \{ n \in \omega : x \in A_n \} \text{ is finite} \} \not= \emptyset.
\]

Using our assumption on \( j_0 \) for \( (A[I] \uparrow C, I \uparrow C) \) and its infinite-fold cover \( (\mathcal{A}_n \cap C)_{n \in H} \) (with a copy of \( j_0 \) on \( H \)), we can choose an infinite \( H' \subseteq H \) such that \( H \setminus H' \) is also infinite and \( (\mathcal{A}_n \cap C)_{n \in H'} \) is an \( I \uparrow C \)-a.e. infinite-fold cover of \( C \). Finally, let \( S_\alpha = S'_\alpha \cup (\omega \setminus H) \cup H' \).

It is easy to see from the construction that \( j \) is as required.

Corollary 5.2.9. If \( t = c \) then there are ideals \( j_0 \) and \( j_1 \) such that \( \not\leq_{\text{min}} j_0 \) and \( \mathcal{E} \mathcal{D} \) has \( j_0 \)-covering property and \( \mathcal{E} \mathcal{D} \) has \( j_1 \)-covering property.
CHAPTER 5. COVERING PROPERTIES OF IDEALS

Without $t = \varepsilon$ we can use a simple forcing construction.

**Theorem 5.2.10.** After adding $\omega_1$ Cohen reals by finite support iteration there is an ideal $\mathcal{I}$ such that $\mathcal{D}_{\text{fin}} \not\subseteq \mathcal{I}$ (in particular, $\mathcal{Z} \not\subseteq \mathcal{I}$) but (Borel($2^\omega$), $\mathcal{N}$) and (Borel($2^\omega$), $\mathcal{M}$) have the $\mathcal{I}$-covering property.

**Proof.** Let $(c_a)_{a < \omega_1}$ be the sequence of generic Cohen-reals in $2^\omega$, $C_a = c_a^{-1}[\{1\}] \subseteq \omega$, and let $\mathcal{I}$ be the ideal generated by these sets. A trivial density argument shows that $\mathcal{I}$ is a proper ideal.

To show that (Borel($2^\omega$), $\mathcal{N}$) has the $\mathcal{I}$-covering property in the extension, it is enough to see that if $(A_n)_{n \in \omega}$ is an infinite cover of $2^\omega$ by Borel sets in a ground model $V$, then $(A_n)_{n \in C}$ is an $\mathcal{N}$-a.e. infinite-fold cover of $2^\omega$ in $V[C]$ where $C \subseteq \omega$ is a Cohen real over $V$. By a simple density argument $V[C] \models \lambda(\bigcup_{n \in C \setminus \mathcal{K}} A_n) = 1$ for each $k \in \omega$, so $V[C] = \lambda(\limsup_{n \in C} A_n) = 1$.

To show that (Borel($2^\omega$), $\mathcal{M}$) has the $\mathcal{I}$-covering property in the extension, it is enough to prove that if $(A_n)_{n \in \omega}$ is an infinite-fold cover of $2^\omega$ by open sets in $V$, then $(A_n)_{n \in C}$ is an $\mathcal{M}$-a.e. infinite-fold cover of $2^\omega$ in $V[C]$. By a simple density argument $V[C] \models \text{"} \bigcap_{n \in C \setminus \mathcal{K}} A_n \text{ is dense open} \text{"}$ for each $k \in \omega$, so $V[C] \models \text{"} \limsup_{n \in C} A_n \text{ is residual.} \text{"}$

To show that $\mathcal{D}_{\text{fin}} \not\subseteq \omega_1 \mathcal{I}$, it is enough to see that if $f \in \Delta^\omega \cap V[(c_\xi)_{\xi < a}]$ is a finite-to-one function for some $a < \omega_1$, then there is an $A \in \mathcal{D}_{\text{fin}} \cap V[(c_\xi)_{\xi \leq a}]$ such that $f^{-1}[A]$ cannot be covered by finitely many of $C_\xi$’s ($\xi < \omega_1$). Simply let $A$ be a Cohen function in $\prod_{n \in \omega}(n + 1)$, i.e. the graph of a Cohen function in $\Delta$, for example if $c_a \in \omega^\omega$ is a Cohen real over $V[(c_\xi)_{\xi < a}]$ then $A = \{(n, k) \in \Delta : c_a'(n) \equiv k \mod (n+1)\}$ is suitable. Using the presentation of this iteration by finite partial functions from $\omega_1 \times \omega$ to $2$, we are done by a simple density argument. \hfill $\square$

**Question 5.2.11.** Does there exist an (analytic) ideal $\mathcal{I}$ in ZFC such that $\mathcal{Z} \not\subseteq \mathcal{I}$ but (Borel($2^\omega$), $\mathcal{N}$) has the $\mathcal{I}$-covering property? Does there exist Katětov-Blass-smallest ideal in the family of all analytic (or Borel) ideals $\mathcal{I}$ such that (Borel($2^\omega$), $\mathcal{N}$) has the $\mathcal{I}$-covering property?

Similarly, one can ask the analogous question for the meager ideal with $\mathcal{D}_{\text{fin}}$ instead of $\mathcal{Z}$; and about the existence of Katětov-Blass-smallest ideal in the corresponding family as well.

### 5.3 When the $\mathcal{I}$-covering property “strongly” fails

In this section we give a positive answer for Question 5.1.5 in a special case but first of all, we present a counterexample:

**Example 5.3.1.** Consider $X = (-1, 1)$ and let an ideal $I$ on $X$ consist of sets $A \subseteq X$ such that $A \cap (-1, 0]$ is meager and $A \cap (0, 1]$ is of Lebesgue measure zero. Using Example 5.1.4 and Corollary 5.2.4 observe that (Borel($X$), $I$) yields the negative answer to Question 5.1.5 with $\mathcal{I} = I_{1/\omega}$. However, this question remains interesting if we restrict it to pairs (Borel($\mathcal{R}$), $I$) where $I$ is a translation invariant ideal on $\mathcal{R}$. 

In the translation invariant case, we describe a class of ideals $I$ which yields a positive answer to Question 5.1.5 provided $\mathcal{J}$ is a $\mathcal{P}$-ideal.

Let $\mathcal{Q}$ stand for the set of rational numbers. For $A, B \subseteq \mathbb{R}$ and $x \in \mathbb{R}$ we write $A + x = \{a + x : a \in A\}$ and $A + B = \{a + b : A \in A$ and $b \in B\}$. We say that an ideal $I$ on a Polish space $X$ is a ccc ideal if every disjoint subfamily of Borel($X$) \ $I$ is countable.

**Theorem 5.3.2.** Assume that $I$ is a translation invariant ccc $\sigma$-ideal on $\mathbb{R}$ fulfilling the condition
\[
\mathcal{Q} + A \in I^* \text{ for each } A \in \text{Borel}(\mathbb{R}) \setminus I.
\]
Fix a $\mathcal{P}$-ideal $\mathcal{J}$ on $\omega$. If $I(\text{Borel}(\mathbb{R}), I)$ does not have the $\mathcal{J}$-covering property, then there exists an infinite-fold Borel cover $(A_n')_{n \in \omega}$ of $\mathbb{R}$ with $\limsup_{n \in \omega} A_n' \in I$ for all $S \in \mathcal{J}$.

**Proof.** Fix an infinite-fold Borel cover $(A_n)_{n \in \omega}$ of $\mathbb{R}$ such that $\limsup_{n \in \omega} A_n \notin I^*$ for all $S \in \mathcal{J}$. We will show that there is a Borel set $B \subseteq \mathbb{R}$ with $B \notin I$ and $(\limsup_{n \in \omega} A_n) \cap B \notin I$ for all $S \in \mathcal{J}$. Suppose it is not the case. So, in particular (when $B = \mathbb{R}$), we find $S_0 \in \mathcal{J}$ with $X_0 := \limsup_{n \in S_0} A_n \notin I$. Then by transfinite recursion we define sequences $(S_n)_{\alpha < \gamma}$ and $(X_n)_{\alpha < \gamma}$, with $S_\alpha \in \mathcal{J}$ and $X_\alpha := (\limsup_{n \in S_\alpha} A_n) \setminus \bigcup_{\beta < \alpha} X_\beta \notin I$ (when $B = \mathbb{R} \setminus \bigcup_{\beta < \alpha} X_\beta \notin I$). Since $I$ is ccc, this construction stops at a stage $\gamma < \omega_1$ with $\bigcup_{\alpha < \gamma} \limsup_{n \in S_\alpha} A_n = \limsup_{n \in S_\gamma} A_n \notin I^*$. Since $I$ is a $\mathcal{P}$-ideal, there is $S \in \mathcal{J}$ which almost contains each $S_\alpha$ for $\alpha < \gamma$. Then $\limsup_{n \in S_\gamma} A_n \in I^*$ which contradicts our supposition.

So, fix a Borel set $B \notin I$ such that $(\limsup_{n \in S_\gamma} A_n) \cap B \in I$ for all $S \in \mathcal{J}$. Let $\mathcal{Q} = \{q_k : k \in \omega\}$. Define $B_0 := B$ and $B_k := (q_k + B) \setminus \bigcup_{i < k} B_i$ for $k \in \omega$. Then put $A_n := \bigcup_{k \in \omega} ((q_k + A_n) \cap B_k)$ for $n \in \omega$. Since $(A_n)_{n \in \omega}$ is an infinite-fold cover of $\mathbb{R}$, we have $\limsup_{n \in \omega} ((q_k + A_n) \cap B_k) = B_k$ for all $k \in \omega$. Note that $(A_n')_{n \in \omega}$ is an $I$-a.e. infinite-fold cover of $\mathbb{R}$ since
\[
\limsup_{n \in \omega} A_n' \supseteq \bigcup_{k \in \omega} \limsup_{n \in \omega} ((q_k + A_n) \cap B_k) = \mathcal{Q} + B
\]
and $\mathcal{Q} + B \in I^*$ by (\text{\textcircled{I}}). Fix an $S \in \mathcal{J}$. Since $I$ is translation invariant and $(\limsup_{n \in S} A_n) \cap B \in I$, we have $\limsup_{n \in \omega} ((q_k + A_n) \cap B_k) \in I$ for all $k \in \omega$. Since $I$ is a $\sigma$-ideal and $B_k$'s are pairwise disjoint, it follows that
\[
\limsup_{n \in S} A_n' = \bigcup_{k \in \omega} \limsup_{n \in \omega} ((q_k + A_n) \cap B_k) \in I.
\]
Of course, we can modify $(A_n')$ to be an infinite-fold cover of $\mathbb{R}$. \hfill $\square$

Theorem 5.3.2 can be generalized to any Polish group $G$ with $\mathcal{Q}$ replaced by a countable dense subset of $G$. Condition (\text{\textcircled{I}}) is related to the Steinhaus property, for details see [2]. Note that $\mathcal{M}$, $\mathcal{N}$, $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$ satisfy (\text{\textcircled{I}}) with $\mathcal{Q}$ replaced by any dense subset of $\mathbb{R}$ (resp. $\mathbb{R}^2$).

**5.4 $\mathcal{J}$-covering properties of $(\mathcal{P}(\omega), \mathcal{J})$**

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on $\omega$ where $\mathcal{J}$ is tall. We will say that $\mathcal{J}$ has the $\mathcal{J}$-covering property whenever the pair $(\mathcal{P}(\omega), \mathcal{J})$ has the $\mathcal{J}$-covering property.
It is trivial that \( \text{non}^{*}(\mathcal{J}) > \omega \) iff \( \mathcal{J} \) is \( \omega \)-hitting, that is, for every sequence \((X_n)_{n \in \omega}\) in \([\omega]^{\omega}\), there is an \( A \in \mathcal{J} \) such that \( |X_n \cap A| = \omega \) for each \( n \in \omega \). We will use a weaker version of this property: An ideal \( \mathcal{J} \) on \( \omega \) is \( \omega \)-hitting if for each sequence \((X_n)_{n \in \omega}\) in \([\omega]^{\omega}\) there is an \( A \in \mathcal{J} \) such that \( \{n \in \omega : |X_n \cap A| = \omega \} \) is infinite.

This property is really weaker than \( \text{non}^{*}(\mathcal{J}) > \omega \) by Lemma 5.4.2(4). Moreover, it is easy to see the following characterization:

**Proposition 5.4.1.** \( \mathcal{J} \) is weakly \( \omega \)-hitting if, and only if \( \mathcal{J} \not\lesssim \text{Fin} \otimes \text{Fin} \).

By the following easy lemma, in the characterization of “\( \mathcal{J} \) has the \( \mathcal{J} \)-covering property” the interesting case is when \( \mathcal{J} \) is weakly \( \omega \)-hitting but not \( \omega \)-hitting.

