

M Ű E G Y E T E M 1 7 8 2

Budapest University of Technology and Economics
Department of Applied Mechanics



Attila KOSSA

**Exact stress integration schemes
for elastoplasticity**

Booklet of PhD Theses

2011, Budapest

Supervisor: László SZABÓ, DSc

Introduction

Understanding the theoretical background of applied mechanics is a crucial basis in engineering practice, within which the modelling of deformations of materials is an important component. In the mechanical and civil engineering fields, the calculations involve mostly the theory of applied solid mechanics.

In order to describe the relationship between the deformation and the applied loads, material models and constitutive equations have been introduced. The most simple form of such models is Hooke's law, which assumes purely linear material behavior. However, Hooke's law can be used only for a limited class of problems, where the application of the linear elastic material model can provide numerical results within the required engineering accuracy. In order to predict more accurately the material behavior, more complicated constitutive models are needed. By combining the modelling of the plastic deformation with the elastic material law, we obtain the elastoplastic models. There are many physical phenomena corresponding to the plastic deformation which can be modelled. Depending on the order of such modelling aspects, various elastoplastic models exist. The most frequently used are usually included in commercial finite element software packages. The most important ingredient of elastoplastic models is the yield criterion employed in the construction of the constitutive equation. The von Mises and the Drucker–Prager yield criteria are two widely adopted approaches in elastoplastic calculations.

The von Mises yield criterion is usually suggested for metals, where the hydrostatic pressure does not exhibit influence on the plastic behavior of the material. By including the effects of the hydrostatic pressure into the definition of the yield criterion, we can arrive at the Drucker–Prager yield criterion, which is applied for pressure-dependent materials such as soils, concrete and some polymers.

Elastoplastic deformations can be described more accurately by extending the models with hardening rules, which can be categorized into the following two classes: isotropic and kinematic hardening rules.

In addition, they can be mixed resulting in the combined hardening rule.

The selection of the appropriate flow rule is also an important component in constitutive modelling. When the plastic strain rate tensor is obtained using the gradient of the yield surface, the flow rule is said to be associative. Whereas, if the plastic strain rate tensor is computed using the gradient of a plastic potential function, then we have the non-associative case.

Elastoplastic problems can be divided into the following two cases: a) strain-driven description; b) stress-driven description. Closed-form solution of elastoplastic problems exist only for a limited class of regular geometries with simple loading. In most cases, we have 3D geometry with complicated loads and boundary conditions. Hence, the corresponding boundary value problem can be so complicated that obtaining an analytical solution is practically impossible. The application of the finite element method is a useful approach in solving these problems numerically. Usually, the finite element codes we use for elastoplastic problems are formulated using displacement-based formulation, which implies the strain-driven description. During the calculation, we have to compute the stress increment in every integration point of the elements. The numerical results are strongly dependent on the numerical technique employed in the calculation of the stress increment. There are various numerical stress update methods proposed for elastoplastic finite element calculations. In order to have more accurate results, the application of exact stress integration schemes is obvious. The efficiency of the finite element calculations is strongly dependent on the stiffness matrix used in the solution of the global nonlinear equilibrium equation. In order to provide quadratic rate on convergence, the construction of the so-called consistent tangent tensor corresponding to the stress update formulae is crucial.

Aims of the work

The main goal of this work is to present exact stress and strain solutions for two widely used elastoplastic models, which are briefly summarized in Table 1. The corresponding yield surfaces in the principal stress space are illustrated in Figure 1.