**Lemma 5.4.2.** Assume \( \mathcal{J} \) is a tall ideal on \( \omega \). Then

(1) \( \mathcal{J} \) is \( \omega \)-hitting iff all ideals have the \( \mathcal{J} \)-covering property.

(2) If \( \mathcal{J} \) is not \( \omega \)-hitting, then there is no ideal with the \( \mathcal{J} \)-covering property.

(3) If \( \mathcal{J} \) is weakly \( \omega \)-hitting and all tall ideals have the \( \mathcal{J} \)-covering property, then \( \mathcal{J} \) is \( \omega \)-hitting.

(4) If \( \mathcal{J} \) is weakly \( \omega \)-hitting but not \( \omega \)-hitting, then there is a tall ideal \( \mathcal{J}_0 \) such that (up to isomorphism) \( \mathcal{J} \) is contained in \( \mathcal{J}_0 \otimes \text{Fin} \). And all ideals of this form are weakly \( \omega \)-hitting but not \( \omega \)-hitting.

**Proof.** (1): It is trivial by Fact 5.1.3 that if \( \mathcal{J} \) is \( \omega \)-hitting, then all ideals have the \( \mathcal{J} \)-covering property. Conversely, clearly \( \text{Fin} \) has the \( \mathcal{J} \)-covering property iff \( \mathcal{J} \) is \( \omega \)-hitting.

(2): Let \((X_n)_{n \in \omega}\) witness that \( \mathcal{J} \) is not \( \omega \)-hitting and let \( F(n) = X_n \). Then \( \{n \in \omega : |F(n) \cap A| = \omega \} \) is finite for each \( A \in \mathcal{J} \), so by Fact 5.1.3, \( F \) shows that \( \mathcal{J} \) does not have the \( \mathcal{J} \)-covering property for all \( \mathcal{J} \) (because finite sets cannot be in \( \mathcal{J} \)).

(3): Assume on the contrary that \( \mathcal{J} \) is not \( \omega \)-hitting witnessed by the sequence \((X_n)_{n \in \omega}\). For each \( A \in \mathcal{J} \) let \( E_A = \{n \in \omega : |X_n \cap A| = \omega \} \). Clearly, \( E_A \) is co-infinite and \( E_{A \cup B} = E_A \cup E_B \) for \( A, B \in \mathcal{J} \), so these sets generate an ideal \( \mathcal{J} \) on \( \omega \). Moreover, it is trivial to see that \( \mathcal{J} = \{E_A : A \in \mathcal{J} \} \). \( \mathcal{J} \) is tall because of the weak \( \omega \)-hitting property of \( \mathcal{J} \). If \( F(n) = X_n \) then \( \{n \in \omega : |F(n) \cap A| = \omega \} \in \mathcal{J} \) for each \( A \in \mathcal{J} \), so \( \mathcal{J} \) does not have the \( \mathcal{J} \)-covering property, a contradiction.

(4): Let \( \mathcal{J}_0 = \mathcal{J} \) from the proof of (3). (Of course, we can assume that \( X_n \)'s are pairwise disjoint and \( \bigcup_{n \in \omega} X_n = \omega \).)

Assume \( \mathcal{J}_0 \) is a tall ideal. The columns in \( \omega \times \omega \) show that \( \mathcal{J}_0 \otimes \text{Fin} \) is not \( \omega \)-hitting. To prove the weak \( \omega \)-hitting property, fix a sequence \( X_n \in [\omega \times \omega]^{\omega} \) \( n \in \omega \). We can assume the followings:

(i) if there is a \( k \) such that \( |(X_n)_k| = \omega \), then \( X_n \subseteq \{k\} \times \omega \);

(ii) if \( n \neq m \) and \( (X_n)_k \) and \( (X_m)_k \) are finite for all \( k \in \omega \) then

\[ \{k \in \omega : (X_n)_k \neq \emptyset\} \cap \{k \in \omega : (X_m)_k \neq \emptyset\} = \emptyset. \]
5.4. \( \mathcal{J} \)-COVERING PROPERTIES OF \((\mathcal{P}(\omega), \mathcal{J})\)

If \( B = \{ n \in \omega : \exists k \in \omega \ X_n \subseteq \{ k \} \times \omega \} \) is finite, then let \( A = \bigcup_{n \in \omega \setminus B} X_n \in \mathcal{J}_0 \otimes \text{Fin} \).

If \( |B| = \omega \) then let \( B' \subseteq B, |B'| = \omega, B' \in \mathcal{J}_0 \), and let \( A = B' \times \omega \in \mathcal{J}_0 \otimes \text{Fin} \). Clearly, the set \( \{ n \in \omega : |X_n \cap A| = \omega \} \) is infinite in both of these cases.

In particular, if \( \mathcal{J} \) is weakly \( \omega \)-hitting but not \( \omega \)-hitting, then there is a tall ideal \( \mathcal{J} \) which does not have the \( \mathcal{J} \)-covering property, so the next natural question is the following: Does there exist an ideal \( \mathcal{J} \) with the \( \mathcal{J} \)-covering property in this case?

In next theorem we characterize ideals with the \( \mathcal{J}_0 \otimes \text{Fin} \)-covering property. First we recall an important notion: Assume \( \mathcal{J} \) is an ideal on \( \omega \). Then a filter \( \mathcal{F} \) is a \( \mathcal{J} \)-(ultra)filter if for each function \( f : \omega \rightarrow \omega \) there is an \( X \in \mathcal{F} \) such that \( f[X] \in \mathcal{J} \) (or equivalently, there is an \( A \in \mathcal{J} \) such that \( f^{-1}[A] \in \mathcal{F} \)). For combinatorial properties of \( \mathcal{J} \)-filters and investigation of their existence see for example [24] and J. Flašková’s other publications.

**Theorem 5.4.3.** Let \( \mathcal{J}_0 \) be a tall ideal. Then an ideal \( \mathcal{J} \) has the \( \mathcal{J}_0 \otimes \text{Fin} \)-covering property iff \( \mathcal{J}^+ \) is a \( \mathcal{J}_0 \)-filter.

**Proof.** Assume that \( \mathcal{J} \) has the \( \mathcal{J}_0 \otimes \text{Fin} \)-covering property and let \( f : \omega \rightarrow \omega \) be arbitrary.

Let \( F(n) = \{ f(n) \} \times \omega \). Then there is an \( A \in \mathcal{J}_0 \otimes \text{Fin} \) such that \( X = \{ n \in \omega : |F(n) \cap A| = \omega \} \in \mathcal{J}^+ \). We can assume that \( A \) is of the form \( A_0 \times \omega \) for some \( A_0 \in \mathcal{J}_0 \). Clearly \( f^{-1}[A_0] = X \in \mathcal{J}^+ \), so \( f[X] \in \mathcal{J}_0 \).

Conversely, assume that \( \mathcal{J}^+ \) is a \( \mathcal{J}_0 \)-filter and let \( F : \omega \rightarrow [\omega \times \omega]^\omega \) be arbitrary. We can assume the followings:

(i) if there is a \( k \) such that \( |F(n)| = \omega \), then \( (F(n))_k = \omega \);

(ii) if \( n \neq m \) and \( (F(n))_k \) and \( (F(m))_k \) are finite for all \( k \in \omega \) then \( \{ k \in \omega : (F(n))_k \neq \emptyset \} \cap \{ k \in \omega : (F(m))_k \neq \emptyset \} = \emptyset \).

Let \( X = \{ n \in \omega : \exists \ k \in \omega \ F(n) = \{ k \} \times \omega \} \). If \( X \in \mathcal{J} \) then we do not have to deal with it. If \( X \in \mathcal{J}^+ \) then let \( f : X \rightarrow \omega, f(n) = k_n \). Clearly \( \langle X, f \rangle^+ \) is also a \( \mathcal{J}_0 \)-filter, so there is an \( A_0 \in \mathcal{J}_0 \) such that \( X \setminus f^{-1}[A_0] \in \mathcal{J} \).

It is easy to see that

\[
A = (A_0 \times \omega) \cup \bigcup_{n \in \omega \setminus X} F(n) \in \mathcal{J}_0 \otimes \text{Fin}
\]

and \( \{ n \in \omega : |F(n) \cap A| = \omega \} \in \mathcal{J}^+ \).

**Question 5.4.4.** Is there any similarly easy and reasonable characterization of ideals with the \( \mathcal{J} \)-covering property for an arbitrary (weakly \( \omega \)-hitting but not \( \omega \)-hitting) \( \mathcal{J} \)?
Chapter 6

Generalizations of Hechler’s theorem

6.1 Hechler’s original theorem

A partially ordered set \((Q, \leq)\) is \(\sigma\)-directed if each countable subset of \(Q\) has a strict upper bound in \(Q\). Hechler’s original theorem is the following statement:

**Theorem 6.1.1.** ([29],[15]) Let \((Q, \leq)\) be a \(\sigma\)-directed partially ordered set. Then there is a ccc forcing notion \(P\) such that in \(V^P\) a cofinal subset of \((\omega^\omega, \leq^*)\) is order isomorphic to \((Q, \leq)\).

This theorem has a very important application. It says that we really cannot prove any more connections between \(b\) and \(d\) than \(\omega < \text{cf}(b) = b \leq \text{cf}(d) \leq d \leq c\).

**Theorem 6.1.2.** ([7, Theorem 2.5]) Assume GCH and let \(b', d',\) and \(c'\) three cardinals satisfying

\[ \omega < \text{cf}(b') = b' \leq \text{cf}(d') \leq d' \leq c' \]

and \(\text{cf}(c') > \omega\). Then there is a ccc forcing extension in which \(b = b', d = d',\) and \(c = c'\).

**Proof.** (Sketch.) Apply Hechler’s theorem to \(Q = [d']^{<b'}\) partially ordered by inclusion. \((Q, \subseteq)\) is \(\sigma\)-directed because any \(<b'\) elements of \(Q\) has a strict upper bound by regularity of \(b'\). There is an unbounded family of size \(b'\). Furthermore, \(b' \leq \text{cf}(d')\) implies that a subset of \(Q\) with cardinality smaller than \(d'\) cannot be cofinal. Using GCH we obtain that \(|Q| = d'\). These observations imply that \(V^P \models b = b', d = d',\) and \(P\) is the ccc forcing notion given by Hechler’s theorem.

Finally, \(P \ast B_{c'}\) has the ccc, \(B_{c'}\) does not change \(b\) and \(d\), and \(V^{P \ast B_{c'}} \models c = c'\).

In Section 6.4, we will present another interesting application of this theorem.

6.2 Hechler’s theorem for \(\mathcal{M}\) and \(\mathcal{N}\)

In [41] L. Soukup asked if Hechler’s theorem hold for classical \(\sigma\)-ideals as partially ordered sets with the inclusion. T. Bartoszyński, M.R. Burke, and M. Kada gave the following positive answers.
Theorem 6.2.1. ([4]) Let $(Q, \leq)$ be a $\sigma$-directed partially ordered set. Then there is a ccc forcing notion $\mathbb{P}$ such that in $V^\mathbb{P}$ a cofinal subset of $(\mathcal{M}, \subseteq)$ is order isomorphic to $(Q, \leq)$.

Theorem 6.2.2. ([16]) Let $(Q, \leq)$ be a $\sigma$-directed partially ordered set. Then there is a ccc forcing notion $\mathbb{P}$ such that in $V^\mathbb{P}$ a cofinal subset of $(\mathcal{N}, \subseteq)$ is order isomorphic to $(Q, \leq)$.

In this section, we recall the construction of forcing notion from 6.2.2 and its main properties.

We will use a special version of the localization forcing (see [16, Definition 3.1]): Let $T = \bigcup_{n \in \omega} \prod_{k < n} [\omega]^{\leq k}$ be the tree of initial slaloms. $p \in \text{LOC}^*$ iff $p = (s^p, w^p, F^p)$ where

1. $s^p \in T$, $w^p \in \omega$, $F^p \subseteq \omega^\omega$,
2. $|F^p| \leq w^p \leq |s^p|$, $q \leq p$ iff
   a. $s^q \supseteq s^p$, $w^q \supseteq w^p$, and $F^q \supseteq F^p$,
   b. $\forall n \in |s^q| \setminus |s^p| \forall f \in F^p f(n) \in s^q(n)$,
   c. $w^q \leq w^p + |s^q| - |s^p|$, $q \leq p$ iff $w^q \leq w^p + |s^q| - |s^p|$.

Lemma 6.2.3. ([16, Lemma 3.2, 3.3, and 3.4]) $\text{LOC}^*$ is $\sigma$-linked (so ccc) and adds a slalom over the ground model.