Table 1: Features of the models under consideration

| | Model 1 | Model 2 |
|-----------------|-----------------|------------------|
| yield criterion | von Mises | Drucker–Prager |
| flow rule | associative | non-associative |
| hardening rule | combined linear | linear isotropic |

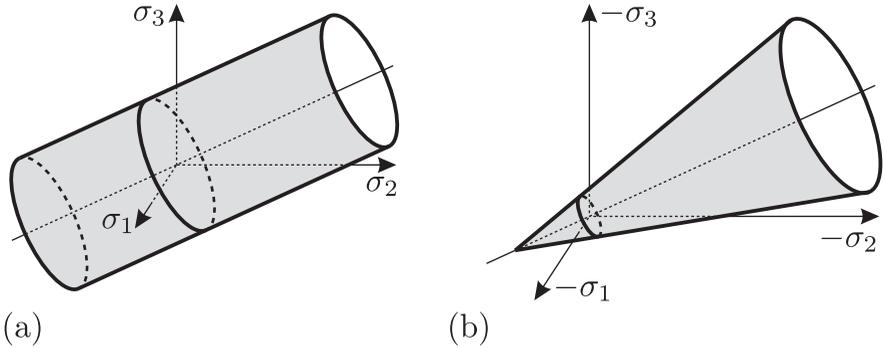


Figure 1: Schematic illustration of the yield surfaces in the principal stress space. (a) von Mises yield surface (b) Drucker–Prager yield surface.

The rate-form constitutive equations can be written in the following form:

$$\boxed{\dot{\boldsymbol{\sigma}} = \mathcal{D}^{\text{ep}} : \dot{\boldsymbol{\varepsilon}}} \quad (1)$$

and its inverse form is

$$\boxed{\dot{\boldsymbol{\varepsilon}} = \mathcal{C}^{\text{ep}} : \dot{\boldsymbol{\sigma}}} \quad (2)$$

The structure of the fourth-order tensor \mathcal{D}^{ep} and \mathcal{C}^{ep} , respectively, appearing in the expressions above are given below.

Model 1:

$$\mathcal{D}^{\text{ep}} = \mathcal{D}^{\text{e}} - \frac{\mathcal{D}^{\text{e}} : \mathbf{N} \otimes \mathbf{N} : \mathcal{D}^{\text{e}}}{2G + h}, \quad (3)$$

$$\mathcal{C}^{\text{ep}} = \mathcal{C}^{\text{e}} + \frac{1}{h} \mathbf{N} \otimes \mathbf{N}, \quad (4)$$

where

$$\mathbf{N} = \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}. \quad (5)$$

Model 2:

$$\mathcal{D}^{\text{ep}} = \mathcal{D}^{\text{e}} - \frac{\mathcal{D}^{\text{e}} : \mathbf{Q} \otimes \mathbf{N} : \mathcal{D}^{\text{e}}}{\tilde{h}}, \quad (6)$$

$$\mathcal{C}^{\text{ep}} = \mathcal{C}^{\text{e}} + \frac{1}{j} \mathbf{Q} \otimes \mathbf{N}, \quad (7)$$

where

$$\mathbf{N} = \frac{\mathbf{s}}{\sqrt{2}\|\mathbf{s}\|} + \alpha\boldsymbol{\delta}, \quad \text{and} \quad \mathbf{Q} = \frac{\mathbf{s}}{\sqrt{2}\|\mathbf{s}\|} + \beta\boldsymbol{\delta}. \quad (8)$$

In the expressions above: \mathcal{D}^{e} denotes the fourth-order elasticity tensor; \mathcal{C}^{e} is the elastic compliance tensor; \mathbf{s} stands for the deviatoric stress tensor; $\boldsymbol{\xi} = \mathbf{s} - \boldsymbol{\alpha}$ denotes the relative stress tensor, where $\boldsymbol{\alpha}$ is the back-stress tensor; $\boldsymbol{\delta}$ stands for the second-order identity tensor; G , h , \tilde{h} , j , α , and β are material constants.

The aim of the dissertation is to solve the differential equation (1) for both elastoplastic models, when the strain-driven case is considered

and when the strain rate is assumed to be constant. These novel solutions can be easily implemented into finite element codes, where it is a widely accepted approach that the strain rate is constant in the strain increment. Further goals are the construction of the discretized stress update algorithms based on the exact solutions and the derivation of the corresponding consistent tangent tensors. In addition, the dissertation is intended to present the exact strain solution of (2) for the stress-driven case assuming constant stress rate input.