Let $(Q, \leq)$ be a partially ordered set such that each countable subset of $Q$ has a strict upper bound in $Q$. Let $Q^* = Q \cup \{Q\}$ and extend the partial order to this set with $x < Q$ for each $x \in Q$.

Fix a well-founded cofinal $R \subseteq Q$ and a rank function on $R^* = R \cup \{Q\}$, $\rho : R^* \rightarrow \text{On}$. Extend $\rho$ to $Q^*$ by letting $\rho(x) = \min\{\rho(y) : y \in R^*, x < y\}$ for $x \in Q \setminus R$. For $x, y \in Q^*$ define $x \prec y$ if $x < y$ and $\rho(x) < \rho(y)$. Further notations:

- $Q_x = \{y \in Q : y \prec x\}$ for $x \in Q^*$,
- $D_\xi = \{x \in D : \rho(x) = \xi\}$ for $D \subseteq Q$ and $\xi \in \text{On}$,
- $D_{<\xi} = \{x \in D : \rho(x) < \xi\}$ for $D \subseteq Q$ and $\xi \in \text{On}$,
- $D_{\leq x} = \{y \in D : \rho(y) = \rho(x), y \leq x\}$ for $D \subseteq Q$ and $x \in Q$.

If $E \subseteq D \subseteq Q$, we say that $E$ is downward closed in $D$, $E \subseteq_{d.c.} D$ in short, if $y \in E$ whenever $y \in D$ and $y \leq x \in E$ for some $x$.

Definition 6.2.4. ([16, Definition 3.1]) The forcing notions $\mathbb{N}_a$ for $a \in Q^*$ are defined by recursion on $\rho(a)$.

$$p = \{(s^p_x, w^p_x, F^p_x) : x \in D^p\} \in \mathbb{N}_a$$ where $D^p \in [Q_a]^{<\omega}$ if the following hold:
6.2. HECHLER’S THEOREM FOR $M$ AND $N$

(I) for $x \in D^p$, $s^p_x \in T$, $w^p_x \in \omega$, and $F^p_x$ is a set of nice $\mathbb{N}_x$-names for elements of $\omega^\omega$ with $|F^p_x| \leq w^p_x$;

(II) for $x \in D^p$, $\sum\{w^p_z : z \in D^p_{\leq x}\} \leq |s^p_x|$;

(III) for each $\xi \in g[D^p]$ there is an $\ell^p_\xi \in \omega$ such that $|s^p_x| = \ell^p_\xi$ for each $x \in D^p$.

If $p \in \mathbb{N}_a$ and $b \in Q_a$, define $p \upharpoonright b \in \mathbb{N}_b$ by letting

$$p \upharpoonright b = \{(s^p_x, w^p_x, F^p_x) : x \in D^p \cap Q_b\}.$$

$p \leq_{\mathbb{N}_a} q$ iff

(A) $D^p \supseteq D^q$;

(B) $\forall x \in D^q \ (s^q_x \supseteq s^q_x \land w^q_x \supseteq w^q_x \land F^q_x \supseteq F^q_x)$;

(C) $\forall x \in D^q \ \forall n \in |s^q_x| \land |s^q_x| \forall f \in F^q_x \ (p \upharpoonright n \upharpoonright \mathbb{N}_a \ f(n) \in s^q_x(n))$;

(D) $\forall \xi \in g[D^q] \ \forall x, y \in D^q_\xi \ (x < y \Rightarrow \forall n \in \ell^p_\xi \ \ell^q_\xi \ s^p_x(n) \leq s^p_y(n))$;

(E) $\forall \xi \in g[D^q]$

$$\sum\{w^p_x : x \in D^p_\xi\} \leq \sum\{w^q_x : x \in D^q_\xi\} + (\ell^p_\xi - \ell^q_\xi)$;

(F) $\forall \xi \in g[D^q] \ \forall E \subseteq_{d.c.} D^q_\xi \ \forall n \in \ell^p_\xi \ \ell^q_\xi$

$$\left|\bigcup\{s^p_x(n) : x \in E\}\right| \leq \sum\{w^q_x : x \in E\} + (n - \ell^q_\xi)$.$$

**Proposition 6.2.5.** ([16, Proposition 4.3])

(a) If $p, q \in \mathbb{N}_a$, $p \leq_{\mathbb{N}_a} q$, and $b \in Q_a$, then $p \upharpoonright b \leq_{\mathbb{N}_b} q \upharpoonright b$.

(b) $\leq_{\mathbb{N}_a}$ is a partial order.

(c) If $a, b \in Q^*$ and $p, q \in \mathbb{N}_a \cap \mathbb{N}_b$, then $p \leq_{\mathbb{N}_a} q \iff p \leq_{\mathbb{N}_b} q$.

From now on we write $\leq (\equiv \leq_{\mathbb{N}_a})$ instead of $\leq_{\mathbb{N}_a}$.

**Definition 6.2.6.** ([16, Definition 4.4]) For an $A \subseteq_{d.c.} Q$, let $\mathbb{N}_A = \{p \in \mathbb{N}_Q : D^p \subseteq A\}$, and for $p \in \mathbb{N}_Q$, let $p \upharpoonright A = \{s^p_x, w^p_x, F^p_x) : x \in D^p \cap A\} \in \mathbb{N}_A$. Furthermore, if $\xi \in \text{On}$ then let $\mathbb{N}_\xi = \mathbb{N}_{Q_{<\xi}}$, $p \upharpoonright \xi = p \upharpoonright Q_{<\xi} \in \mathbb{N}_\xi$, and $p \upharpoonright [\xi, \infty) = \{s^p_x, w^p_x, F^p_x) : x \in D^p \setminus Q_{<\xi}\}$.

So we have $\mathbb{N}_a = \mathbb{N}_{Q_a}$ for each $a \in Q^*$, and $\mathbb{N}_Q$ has the same meaning if we consider $Q$ either as an element of $Q^*$ or as a subset of $Q$.

**Lemma 6.2.7.** ([16, Lemma 4.6]) If $A, B \subseteq_{d.c.} Q$ and $A \subseteq B$, then $\mathbb{N}_A \subseteq_{c} \mathbb{N}_B$.

**Remark 6.2.8.** In [16] Lemma 4.6, exactly the following stronger result was proved: If $p \in \mathbb{N}_B$, $r \in \mathbb{N}_A$, and $r \leq p \upharpoonright A$ then there is a $q \in \mathbb{N}_B$ satisfying $q \leq p, r$. 
Lemma 6.2.9. ([16, Lemma 4.10]) $\mathbb{N}_Q$ has ccc.

We will use the following density arguments.

Lemma 6.2.10. ([16, Lemma 5.1, 5.2, 5.3, and 5.4]) If $a \in A \subseteq \text{d.c.} \mathbb{Q}$, $\xi \in \text{On}$, $N \in \omega$, and $\dot{f}$ is a nice $\mathbb{N}_a$-name for an element of $\omega^{\omega}$, then the following sets are dense in $\mathbb{N}_A$:

(i) $\{ p \in \mathbb{N}_A : a \in D^p \}$;

(ii) $\{ p \in \mathbb{N}_A : \xi \in \dot{q}[D^p] \land \ell^p_\xi \geq N \}$;

(iii) $\{ p \in \mathbb{N}_A : a \in D^p \land w^p_a \geq |F^p_a| + 1 \}$;

(iv) $\{ p \in \mathbb{N}_A : a \in D^p \land \dot{f} \in F^p_a \}$.

6.3 Hechler’s theorem for tall analytic P-ideals

Using the model of [16] we prove the following theorem (see also in [21]).

Theorem 6.3.1. Let $(Q, \leq)$ be a $\sigma$-directed partially ordered set. Then there is a ccc forcing notion $\mathbb{P}$ such that in $V^{\mathbb{P}}$ for each tall analytic P-ideal $I$ coded in $V$ a cofinal subset of $(I, \subseteq^*)$ is order isomorphic to $(Q, \leq)$.

Remark 6.3.2. Tallness is not really necessary in Theorem 6.3.1. It is enough to assume that $I$ can be represented by $\text{Exh}(\vartheta)$ such that $\{ n \in \omega : \vartheta(\{ n \}) < \varepsilon \} \notin I$ for each $\varepsilon > 0$. This property of $\text{Exh}(\vartheta)$ is really weaker than tallness.

For an $a \in Q$, let $\dot{S}_a$ be an $\mathbb{N}_Q$-name such that

$$\models_{\mathbb{N}_Q} \dot{S}_a = \bigcup \{ s^p_a : p \in \dot{G} \}.$$  

Using (i) and (ii) from Lemma 6.2.10, $\models_{\mathbb{N}_Q} \dot{S}_a \in \text{Slm}$ for each $a \in Q$. Furthermore using (iv) and the definition of $\mathbb{N}_Q$ we know that $\dot{S}_a$ is a slalom over $V[\dot{G} \cap \mathbb{N}_a]$, i.e.

$$\models_{\mathbb{N}_Q} \forall f \in \omega^{\omega} \cap V[\dot{G} \cap \mathbb{N}_a] \ f \preceq^* \dot{S}_a. \quad (\sharp)$$

At last, using the definition of $\mathbb{N}_Q$ it is clear that if $\varrho(a) = \varrho(b)$ and $a < b$ then

$$\models_{\mathbb{N}_Q} \forall \infty n \in \omega \ S_a(n) \subseteq S_b(n). \quad (\dagger)$$

Let $J = \text{Exh}(\vartheta)$ be a tall analytic P-ideal. We will use Corollary 4.2.5 (2b): Fix a bijection $e : \omega \rightarrow [\omega]^{<\omega}$ and for a slalom $S \in \text{Slm}$, let

$$I(S) = \bigcup_{n \in \omega} \{ e(k) : k \in S(n) \land \vartheta(e(k)) < 2^{-n} \} \subseteq J.$$ 

We prove that in $V^{\mathbb{N}_Q}$ the set $\{ I(\dot{S}_a) : a \in Q \} \subseteq J$ is

(i) cofinal, i.e. $\forall I \in J \cap V^{\mathbb{N}_Q} \exists a \in Q \ I \subseteq^* I(\dot{S}_a)$;
(ii) order isomorphic to \((Q, \leq)\), i.e. \(I(\dot{S}_a) \subseteq^* I(\dot{S}_b)\) iff \(a \leq b\).

The only difficult step is to show that \(a \not\leq b\) implies \(I(\dot{S}_a) \not\subseteq^* I(\dot{S}_b)\).

It is clear from (2.1) and from Corollary 4.2.5 (2b) that for each \(a \in Q\)
\[
\forces_{\mathbb{N}_Q} \bigwedge \mathcal{N} \cap V[\dot{G} \cap \mathbb{N}_a] \subseteq^* I(\dot{S}_a).
\] (\(\sharp\))

**Lemma 6.3.3.** \(\forces_{\mathbb{N}_Q} \{I(\dot{S}_a) : a \in Q\} \text{ is cofinal in } (\mathcal{J}, \subseteq^*)\).

**Proof.** Let \(\dot{I}\) be a nice \(\mathbb{N}_Q\)-name for an element of \(\mathcal{J}\). Using that \(\mathbb{N}_Q\) is ccc and that each countable subset of \(Q\) is (strictly) bounded in \(Q\), there is an \(a \in Q\) such that \(\dot{I}\) is an \(\mathbb{N}_a\)-name. Then \(\forces_{\mathbb{N}_Q} I(\dot{S}_a) \subseteq^* I(\dot{S}_b)\) by (\(\sharp\)). \(\square\)

**Lemma 6.3.4.** Assume \(a, b \in Q\) and \(a \leq b\). Then \(\forces_{\mathbb{N}_Q} I(\dot{S}_a) \subseteq^* I(\dot{S}_b)\).

**Proof.** If \(a \ll b\) then \(\forces_{\mathbb{N}_Q} I(\dot{S}_a) \subseteq^* I(\dot{S}_b)\) so we are done by (\(\sharp\)). If \(\varrho(a) = \varrho(b)\) then we are done by (\(\dagger\)). \(\square\)

We will need the following version of Lemma 6.2.10 (ii) which says that we can extend conditions in a natural way.

**Lemma 6.3.5.** Assume \(p \in \mathbb{N}_Q\), \(\xi \in \check{\varrho}[D^p]\), and \(m \geq \ell^p_{\xi}\). Then there is a \(q \leq p\) such that \(D^q_{\xi} = D^p_{\xi}\) and \(q \upharpoonright \xi\) forces that \(\forall b \in D^q_{\xi} \forall n \in [\ell^p_{\xi}, m]\)
\[
\mathcal{s}^q_{\xi}(n) = \left\{ \hat{f}(n) : \hat{f} \in \bigcup \{ F^p_{b'} : b' \in D^p_{\leq b} \} \right\}.
\]

**Proof.** First we choose an \(r \in \mathbb{N}_{\xi}\), \(r \leq p \upharpoonright \xi\) which decides \(\hat{f} \upharpoonright [\ell^p_{\xi}, m]\) for each \(\hat{f} \in \bigcup \{ F^p_{b'} : b' \in D^p_{\leq b} \}: r \forces_{\mathbb{N}_{\xi}} \hat{f} \upharpoonright [\ell^p_{\xi}, m] = g_f\) for some \(g_f \in \omega[\ell^p_{\xi}, m]\).

Now let \(q\) be the following condition:

(i) \(q \upharpoonright \xi = r\), \(q \upharpoonright [\xi + 1, \infty) = p \upharpoonright [\xi + 1, \infty)\), and \(D^q_{\xi} = D^p_{\xi}\);

(ii) if \(b \in D^q_{\xi}\) then let \(s^q_{b} = m + 1\), \(s^q_{b} \upharpoonright \ell^p_{\xi} = s^p_{b}\), \(w^q_{b} = w^p_{b}\), and \(F^q_{b} = F^p_{b}\);

(iii) if \(b \in D^q_{\xi}\) and \(n \in [\ell^p_{\xi}, m]\) then let
\[
\mathcal{s}^q_{\xi}(n) = \left\{ g_f(n) : \hat{f} \in \bigcup \{ F^p_{b'} : b' \in D^p_{\leq b} \} \right\}.
\]

Clearly \(q \in \mathbb{N}_Q\). We have to show that \(q \leq p\). (A), (B), (C), (D), and (E) hold trivially.

To see (F) assume \(E \subseteq_{d.c.} D^p_{\xi}\) and \(n \in [\ell^p_{\xi}, m]\) \(m + 1 = \ell^q_{\xi}\). Then
\[
\left| \bigcup \{ \mathcal{s}^q_{\xi}(n) : x \in E \} \right| = \left| \left\{ g_f(n) : \hat{f} \in \bigcup \{ F^p_{b'} : x \in E \} \right\} \right| \leq \sum \{ |F^p_{b'}| : x \in E \} \leq \sum \{ w^p_{b} : x \in E \} + (n - \ell^p_{\xi}).
\] \(\square\)
In Lemma 6.3.6 we will use the following notation: if \( s \in \mathcal{T} \) is an initial slalom then let

\[
I(s) = \bigcup_{n < |s|} \{ e(k) : k \in s(n) \land \vartheta(e(k)) < 2^{-n} \} \subseteq [\omega]^{<\omega}.
\]

Clearly, if \( p \in \mathbb{N}_Q \) and \( a \in D^p \), then \( p \upharpoonright_{\mathbb{N}_Q} I(s^a) \subseteq I(\hat{S}_a) \).

**Lemma 6.3.6.** Assume \( a, b \in Q \) and \( a \notin b \). Then \( p \upharpoonright_{\mathbb{N}_Q} I(\hat{S}_a) \not\subseteq I(\hat{S}_b) \).

**Proof.** Let \( p \in \mathbb{N}_Q \) and \( N \in \omega \). We have to find a \( q \leq p \) such that \( q \upharpoonright_{\mathbb{N}_Q} I(\hat{S}_a) \not\subseteq I(\hat{S}_b) \).

Using Lemma 6.2.10 (i) and (iii) we can assume that \( a, b \in D^p \) and \( |w^p_a| \geq |F^p_{a,b}| + 1 \).

Let \( N = \max\{|s^a|, |s^b|\} \). Using Lemma 6.2.10 we can assume that \( M \) is large enough such that \( \vartheta(|\{k\}|) \geq 2^{-M} \) for each \( k < N \). For each \( m \in \omega \) let

\[
X_m = \{ k \in \omega : 2^{-m-1} \leq \vartheta(|\{k\}|) < 2^{-m} \}.
\]

Let \( \xi = \varrho(b) \). Using that \( \mathbb{N}_{b'} \leq^* \mathbb{N}_b \) if \( b' \in D^p_{\leq b} \) by Lemma 6.2.7, we can define a descending sequence in \( \mathbb{N}_b : p \upharpoonright b \geq r_M \geq r_{M+1} \geq \ldots \) such that \( r_m \) decides \( \dot{f} \upharpoonright [\ell^p_\xi, m] \) for each \( \dot{f} \in \bigcup \{ F^p_{b'} : b' \in D^p_{\leq b} \} \). Let \( I_m : [\ell^p_\xi, m] \to [\omega]^{<\omega} \) be defined by

\[
r_m \upharpoonright_{\mathbb{N}_b} I_m(n) = \bigcup \{ e(\dot{f}(n)) : \dot{f} \in \bigcup \{ F^p_{b'} : b' \in D^p_{\leq b} \} \land \vartheta(e(\dot{f}(n))) < 2^{-n} \}.
\]

**Claim.** There is an \( m \geq M \) such that \( X_m \not\subseteq I(s^b) \cup \bigcup \{ I_m(n) : n \in [\ell^p_\xi, m] \} \).

**Proof of the Claim.** Assume on the contrary that there is no such an \( m \). Then

\[
X_m \subseteq I(s^b) \cup \bigcup \{ I_m(n) : n \in [\ell^p_\xi, m] \}
\]

for each \( m \geq M \). Clearly \( \omega \not\subseteq^* \bigcup_{m \geq M} X_m \) by tallness\(^1\), the sets \( I(s^b) \) and \( I_m(n) \) are finite, and if \( n \leq m_1 \leq m_2 \) then \( I_{m_1}(n) = I_{m_2}(n) \) so we have

\[
\omega \not\subseteq^* I(s^b) \cup \bigcup_{m \geq M} \bigcup_{n = \ell^p_\xi}^m I_m(n) \subseteq^* \bigcup_{m \geq M} I_m(n).
\]

Using that \( \vartheta(I_m(n)) \leq |D^p_{\leq b}| \frac{m}{2^m} \) we obtain that \( \omega \in \mathcal{T} \), a contradiction. \( \Box \)

Assume \( m \) is suitable in the Claim and let \( r = r_m \). Fix a \( k \in X_m \setminus (I(s^b) \cup \bigcup \{ I_m(n) : n \in [\ell^p_\xi, m] \}) \). Then there is a \( \tilde{k} \) such that \( e(\tilde{k}) = \{k\} \). Let \( \tilde{g} \) be the canonical \( \mathbb{N}_a \)-name for the constant function with value \( \tilde{k} \). Let \( p' \in \mathbb{N}_Q \) be the condition which extends \( p \) by putting \( \tilde{g} \) into \( F^p_{a,b} \) (this is really a condition extending \( p \) because of our assumption \( |w^p_a| \geq |F^p_{a,b}| + 1 \)). We know that \( p \upharpoonright b = p' \upharpoonright b \) because \( a \notin Q_b \) so \( r \leq p' \upharpoonright b \).

Using Remark 6.2.8 for \( Q_b \subseteq Q_{\xi, \xi} \), \( r \in \mathbb{N}_b \), and \( p' \upharpoonright \xi \in \mathbb{N}_\xi \) we can find a \( q' \in \mathbb{N}_\xi \) with \( q' \leq r, p' \upharpoonright \xi \). Let \( p'' = q' \cup p' \upharpoonright [\xi, \infty) \) and \( p'' \upharpoonright [\xi, \infty) \leq p \).

---

\(^1\)This is the only point in the proof where we used tallness of the ideal. As we mentioned in Remark 6.3.2, it would be enough to assume that \( \bigcup_{m \geq M} X_m = \{ k \in \omega : \vartheta(|\{k\}|) < 2^{-M} \} \not\subseteq \mathcal{T} \).
Finally, using Lemma 6.3.5 we can extend $p''$ to a $q$ such that $D^q_\xi = D^{p''}_\xi$ ($= D^p_\xi$) and
$q \upharpoonright \xi \Vdash_{N_1} \forall n \in [\ell^p_\xi, m) s^q_\xi(n) = \{ \bar{f}(n) : \bar{f} \in \bigcup \{ P^p_{\ell^p_\xi} : \bar{b}' \in D^p_{\ell^p_\xi} \} \}.$
Because $q \upharpoonright b \leq r$ we obtain that
$q \Vdash_{N_1} I(s^q_b) \subseteq I(s^p_b) \cup \bigcup \{ I_m(n) : n \in [\ell^p_\xi, m) \}.$

By the choice of $k$ and $p'$ it is clear that $\bar{k} \in s^q_b(m)$ and $\vartheta(e(k)) = \vartheta(\{k\}) < 2^{-m}$ so $k \in I(s^q_b)$ which implies that $q \Vdash_{N_1} k \in I(\tilde{S}_a) \setminus N.$

To show that $q \Vdash_{N_2} k \notin I(\tilde{S}_b)$ we know that $k \notin I(s^q_b)$ and if there would be a $\bar{q} \leq q$ such that $k \in I(s^\bar{q}_b),$ then there would be an $n > m$ and a $k' \in s^\bar{q}_b(n)$ such that $k \in e(k') \subseteq I(s^\bar{q}_b)$ but then $2^{-n} > \vartheta(e(k')) \geq \vartheta(\{k\}) \geq 2^{-m-1}$ would give a contradiction because $n \geq m + 1.$ The proof of Lemma 6.3.6 is done.

Now we have finished the proof of Theorem 6.3.1.

**Problem 6.3.7.** Are there any connections between known versions of Hechler’s theorem, more precisely: can we (easily) deduce one of them from another?

### 6.4 An application: coding with spectrum

J. D. Hamkins asked the following:

**Question 6.4.1.** Is it true that for each set $x$ there is a cardinal preserving forcing notion $\mathbb{P}$ such that $x$ is (first order) definable in $V^\mathbb{P}$ without any parameters?

L. Soukup noticed that spectrums of cardinal invariants and versions of Hechler’s theorem can be used for coding reals in generic extensions. For a given pre-ordered set $(P, \leq)$ S. Fuchino and L. Soukup defined (unpublished note) the *unbounded chain spectrum* of $(P, \leq)$: $Sp^1(P, \leq)$ is the set of all regular cardinals $\kappa$ such that there is an unbounded increasing chain of length $\kappa$ in $(P, \leq)$.

We refer the reader to [11] and [41] for other notions of spectra, their closedness properties, and for related consistency results.

Let $(B, \sqsubseteq)$ be a $\sigma$-directed Borel pre-ordered set, that is, $B \subseteq \omega^\omega$ and $\sqsubseteq$ as a subset of $\omega^\omega \times \omega^\omega$ are Borel-sets. We say that Hechler’s theorem holds for $(B, \sqsubseteq),$ if for all $\sigma$-directed partially ordered set $(Q, \leq)$ there is a ccc forcing notion $\mathbb{P}$ such that in $V^\mathbb{P}$ a cofinal subset of $(B, \sqsubseteq)$ is order isomorphic to $(Q, \leq)$.

Theorems presented in this chapter say that Hechler’s theorem holds for $(\omega^\omega, \leq^*)$, $(N, \subseteq)$, $(M, \subseteq),$ and for $(J, \subseteq^*)$ if $J$ is a tall analytic $\mathbb{P}$-ideal.

**Theorem 6.4.2.** (L. Soukup) Assume Hechler’s theorem holds for $(B, \sqsubseteq)$ and $X \subseteq \omega$ is a real. Then there is a ccc forcing notion $\mathbb{P}$ such that $V^\mathbb{P} \models X = \{ n \in \omega : \mathcal{K}_{n+1} \in Sp^1(B, \sqsubseteq) \}.$
Proof. Applying Hechler’s theorem to \((Q, \leq) = \prod_{n \in X} (\kappa_{n+1}, \leq)\) we have a ccc forcing notion \(P\) such that \(V^P \models \langle (Q, \leq) \rangle\). We will work in this extension.

Assume \(k \in X\). For each \(\alpha < \kappa_{k+1}\) let \(f^k_\alpha \in \prod_{n \in X} \kappa_{n+1}, f^k_\alpha(k) = \alpha\), and \(f^k_\alpha(\ell) = 0\) if \(\ell \neq k\). Clearly, \(\{f^k_\alpha : \alpha < \kappa_{k+1}\}\) shows that \(\kappa_{k+1} \in \text{Sp}^\uparrow (B, \subseteq)\).

Assume \(k \in \omega \setminus X\) and assume on the contrary that \(\langle g_\alpha : \alpha < \kappa_{k+1}\rangle\) is a \(\leq\)-increasing and \(\leq\)-unbounded sequence in \(\prod_{n \in X} \kappa_{n+1}\). If \(n \in X \cap k\) then \(\langle g_\alpha(n) : \alpha < \kappa_{k+1}\rangle\) is a nondecreasing sequence \(\kappa_{n+1}\) so it is constant from an \(\alpha_n < \kappa_{k+1}\).

If \(n \in X \setminus k\) then \(n > k\) and so \(\{g_\alpha(n) : \alpha < \kappa_{k+1}\}\) is bounded in \(\kappa_{n+1}\), let \(\alpha_n < \kappa_{n+1}\) be an upper bound of this set. The function \(g \in \prod_{n \in X} \kappa_{n+1}, g(n) = \alpha_n\) shows that \(\langle g_\alpha \rangle_{\alpha < \kappa_{k+1}}\) \(\leq\)-bounded, a contradiction.

If \((B, \subseteq) = (\omega^\omega, \leq^*)\), \((N, \subseteq)\), \((M, \subseteq)\) or \((I, \leq^*)\) for some \(\Sigma^1_1\) tall analytic \(P\)-ideal, then \(\text{Sp}^\uparrow (B, \subseteq)\) is first order definable without parameters so we obtain the following:

**Corollary 6.4.3.** If \(X \subseteq \omega\) is a real, then there is a ccc forcing extension in which \(X\) is first order definable without parameters.

It seems that this method (using spectrums of pre-ordered sets) does not work for coding subsets of uncountable cardinals because it turned out (see [41]) that some notions of spectrum have certain closedness properties so they cannot be arbitrary.
Chapter 7

Generalizations of the pseudo-intersection number

7.1 Intersection numbers of ideals

In the definition of the pseudo-intersection number one can consider other ideals than \( \text{Fin} \). It was done in various ways in many papers:

- \( \text{add}^*(\mathcal{I}) \) is the minimal cardinality of a family of elements of the dual filter \( \mathcal{I}^* \) which does not have a pseudo-intersection in \( \mathcal{I}^* \) (other notation: \( \text{p}(\mathcal{I}^*) \), see [13]);

- \( \text{cov}^*(\mathcal{I}) \) is the minimal cardinality of a family of elements of the dual filter \( \mathcal{I}^* \) which does not have a pseudo-intersection (other notation: \( \chi \text{p}(\mathcal{I}^*) \), see [13]);

- the minimal cardinality of a family \( \mathcal{A} \) with the \( \mathcal{I} \)-strong finite intersection property (every finite subfamily has an intersection outside \( \mathcal{I} \)) without a set outside \( \mathcal{I} \) which is almost included (in the sense of \( \mathcal{I} \)) in every member of \( \mathcal{A} \) (\( \text{p}_{\mathcal{I}} \), see [31]; \( \text{p}(\mathcal{I}) \), see [12]).

The aim of this chapter is to present another way of generalizing the pseudo-intersection number. This generalization is quite natural in the context of topological motivations. All results in this chapter are from [8].

We will use one more pre-order on ideals:

**Definition 7.1.1.** (One-to-on order) For ideals \( \mathcal{I} \) and \( \mathcal{J} \) let \( \mathcal{I} \leq_{1,1} \mathcal{J} \) if there is a one-to-one function \( f : \omega \to \omega \) such that \( A \in \mathcal{I} \Rightarrow f^{-1}[A] \in \mathcal{J} \).

Just like classical pre-orders on ideals (see Section 2.3), the one-to-one order on Borel ideals is also absolute between transitive models \( M \subseteq N \) with \( \omega_1^N \subseteq M \).

One can think about one more natural pre-order here, defined in the same way as \( \leq_{1,1} \) but with “bijection” instead of “one-to-one function”. However, the following fact shows that it would not give us anything new.

**Proposition 7.1.2.** If \( \mathcal{J} \) strictly extends \( \text{Fin} \) then \( \mathcal{I} \leq_{1,1} \mathcal{J} \) if, and only if there is a permutation \( g : \omega \to \omega \) such that \( A \in \mathcal{I} \Rightarrow g^{-1}[A] \in \mathcal{J} \).
Proof. The “if” part is trivial. Conversely, assume \( f \) is one-to-one and \( A \in \mathcal{I} \Rightarrow f^{-1}[A] \in \mathcal{J} \). We can modify \( f \) on an infinite element \( B \) of \( \mathcal{J} \) to be a permutation \( g \) such that \( g \downarrow (\omega \setminus B) \equiv f \downarrow (\omega \setminus B) \) and \( g[B] = f[B] \cup (\omega \setminus \text{ran}(f)) \). Then \( g \) is as required. \( \square \)

Using the above proposition, \( \mathcal{J} \leq_{1,1} \mathcal{J} \neq \text{Fin} \) means that \( \mathcal{J} \) can be permuted into \( \mathcal{J} \) (by \( g^{-1} \)). Clearly, \( \text{Fin} \leq_{1,1} \mathcal{J} \) for each \( \mathcal{J} \), and \( \mathcal{J} \) is maximal in this pre-order iff \( \mathcal{J} \) is a prime ideal. There is no largest element in this pre-order because there are \( 2^\omega \) many prime ideals but only \( \omega \) many permutations. Furthermore, \( \leq_{1,1} \) is \( \omega^+ \)-downward directed because the proof of the first part of Proposition 2.3.2 gives this stronger result.

**Definition 7.1.3.** For a pre-order (or simply a relation) \( \subseteq \) on ideals on \( \omega \) the \( \subseteq \)-intersection number of an ideal \( \mathcal{J} \) on \( \omega \) is

\[
p_{\subseteq}(\mathcal{J}) = \min \{ \chi(\mathcal{J}) : \mathcal{J} \not\subseteq \mathcal{I} \}
\]

provided there is an ideal \( \mathcal{I} \) such that \( \mathcal{J} \not\subseteq \mathcal{I} \).

The \( \subseteq \)-intersection number of a filter \( \mathcal{F} \) is \( p_{\subseteq}(\mathcal{F}) = p_{\subseteq}(\mathcal{F}^c) \).

We will be interested in the Katětov-intersection number (which we will denote for simplicity by \( p_{K}(\mathcal{J}) \)), the Katětov-Blass-intersection number (\( p_{KB}(\mathcal{J}) \)), and the one-to-one-intersection number (\( p_{1,1}(\mathcal{J}) \)).

We list some immediate facts.

**Fact 7.1.4.**

(a) \( p \leq p_{1,1}(\mathcal{J}) \leq p_{KB}(\mathcal{J}) \leq p_{1}(\mathcal{J}) \);

(b) if \( \mathcal{J} \subseteq \mathcal{I} \), then \( p_{\subseteq}(\mathcal{J}) \leq p_{\subseteq}(\mathcal{I}) \) for any pre-order \( \subseteq \);

(c) \( p_{1,1}(\mathcal{J}) = p \) for any \( \mathcal{J} \) which is not tall.

The last part of the above proposition explains why \( p_{K}(\mathcal{J}) \) and the cardinal invariants defined above generalize the pseudo-intersection number. In fact, we can indicate also the generalization of the notion of pseudo-intersection itself in this context. Assume \( \mathcal{J} \) is an ideal on \( \omega \). For an injective sequence \( x = (x_n) \in \omega^\omega \), the copy of \( \mathcal{J} \) on \( x \) is the ideal

\[
\mathcal{J}(x) = \{ A \subseteq \omega : \{ n \in \omega : x_n \in A \} \in \mathcal{J} \}.
\]

Let \( \mathcal{F} \) be a family with SFIP (or simply a filter) and assume that \( \mathcal{J} \) is an ideal. We say that an injective sequence \( x = (x_n) \in \omega^\omega \) is a \( \mathcal{J} \)-intersection of \( \mathcal{F} \) if \( \omega \setminus F \in \mathcal{J}(x) \) for each \( F \in \mathcal{F} \). In other words, a set \( X = \text{ran}(x) \) is a \( \mathcal{J} \)-intersection of \( \mathcal{F} \) if we can reorder the elements of \( X \) in such a way that elements of \( \mathcal{F} \) are in the copy of \( \mathcal{J}^c \) on the rearranged \( X \). Notice that \( x \) is a Fin-intersection of \( \mathcal{F} \) iff \( \text{ran}(x) \) is a pseudo-intersection of \( \mathcal{F} \). Plainly, \( p_{1,1}(\mathcal{J}) \) is the minimal cardinality of a family with SFIP and without an \( \mathcal{J} \)-intersection.

The cardinal \( \sup \{ p_{\subseteq}(\mathcal{J}) : \mathcal{J} \text{ is an ideal on } \omega \} \leq \omega \) is the smallest cardinal \( \kappa \) such that there is no \( \leq_{\omega} \)-upper bound of all ideals generated by at most \( \kappa \) many elements. First we show that this supremum is determined by cardinal exponentiation.
Proposition 7.1.5. \( p_\kappa(J) \leq \kappa \) for each ideal \( J \) if and only if \( 2^\kappa > 2^\omega \).

Proof. First, we prove the “if” part. Let \( \{(A^\alpha_n, A^\alpha_n): \alpha < \kappa\} \subseteq ([\omega]^\omega)^2 \) be an independent system, that is \( A^\alpha_n = \omega \setminus A^\alpha_{n+1} \) for each \( \alpha \), and if \( D \in [\kappa]^{<\omega} \) and \( f: D \to 2 \) then \( \bigcap\{A^\alpha_n: \alpha \in D\} = \omega \) (see [7, Proposition 8.9]).

For an \( F \in 2^\kappa \) let \( J_F \) be the ideal generated by \( \{A^\alpha_n: \alpha < \kappa\} \). Observe that \( \chi(J_F) = \kappa \) for every \( F \). Suppose for a contradiction that there is an ideal \( J \) such that \( J_F \leq_K J \) for each \( F \) witnessed by \( g_F \in \omega^\omega \). Since \( 2^\kappa > \epsilon \) there are distinct \( F_0, F_1 \in 2^\kappa \) such that \( g_{F_0} = g_{F_1} \). Let \( \alpha \) be such that \( F_0(\alpha) \neq F_1(\alpha) \). Then \( A^{F_0(\alpha)}_\alpha \cup A^{F_1(\alpha)}_\alpha = \omega \). Consequently
\[
g_{F_0}^{-1}[A^{F_0(\alpha)}_\alpha] \cup g_{F_1}^{-1}[A^{F_1(\alpha)}_\alpha] = g_{F_0}^{-1}[A^{F_0(\alpha)}_\alpha] \cup g_{F_0}^{-1}[A^{F_0(\alpha)}_\alpha] = g_{F_0}^{-1}[A^{F_0(\alpha)}_\alpha] \cup A^{F_1(\alpha)}_\alpha = \omega
\]
so \( \omega \in J \), a contradiction.

Now, we prove the converse implication. Assume that \( 2^\kappa = 2^\omega \). Then the family of ideals generated by at most \( \kappa \) elements has cardinality \( c \). Using Proposition 2.3.2, this family has a \( \leq_\kappa \)-upper bound \( J \) and so \( p_\kappa(J) > \kappa \), a contradiction. \( \square \)

At last we show an easy upper bound for \( p_{\text{kn}}(J) \) if \( J \) is meager:

Proposition 7.1.6. If \( J \) is meager, then \( p_{\text{kn}}(J) \leq \beta \).

Proof. It is enough to show that if \( J \) is meager and \( J \leq_{\text{kn}} J \), then \( J \) is also meager because then we can use Theorem 2.1.4 (there is a nonmeager ideal of character \( \beta \)).

Assume that the partition \( (P_n) \) witnesses that \( J \) is meager, i.e., \( \forall A \in J \forall n \in \omega \ P_n \not\subseteq A \) (see Theorem 2.1.3), and assume \( f: \omega \to \omega \) is finite-to-one and witnesses \( J \leq_{\text{kn}} J \). We can define a partition \( (P'_n) \) of ran(\( f \)) into finite sets by recursion on \( n \) such that \( \forall n \in \omega \exists k \in \omega \ P_k \subseteq f^{n-1}[P'_n] \). Then \( (P'_n) \) witnesses that \( J \upharpoonright \text{ran}(f) \) is meager and it clearly implies that \( J \) is meager too because of the natural homeomorphism \( \mathcal{P}(\omega) \to \mathcal{P}(\text{ran}(f)) \times \mathcal{P}(\omega \setminus \text{ran}(f)) \). \( \square \)

The assumption on meagerness of the ideal and the use of finite-to-one functions are necessary in the previous proposition because of part (a) and part (b) of Theorem 7.3.4.

7.2 The convex Fréchet-Urysohn property

In this section we will show that the Katětov-intersection number has applications in certain topological and analytical considerations. All topological spaces in what follows are Hausdorff. The weight of a space \( X \) (denoted by \( w(X) \)) is the minimal cardinality of a base of topology of \( X \). Recall that a topological space is Fréchet-Urysohn (FU) if for every subset \( A \) of this space and every \( x \in \overline{A} \) there is a sequence in \( A \) converging to \( x \).

The definition of the pseudo-intersection number can be reformulated in topological terms: it is the smallest weight of a (locally) countable (even completely regular or normal) space which is not FU (it is a special case of Theorem 7.2.2).

We can generalize the FU property for ideals using the notion of \( J \)-convergency. A sequence \( (x_n) \) in a space \( X \) \( J \)-converges to \( x \) if
\[
\forall U \subseteq X \text{ open } (x \in U \Rightarrow \{n \in \omega: x_n \notin U\} \in J).
\]
Definition 7.2.1. Let $\mathcal{I}$ be an ideal on $\omega$. A space $X$ satisfies the $\mathcal{I}$-Fréchet-Urysohn ($\mathcal{I}$-FU) condition if for every $A \subseteq X$ and every $x \in \overline{A}$ there is a sequence in $A$ $\mathcal{I}$-converging to $x$.

Clearly, if $\mathcal{I}$ is not tall, then the $\mathcal{I}$-FU condition is equivalent to the (Fin-)FU condition.

Theorem 7.2.2. $p_\kappa(\mathcal{I})$ is the smallest weight of a countable space which is not $\mathcal{I}$-FU.

Proof. Let $\mathcal{F}$ be a filter, $\chi(\mathcal{F}) = p_\kappa(\mathcal{I})$, and $\mathcal{F} \nsubseteq \mathcal{I}^\kappa$. Let $X = \omega \cup \{\mathcal{F}\}$ be equipped with the topology inherited from the Stone space of the Boolean algebra generated by $\mathcal{F}$, that is

(a) subsets of $\omega$ are open;

(b) $U \cup \{\mathcal{F}\}$ is an open neighborhood of $\mathcal{F}$ iff $U \cap \omega \in \mathcal{F}$.

Clearly, $w(X) = \chi(\mathcal{F})$ and $\mathcal{F} \in \overline{\omega}$. We claim that there is no sequence $(x_n)$ in $\omega$ which $\mathcal{I}$-converges to $\mathcal{F}$. Assume $(x_n)$ is a sequence in $\omega$ and let $f \in \omega^\omega$, $f(n) = x_n$. Using the assumption $\mathcal{F} \nsubseteq \mathcal{I}^\kappa$ we deduce that there is a $V \in \mathcal{F}$ such that $f^{-1}[V] \notin \mathcal{I}^\kappa$. Consider the open neighborhood $U = V \cup \{\mathcal{F}\}$ of $\mathcal{F}$. Then $\{n : x_n \notin U\} = \omega \setminus f^{-1}[V] \notin \mathcal{I}$ so $(x_n)$ does not $\mathcal{I}$-converge to $\mathcal{F}$.

Conversely, let $X$ be a countable space with $w(X) < p_\kappa(\mathcal{I})$, let $A \subseteq X$, and $x \in \overline{A} \setminus A$. Then the family $\{A \cap U : U$ is an open neighborhood of $x\}$ forms a filter-base on $A$. Let $\mathcal{F}$ be the generated filter. Since $\chi(\mathcal{F}) \leq w(X)$ we know that $\mathcal{F} \leq \mathcal{I}^\kappa$ is witnessed by a function $f \in \omega^\omega$.

We claim that the sequence defined by $x_n = f(n)$ $\mathcal{I}$-converges to $x$. Let $U$ be an open neighborhood of $x$. Then $\{n : x_n \notin U\} = \omega \setminus f^{-1}[A \cap U] \in \mathcal{I}$ and we are done. \qed

We can give another characterization of $p_\kappa(\mathcal{I})$ in the special case when $\mathcal{I} = \mathcal{Z}$. Recall that a completely regular space $X$ can be seen as a closed subspace of the space of Borel probability measures $P(X)$ on $X$ with the weak* topology.

The subbase of this topology is given by the following sets:

$$\mathcal{U}_{f,\varepsilon}(\mu) = \left\{ v \in P(X) : \left| \int_X f \, d\mu - \int_X f \, dv \right| < \varepsilon \right\}$$

where $\mu \in P(X)$, $f \in C_b(X) = \{\text{bounded continuous real-valued functions on } X\}$, and $\varepsilon > 0$. Recall that in this (so called weak*) topology $(\mu_n)$ converges to $\mu$ if, and only if

$$\int_X f \, d\mu_n \to \int_X f \, d\mu \text{ for every } f \in C_b(X).$$

The embedding $X \to P(X)$ is given by $x \mapsto \delta_x$ where $\delta_x$ is the Dirac-measure concentrated on $x$. We will use the notation $A^\delta = \{\delta_y : y \in A\}$ for $A \subseteq X$.

Denote by $\text{conv}(M)$ the convex hull of $M$ for $M \subseteq P(X)$. We will be interested in $\text{conv}(X^\delta)$, i.e. in the probability measures with finite support.

Definition 7.2.3. We say that $X$ satisfies the convex Fréchet-Urysohn condition if for every $A \subseteq X$, if $x \in \overline{A}$ then there is a sequence in $\text{conv}(A^\delta)$ which converges to $\delta_x$. 

In [39, Theorem 1] the following result was proved:

**Theorem 7.2.4.** Assume $X$ is compact. A measure $\mu \in P(X)$ is a weak* limit of measures of finite support if and only if $\mu$ has a uniformly distributed sequence $(x_k)$ in $X$, that is

$$
\mu = \lim_{n \to \infty} \frac{1}{n} \sum_{k<n} \delta_{x_k}.
$$

**Remark 7.2.5.**

1. In [39] this theorem was formulated only for compact spaces but the proof presented there does not use the assumption of compactness and the assertion is true for every (completely regular) topological space.

2. It is clear from the proof of this theorem that if $\mu_n \in \text{conv}(X^\delta)$ and $\mu_n \to \mu$, then the sequence $(x_k)$ can be chosen from $\bigcup_{n<\omega} \text{supp}(\mu_n)$.

3. Using this theorem, a space $X$ is convex FU iff if for every $A \subseteq X$, if $x \in \overline{A}$ then there is a sequence $(x_k)$ in $A$ such that $\delta_x = \lim_{n \to \infty} \frac{1}{n} \sum_{k<n} \delta_{x_k}$.

**Theorem 7.2.6.** Assume $X$ is completely regular. Then $X$ satisfies the convex FU condition if and only if $X$ satisfies the $\mathcal{Z}$-FU condition.

**Proof.** Assume $X$ satisfies the convex FU condition and let $A \subseteq X$, $x \in A \setminus A$. Then, according to Remark 7.2.5 there is a sequence $(x_k)$ in $A$ such that $\delta_x = \lim_{n \to \infty} \frac{1}{n} \sum_{k<n} \delta_{x_k}$.

We claim that $(x_k)$ $\mathcal{Z}$-converges to $x$. Assume on the contrary that there is an open neighborhood $U$ of $x$ such that $H = \{n: x_n \notin U\} \notin \mathcal{Z}$. Using complete regularity of $X$, there is a continuous $f : X \to [0, 1]$ such that $f(x) = 0$ but $f \lceil \{x_n : n \in H\} \equiv 1$. Then by the assumption on $(x_k)$ we have

$$
\int_X f \text{ d} \left( \frac{1}{n} \sum_{k<n} \delta_{x_k} \right) = \frac{1}{n} \sum_{k<n} f(x_k) \to \int_X f \text{ d} \delta_x = f(x) = 0
$$

but $\frac{1}{n} \sum_{k<n} f(x_k) \geq \frac{|H \cap n|}{n}$ for each $n$, a contradiction because $H \notin \mathcal{Z}$.

Conversely, assume that $X$ is $\mathcal{Z}$-FU and let $A \subseteq X$, $x \in A \setminus A$. Then there is a sequence $(x_k)$ in $A$ which $\mathcal{Z}$-converges to $x$.

We claim that $\delta_x = \lim_{n \to \infty} \frac{1}{n} \sum_{k<n} \delta_{x_k}$. Let $f \in C_b(X)$, $|f| \leq c$, $\varepsilon > 0$ and $x \in U$ be an open set such that $|f(x) - f(y)| < \varepsilon$ for each $y \in U$. If $H = \{k : x_k \in U\} \in \mathcal{Z}^*$, then

$$
\left| f(x) - \frac{1}{n} \sum_{k<n} f(x_k) \right| = \frac{1}{n} \sum_{k \in n \cap H} (f(x) - f(x_k)) + \sum_{k \in n \setminus H} (f(x) - f(x_k)) \leq \frac{1}{n} (|H \cap n| \cdot \varepsilon + |n \setminus H| \cdot 2c) \to \varepsilon \text{ if } n \to \infty.
$$

Because $\varepsilon$ was arbitrary, we are done.
So, in a sense we can call $p_K(Z)$ the \textit{convex pseudo-intersection number}.

The idea of the cardinal invariant $p_K(I)$ came from certain analytic considerations contained in [9], where authors were exploring a problem if there is a Mazur space without the Gelfand-Phillips property. The Gelfand-Phillips condition is widely used in functional analysis. The Mazur property is a certain condition weaker than reflexivity used in the theory of Pettis integrability (for the detailed discussion about these properties, confront [9]). It is known that there is a Gelfand-Phillips space without the Mazur property. It is still open if every Mazur space is Gelfand-Phillips.

In [9] the following question connected to the above considerations was raised.

Problem 7.2.7. Is there a minimally generated Boolean algebra $A \subseteq \mathcal{P}(\omega)$ such that no ultrafilter on $A$ has a pseudo-intersection but for every ultrafilter $F$ on $A$ we have $F \leq_K Z^*$?

In [9] it was shown that if there is such a Boolean algebra and this Boolean algebra is dense in $\mathcal{P}(\omega)$, then there is a space which is Mazur but not Gelfand-Phillips\textsuperscript{1}. Briefly speaking, the minimal generation implies that every measure on $A$ is in the sequential closure of measures of finite support. Since $F \leq_K Z^*$ for every ultrafilter $F$ on $A$, every measure of finite support is a limit of measures finitely supported on $\omega$ (cf. Remark 7.2.5 and Theorem 7.2.6 above) and so every measure is in the sequential closure of measures finitely supported on $\omega$. This property simplifies the form of functionals on the space of measures on $A$ and in this way it can be used to achieve Mazur property.

In [9] it was also proved that if $p_K(Z) > h$, then there is a Boolean algebra as described above.

In the next section we will show that consistently there is an ideal (unfortunately, not $\mathcal{Z}$) with the above property (see Remark 7.3.8).

Question 7.2.8. Do there exist reasonable topological characterizations of $p_{K}(J)$ or $p_{1,1}(J)$?

### 7.3 Consistency results

Let us denote $\text{Sym}(\omega)$ the set of permutations on $\omega$ with the usual Polish space topology and let $R$ be the following relation on $([\omega]^\omega \times [\omega]^\omega) \times \text{Sym}(\omega)$: $(X, Y, f) \in R$ iff $|f[X] \cap Y| = \omega$. We will need the following lemma:

**Lemma 7.3.1.** $([\omega]^\omega \times [\omega]^\omega, R, \text{Sym}(\omega)) \not\preccurlyeq_{\text{GT}} (\text{Sym}(\omega), \in, \mathcal{M})$. In particular, after adding a Cohen real there is an $f \in \text{Sym}(\omega)$ such that $|f[X] \cap Y| = \omega$ for all $X, Y \in [\omega]^\omega$.

**Proof.** If $X, Y \in [\omega]^\omega$ and $f \in \text{Sym}(\omega)$, then let $\phi(X, Y) = \{f \in \text{Sym}(\omega) : |f[X] \cap Y| < \omega\}$ and $\psi(f) = f$. Then $\phi(X, Y)$ is meager because its complement

$\text{Sym}(\omega) \setminus \phi(X, Y) = \bigcap_{n \in \omega} \{f \in \text{Sym}(\omega) : |f[X] \cap Y| \geq n\}$

\textsuperscript{1}In fact, in [9] it was shown for the case of one-to-one order but it can be immediately generalized.
7.3. CONSISTENCY RESULTS

and it is a countable intersection of dense open sets. It implies that $(\phi, \psi) : ([\omega]^\omega \times [\omega]^\omega, R, \text{Sym}(\omega)) \preccurlyeq_{\text{lat}} \langle \text{Sym}(\omega), \in, \mathcal{M} \rangle$.

Furthermore, it is easy to see that $\langle \text{Sym}(\omega), \in, \mathcal{M} \rangle \equiv_{\text{lat}} \langle 2^\omega, \in, \mathcal{M} \rangle$ so if a forcing notion $\mathbb{P}$ adds a Cohen real, then $\mathbb{P}$ is $(\text{Sym}(\omega), \in, \mathcal{M})$-dominating and so it is $(\omega^\omega, R, \text{Sym}(\omega))$-dominating as well.

Corollary 7.3.2. $\mathit{d}(\omega^\omega \times \omega^\omega, R, \text{Sym}(\omega)) \leq \mathit{non}(\mathcal{M})$ and $\mathit{cov}(\mathcal{M}) \leq \mathit{b}(\omega^\omega \times \omega^\omega, R, \text{Sym}(\omega))$. 

Question 7.3.3. Are there any other inequalities between these and classical cardinal invariants of the continuum?

It is well-known that if $\kappa > \omega$, then $V^{\mathbb{C}_x} \models b = \omega_1$.

Theorem 7.3.4. Assume GCH. Then in $V^{\mathbb{C}_{\omega_2}}$ the following hold:

(a) there is a filter $\mathcal{F}$ with $p_{\mathcal{F}}(\mathcal{F}) = \omega_2$;

(b) there is a meager filter $\mathcal{G}$ with $p_{\mathcal{G}}(\mathcal{G}) = \omega_2$;

(c) $p_{\mathcal{G}}(\mathcal{J}) = \omega_1$ for all $\mathcal{F}$, ideals and analytic $\mathbb{P}$-ideals.

Proof. (a): We interpret $\mathbb{C}_{\omega_2}$ as the $\omega_2$ stage finite support iteration of $\mathbb{C}$. Notice that for every subfamily of $[\omega]^\omega$ in $V^{\mathbb{C}_{\omega_2}}$ of size $\omega_1$, a nice name of it appears already in some $V^{\mathbb{C}_\alpha}$ for $\alpha < \omega_2$. Additionally, there are $\omega_2^\omega = \omega_2$ such families in $V^{\mathbb{C}_{\omega_2}}$ so we can fix an enumeration $\{\mathcal{F}_\alpha : \alpha < \omega_2\}$ of all names of bases of filters of cardinality $\omega_1$ in $V^{\mathbb{C}_{\omega_2}}$ in such a way that $\mathcal{F}_\alpha \in V^{\mathbb{C}_\omega}$ for each $\alpha$.

Using Lemma 7.3.1 we know that in $V^{\mathbb{C}_{\omega_2}}$ there is a sequence $(f_\alpha)_{\alpha < \omega_2}$ of permutations on $\omega$ such that $\dot{f}_\alpha \in V^{\mathbb{C}_{\omega_2+1}}$ for each $\alpha < \omega_2$ and $|\dot{f}_\alpha[X] \cap Y| = \omega$ for all $X, Y \in [\omega]^\omega \cap V^{\mathbb{C}_\omega}$.

We show that $V^{\mathbb{C}_{\omega_2}} \models "\bigcup \{\dot{f}_{\alpha'}[\dot{f}_{\alpha}] : \alpha < \omega_2\} \text{ forms a base of a filter}.$”

Indeed, consider $\alpha < \beta < \omega_2$ and $F \in \dot{\mathcal{F}}_{\alpha}$, $G \in \dot{\mathcal{F}}_{\beta}$. Since $\dot{f}_\alpha[F] \in V^{\mathbb{C}_{\omega_2+1}} \subseteq V^{\mathbb{C}_\beta}$ is infinite, the set $\dot{f}_{\alpha'}[G] \cap \dot{f}_\alpha[F]$ is also infinite. By induction we can show that every finite subfamily of this family has infinite intersection.

Clearly, the filter $\mathcal{F}$ generated by this family satisfies $p_{\mathcal{F}}(\mathcal{F}) = \omega_2$.

(b) follows from part (a), Fact 7.1.4 and Proposition 2.3.3.

(c): Now let $\mathbb{C}_{\omega_2}$ be the set of finite functions from $\omega_2 \times \omega$ to 2 ordered by reverse inclusion. Let $\mathcal{J}$ be the ideal generated by the first $\omega_1$ Cohen reals, i.e. by $\{c_{\alpha'}^{-1}(\{1\}) : \alpha < \omega_1\}$ where $c_{\alpha'} : \omega \rightarrow 2$ is the $\alpha$th Cohen real. We show that $\mathcal{J}$ witnesses part (c), i.e. for each $\mathcal{J}$ as in the theorem $\mathcal{J} \not\subseteq \mathcal{J}$.

Case 1: Let $\mathcal{J} = \text{Fin}(\phi)$ be an $\mathcal{F}_\alpha$ ideal. Assume $G$ is a $(V, \mathbb{C}_{\omega_2})$-generic filter and $f \in \omega^\omega \cap V[G]$. We can assume that $\phi(f^{-1}[E]) < \infty$ for each $E \in [\omega]^c_\omega$, because else $f$ cannot show any Katětov-reduction. There is a countable $H \subseteq \omega_2$ such that both $\phi \upharpoonright \text{Fin}$ and $f$ are in $V[G \cap \mathbb{C}_H]$ where $\mathbb{C}_H$ is the Cohen forcing which adds Cohen
reals indexed by elements of \( H \). If \( \alpha \in \omega_1 \setminus H \) then \( c_\alpha \) is Cohen over \( V[G \cap C_H] \) so it is enough to show that

\[
D_n = \{ p \in C : \varphi \left( f^{-1}[p^{-1}\{1\}] \right) > n \}
\]

is dense in \( C \) for each \( n \in \omega \) because then \( f \) cannot witness \( J \leq \kappa \) in the extension. Assume \( q \in C \) is defined on an initial segment. Because of our assumption on \( f \) we know that \( f^{-1}\{q\} \in \text{Fin}(\varphi) \) so we can choose a large enough \( \ell > |q| \) such that \( \varphi(f^{-1}[\ell \setminus |q|]) > n \). Define \( p \in C \) by \( p \upharpoonright |q| = q \) and \( p \upharpoonright [\ell, \ell) \equiv 1 \). Then \( p \leq q \) and \( p \in D_n \).

Case 2: Assume \( J = \text{Exh}(\varphi) \) is an analytic \( P \)-ideal, \( \|\omega\|_\varphi = 1 \). Similarly to the previous case, we have an \( f \in \omega^\omega \), we can assume that \( \|f^{-1}[E]\|_\varphi = 0 \) for each \( E \in [\omega]^\omega \), and it is enough to show that

\[
D_n = \{ p \in C : \varphi \left( f^{-1}[p^{-1}\{1\}] \right) \cap n > 0.5 \}
\]

is dense in \( C \) for each \( n \). Assume \( q \in C \) is defined on an initial segment. We can choose a large enough \( \ell > |q| \) such that \( \varphi(f^{-1}[\ell \setminus |q|]) \cap n > 0.5 \). Let \( p \) be chosen as in Case 1. Then \( p \leq q \) and \( p \in D_n \).

We list here some related questions:

**Problem 7.3.5.**

- Is \( p_\kappa(J) \leq b \) for each analytic (\( P \))-ideal \( J \)?
- Is \( p_{\alpha,1}(J) = p_{\kappa\alpha}(J) \) for each ideal \( J \)?
- Is \( p < p_{\alpha,1}(J) \) (or at least \( p < p_{\kappa\alpha}(J) \)) consistent for some meager (or even analytic (\( P \))) ideal \( J \)? Also, for the purposes described in Section 7.2, the consistency of \( h < p_\kappa(\emptyset) \) is particularly interesting.
- Is \( p_{\kappa\alpha}(J) < b \) (or at least \( p_{\alpha,1}(J) < b \)) consistent for some tall ideal \( J \)?

In Proposition 7.1.5 we showed that \( 2^\kappa > 2^{\omega_1} \) implies \( p_\kappa(J) \leq \kappa \) and thus \( p_{\kappa\alpha}(J) \leq \kappa \) for each ideal \( J \), and that the converse implication for the Katětov-order also holds. Now, we show that \( \forall J p_{\kappa\alpha}(J) \leq \omega_1 \) does not imply \( 2^{\omega_1} > 2^\omega \).

**Theorem 7.3.6.** It is consistent with \( ZFC \) that \( 2^{\omega_1} = 2^\omega \) is arbitrary large and the Katětov-Blass-order is not upward directed on filters generated by \( \omega_1 \) sets. In particular, \( p_{\kappa\alpha}(J) \leq \omega_1 \) for each filter \( J \).

**Proof.** Let \( 2^\omega = 2^{\omega_1} \) be arbitrary. We will construct two filters \( F \) and \( G \) generated by \( \leq^* \)-descending sequences \( \{X_\alpha : \alpha < \omega_1\} \) and \( \{Y_\alpha : \alpha < \omega_1\} \) inductively in a model obtained by an \( \omega_1 \) stage finite-support iteration of \( \sigma \)-centered forcing notions \( (\mathbb{P}_\alpha, \hat{Q}_\beta : \alpha < \omega_1, \beta < \omega_1) \).

It is well-known that a \( \leq^* \)-step finite-support iteration of \( \sigma \)-centered forcing notions is \( \sigma \)-centered, in particular \( ccc \), so it does not collapse cardinals. It will be trivial that \( |\mathbb{P}_{\omega_1}| = \mathfrak{c} \) so \( V^{\mathbb{F}_{\omega_1}} = (2^\omega)^V \) and \( 2^{\omega_1} = (2^\omega)^V \).
At the $\alpha$th stage (in $V^{\alpha}$) we have initial segments $\{X_\xi : \xi < \alpha\}$ and $\{Y_\xi : \xi < \alpha\}$. Let $X_\alpha'$ and $Y_\alpha'$ be pseudo-intersections of these sequences. We want to add $X_\alpha' \in [X_\alpha']^\omega$ and $Y_\alpha' \in [Y_\alpha']^\omega$ such that $|f^{-1}[X_\alpha] \cap g^{-1}[Y_\alpha]| < \omega$ for each pairs $(f,g) \in V^{\alpha}$ of finite-to-one functions from $\omega$ to $\omega$. Then in the final extension $\mathcal{F}$ and $\mathcal{G}$ cannot have a common upper bound in $\leq_{\text{sb}}$.

Let $X = X_\alpha'$ and $Y = Y_\alpha'$, and let us denote $\text{FO}$ the set of all finite-to-one functions from $\omega$ to $\omega$. Let $Q = Q_\alpha$ be the following forcing notion: $(s,t,q) \in Q$ iff

1. $s \in [X]^{<\omega}$ and $t \in [Y]^{<\omega}$;
2. $q$ is a finite partial function from $\text{FO} \times \text{FO}$ to $\omega$ such that

\[ \forall (f,g) \in \text{dom}(q) \left[ f^{-1}[s] \cap g^{-1}[t] \right] \leq q(f,g). \]

Define the order in the following way: $(s,t,q) \leq (s',t',q')$ iff $s \supseteq s'$, $t \supseteq t'$, $\text{dom}(q) \supseteq \text{dom}(q')$, and $q(f,g) \leq q'(f,g)$ for each $(f,g) \in \text{dom}(q')$.

Clearly, it is a partial order. It is also $\sigma$-centered: fix $s \in [X]^{<\omega}$, $t \in [Y]^{<\omega}$, and consider conditions $(s,t,q) \in Q$ for $i < \kappa$. Let $q$ be the following partial function:

\[ \text{dom}(q) = \bigcup \{ \text{dom}(q_i) : i < \kappa \} \]

and $q(f,g) = \min \{ q_i(f,g) : i < \kappa, (f,g) \in \text{dom}(q_i) \}$. Then $(s,t,q) \leq (s,t,q_i)$ for every $i < \kappa$.

Let $A$ and $B$ be $\mathbb{Q}$-names for the union of the first and respectively second coordinates of the conditions in the generic filter. We claim that these sets can serve as $X_\alpha$ and $Y_\alpha$.

$\models_{\mathbb{Q}} \text{"} A$ is infinite": The set $D_n = \{ p \in \mathbb{Q} : |s^p| > n \}$ is dense in $\mathbb{Q}$ for each $n$ (where $p = (s^p, t^p, g^p)$) because if $p \in \mathbb{Q}$ is arbitrary, then $s^p$ can be extended by any elements of the infinite set $\omega \setminus \bigcup \{ f''[g^{-1}[t^p]] : (f,g) \in \text{dom}(g^p) \}$.

Similarly, $\models_{\mathbb{Q}} \text{"} B$ is infinite."

At last, we have to show that $E_{f,g} = \{ p \in \mathbb{Q} : (f,g) \in \text{dom}(g^p) \}$ is dense in $\mathbb{Q}$ for each pair $f,g \in \text{FO}$. Any $p \in \mathbb{Q}$ can be extended by adding $(f,g)$ to $\text{dom}(g^p)$ and choosing $g(f,g)$ to be large enough. \hfill $\Box$

**Problem 7.3.7.** Is it consistent with ZFC that $2^{\omega_1} = 2^\omega$ and $p_{\text{sb}}(\mathcal{J}) \leq \omega_1$ (or at least $p_{\text{sb}}(\mathcal{J}) \leq \omega_1$) for each ideal $\mathcal{J}$ but $\leq_{\text{sb}}$ (respectively $\leq_{1,1}$) is upward directed on ideals generated by $\omega_1$ elements?

Note that if it is possible for $\leq_{1,1}$, then in such a model $\mathbb{M}$ is $\omega_2$. It is easy to see that if $\leq_{1,1}$ is upward directed on ideals generated by $\omega_2$ sets, then any $\omega_2$ ideals with character $\omega_1$ have an $\leq_{1,1}$-upper bound. If there would be only $2^{\omega} = 2^{\omega_1} = \omega_2$ many ideals with character $\omega_1$, then they would form a $\leq_{1,1}$-bounded set, i.e. there would be an ideal $\mathcal{J}$ with $p_{\text{sb}}(\mathcal{J}) > \omega_1$.

**Remark 7.3.8.** In [9, Theorem 7.4] the authors proved that consistently there is a Boolean algebra $\mathfrak{A} \subseteq \mathcal{P}(\omega)$ such that all ultrafilters on $\mathfrak{A}$ are meager but none of them has a pseudo-intersection. Using Theorem 7.3.4 (a) and the fact that $h = \omega_1$ in the Cohen model, we can mimic this proof to show a similar result. Namely, we can prove that, consistently, there is an ideal $\mathcal{J}$ and a Boolean algebra $\mathfrak{A} \subseteq \mathcal{P}(\omega)$ such that for each ultrafilter $\mathcal{F}$ on $\mathfrak{A}$, there is no pseudo-intersection of $\mathcal{F}$ but $\mathcal{F} \not\subseteq \mathcal{J}$. 

7.3. CONSISTENCY RESULTS 67
7.4 Permuting MAD families into ideals

The pseudo-intersection number is not the only cardinal invariant which can be generalized in the way presented above. We can easily define the analog of the tower number:

\[ t_{i,1}(I) = \min \{ \chi(\langle I \rangle_{B}) : I \text{ is a tower and } \langle I \rangle_{B} \not\preceq_{i,1} J^{+} \}, \]

where \( I \) is an ideal and \( \langle I \rangle_{B} \) is the filter generated by \( I \). As in the case of \( p_{i,1} \), we have \( t_{i,1}(\text{Fin}) = t \). However, in general this cardinal may not be well-defined, i.e. maybe the family of all (filters generated by) towers are \( \leq_{i,1} \)-bounded. E.g. consider the filter \( I \) from Theorem 7.3.4 (a). Since in the Cohen-model there are no towers of character \( \omega_{2} \) [Kunen, unpublished] and every filter generated by \( \omega_{1} \) sets is \( \leq_{i,1} \)-below \( I \), every tower is \( \leq_{i,1} \)-below \( I \), and \( t_{i,1}(I^{+}) \) is undefined.

Similarly, we can define the cardinal invariant \( a_{i,1}(I) \) analogous to the almost-disjointness number \( a \):

**Definition 7.4.1.** Let \( I \) be an ideal on \( \omega \). An almost disjoint family \( A \) is \( I \)-maximal if id(\( A \)) \( \not\preceq_{i,1} I \).

Using Proposition 7.1.2, if \( I \neq \text{Fin} \) then an AD family is \( I \)-maximal iff it cannot be permuted into \( I \).

Clearly, an AD family is Fin-maximal iff it is a MAD family. Furthermore, if \( I \leq_{i,1} J \) and an AD family is \( J \)-maximal, then it is \( I \)-maximal as well. In particular, each \( I \)-maximal AD family is a MAD family. From now on we will use the phrase “\( I \)-maximal MAD family.” It is trivial that if \( I \) is not tall, then each MAD family is \( I \)-maximal.

As before, let

\[ a_{i,1}(I) = \min \{ \chi(\langle A \rangle_{id}) : A \text{ is } I \text{-maximal} \}. \]

This section is devoted to study when this cardinal invariant is well-defined, i.e. when there is an \( I \)-maximal MAD family.

**Proposition 7.4.2.** \( \text{add}^{*}(I) = c \) implies that there is an \( I \)-maximal MAD family.

*Proof.* Fix an enumeration \( (f_{a} : \omega \leq a < c) \) of injective sequences of natural numbers. We will construct the desired MAD family inductively. Start with a disjoint partition \( (A_{a}) \) of \( \omega \) into infinite sets and assume we have constructed all \( A_{a} \)'s for \( \xi < a < c \).

If for some \( \xi < a \) we have \( f_{a}^{-1}[A_{\xi}] \notin I \), then take \( A_{a} = A_{\xi} \).

If not, then consider the family \( \{ f_{a}^{-1}[A_{\xi}] : \xi < a \} \subseteq J \). Using the assumption \( \text{add}^{*}(I) = c \) we can find a set \( B \in I \) such that \( f_{a}^{-1}[A_{\xi}] \preceq^{*} B \) for every \( \xi < a \). Let \( A_{a} = f_{a}^{*}[\omega \setminus B] \). Then \( \{ A_{\xi} : \xi \leq a \} \) is an AD family, and \( f_{a}^{-1}[A_{a}] \in J^{+} \).

In this way we will construct an AD family \( A = \{ A_{a} : \alpha < c \} \) such that id(\( A \)) \( \not\preceq_{i,1} I \).

We will show that under Martin’s Axiom for \( \sigma \)-centered posets (i.e. \( p = c \), see [6] or [7, Theorem 7.12]) there are \( I \)-maximal MAD families for each \( F_{\sigma} \) ideal and analytic \( P \)-ideal \( I \). Recall that this axiom does not imply that \( \text{add}(N) = c \) (see [25, 522S]) and \( \text{add}^{*}(I) = \text{add}(N) \) for a lot of tall analytic \( P \)-ideals (e.g. for tall summable and tall density ideals, see Corollary 4.2.5 and Remark 4.2.6).
Theorem 7.4.3. Let $\mathcal{I}$ be a tall $F_\sigma$ ideal or a tall analytic $P$-ideal. Then $\text{MA}(\sigma\text{-centered})$ implies that there is an $\mathcal{I}$-maximal MAD family.

Proof. Let $\mathcal{I}$ be a tall analytic $P$-ideal and fix an enumeration $(f_\alpha : \omega \leq \alpha < \epsilon)$ of injective sequences of natural numbers. As in the proof of Proposition 7.4.2, we will construct the desired MAD family inductively. Start with a disjoint partition $(A_n)$ of $\omega$ into infinite sets and assume we have constructed all $A_\zeta$'s for $\xi < \alpha < \epsilon$.

As before, if for some $\xi < \alpha$ we have $f_\alpha^{-1}[A_\xi] \notin \mathcal{I}$, then let $A_\alpha = A_\xi$. If not, consider the almost disjoint family $\mathcal{A}$ of sets $f_\alpha^{-1}[A_\xi] \in \mathcal{I}$ for $\xi < \alpha$.

We claim that $\text{MA}(\sigma\text{-centered})$ implies that $\mathcal{A}$ can be extended to an $\mathcal{I}$ family by a set $C$ from $\mathcal{I}^+$. It is enough because then we can choose $A_\alpha = f_\alpha''[C]$ and proceed as in the proof of Proposition 7.4.2.

Let $\mathcal{I}$ be a tall $F_\sigma$ ideal or a tall analytic $P$-ideal and assume that $\mathcal{A} \subseteq \mathcal{I}$ is an $\mathcal{I}$ family. Then we have to find a $\sigma\text{-centered}$ forcing notion $\mathbb{P}(\mathcal{A})$ such that in $V^{\mathbb{P}(\mathcal{A})}$ the family $\mathcal{A}$ can be extended by an $\mathcal{I}$-positive set.

Let $\mathbb{P}(\mathcal{A})$ be the natural forcing notion that extends $\mathcal{A}$ with a new element. Namely, let $p = (n^p, s^p, B^p) \in \mathbb{P}(\mathcal{A})$ iff $n^p \in \omega$, $s^p \subseteq n$, and $B^p \in [\mathcal{A}]^{\omega}$. We say that $p \leq q$ iff $n^p \geq n^q$, $s^p \cap n^q = s^q$, and $(s^p \setminus s^q) \cap \bigcup B^q = \emptyset$.

It is easy to see that $\mathbb{P}(\mathcal{A})$ is $\sigma$-centered and that the sets $D_k = \{ p \in \mathbb{P}(\mathcal{A}) : |s^p| > k \}$ and $D_A = \{ p \in \mathbb{P}(\mathcal{A}) : A \in B^p \}$ are dense in $\mathbb{P}(\mathcal{A})$ for each $k \in \omega$ and $A \in \mathcal{A}$. Consequently, if $\hat{S}$ is a $\mathbb{P}(\mathcal{A})$-name such that $\Vdash_{\mathbb{P}(\mathcal{A})} \hat{S} = \bigcup \{ s^p : p \in \hat{G} \}$ (where $\hat{G}$ is the canonical name of the generic filter), then $\Vdash_{\mathbb{P}(\mathcal{A})} \hat{S} \in [\omega]^\omega$ and $\Vdash_{\mathbb{P}(\mathcal{A})} |\hat{S} \cap A| < \omega$ for each $A \in \mathcal{A}$.

We have to show that $\Vdash_{\mathbb{P}(\mathcal{A})} \hat{S} \in \mathcal{I}^+$. We have two cases:

Case 1: $\mathcal{I} = \text{Fin}(\varphi)$ is a tall $F_\sigma$ ideal. We will show that $\Vdash_{\mathbb{P}(\mathcal{A})} \varphi(\hat{S}) = \infty$. It is enough to prove that $E_k = \{ p \in \mathbb{P}(\mathcal{A}) : \varphi(s^p) > k \}$ is dense in $\mathbb{P}(\mathcal{A})$ for each $k \in \omega$. Fix a $p \in \mathbb{P}(\mathcal{A})$. Since $\bigcup B^p \in \mathcal{I}$ we have $\varphi(\omega \setminus (n^p \cup \bigcup B^p)) = \infty$ so by the lsc property of $\varphi$ we can find a finite $F \subseteq \omega \setminus (n^p \cup \bigcup B^p)$ such that $\varphi(F) > k$. If $q = (\max(F) + 1, s^p \cup F, B^p)$, then $q \in E_k$ and $q \leq p$. We are done.

Case 2: $\mathcal{I} = \text{Exh}(\varphi)$ is a tall analytic $P$-ideal, $\|\omega\|_{\varphi} = 1$. We will show that $\Vdash_{\mathbb{P}(\mathcal{A})} \|\hat{S}\|_{\varphi} = 1$. It is enough to prove that $H^e_k = \{ p \in \mathbb{P}(\mathcal{A}) : \varphi(s^p \setminus k) > e \}$ is dense in $\mathbb{P}(\mathcal{A})$ for each $e < 1$ and $k \in \omega$. To see this, fix a condition $p \in \mathbb{P}(\mathcal{A})$. Since $\mathcal{A} \subseteq \mathcal{I}$ and $\| \cdot \|_{\varphi}$ is subadditive we conclude that

$$\varphi(\omega \setminus (n^p \cup k \cup \bigcup B^p)) \geq \| \omega \setminus (n^p \cup k \cup \bigcup B^p) \|_{\varphi} = 1$$

so by the lsc property of $\varphi$ we can find a finite $F \subseteq \omega \setminus (n^p \cup k \cup \bigcup B^p)$ such that $\varphi(F) > e$. If $q = (\max(F) + 1, s^p \cup F, B^p)$, then $q \in H^e_k$ and $q \leq p$. The proof is complete.

We finish with some related questions:

Question 7.4.4.

- Does there exist an $\mathcal{I}$-maximal MAD family for a tall (analytic) ideal $\mathcal{I}$ in $\text{ZFC}$?
- Is it consistent with $\text{ZFC}$ that there is no $\mathcal{I}$-maximal MAD family for some (nice) $\mathcal{I}$?
• Is it consistent with ZFC + ¬CH that there are $\mathcal{I}$-maximal MAD families for each ideal $\mathcal{I}$?
Bibliography