Theses

Thesis 1

I have derived the exact stress solution corresponding to the associative von Mises elastoplasticity model governed by the combined linear hardening rule.

I have obtained the exact solution of the differential equation describing the relationship between the stress rate and the strain rate tensors. The new stress solution is valid under the constant strain rate assumption and it take into account both the linear isotropic and linear kinematic hardening rules. The new solution method is based on the introduction of an angle-like parameter in the deviatoric planes. I have solved the differential equation defining the evaluation of this angle-like variable using an incomplete beta function.

■ Related publications: [2], [1], [3], [6], [9].

Thesis 2

I have developed a complete stress update algorithm for the exact stress solution considered in Thesis 1. Furthermore, I have constructed the explicit expression of the corresponding consistent tangent tensor.

I have derived the discretized stress update formulae for the exact stress solution discussed in Thesis 1. Besides the general loading case, I have presented the stress update formulae for proportional loading.

The new stress update algorithm is applicable to all possible loading scenarios that can occur during the loading. I have developed an efficient numerical technique to invert an incomplete beta function appearing in the stress update formulae. In addition, I have constructed the consistent tangent tensor, which is crucial for finite element implementation in order to have a quadratic rate of convergence. I have implemented the new stress update algorithm with the consistent tangent tensor into the commercial finite element software ABAQUS via its user material interface. The accuracy and efficiency of the new method has been proven by performing numerical test examples.

■ Related publications: [2], [3], [4].

Thesis 3

I have obtained the exact strain solution for the associative von Mises elastoplasticity model with combined linear hardening.

I have derived the exact solution of the differential equation corresponding to the inverse form of the constitutive equation discussed in Thesis 1. The new strain solution has been obtained assuming constant stress rate input and it takes into account the linear isotropic and the linear kinematic hardening rules. The solution method utilizes the introduction of an angle-like variable in the deviatoric planes. I have solved the evolutionary equation of this angle-like variable by utilizing an incomplete beta function.

■ Related publication: [2], [1], [9].

Thesis 4

I have derived the exact stress solution for the non-associative Drucker–Prager elastoplastic model governed by linear isotropic hardening.

I have solved the differential equation describing the relation between the strain rate and the stress rate tensors. The new stress solution is valid under a constant strain rate assumption and it takes into account the linear isotropic hardening mechanism. The solution method utilizes an angle-like variable introduced in the deviatoric planes. I have

obtained the solution of this angle-like parameter using an incomplete beta function. I have derived the analytical stress solution for deviatoric radial loading case. Furthermore, I have proposed an approach to solve the singularity problem appearing at the apex of the yield surface.

■ Related publications: [5], [10], [12].

Thesis 5

I have developed a complete stress update algorithm based on the exact stress solution discussed in Thesis 4. I have obtained the explicit expression of the corresponding consistent tangent tensor.

I have constructed the discretized stress update procedure based on the exact stress solution considered in Thesis 4. Besides the general loading case, I have presented the stress update formulae for the deviatoric radial loading case and for the special loading scenario, when the stress state is located at the apex of the yield surface. I have derived a condition to determine whether the updated stress will leave the apex or will remain at that point. By exact linearization of the stress update formulae, I have obtained the consistent tangent tensors for all loading cases.

■ Related publications: [5], [10], [12].

Thesis 6

I have obtained the analytical strain solution for the non-associative Drucker–Prager elastoplastic model governed by linear isotropic hardening.

I have derived the analytical solution of the differential equation corresponding to the inverse form of the constitutive equation considered in Thesis 4. The new solution is valid for linear isotropic hardening under constant stress rate assumption. The solution method is based on the introduction of an angle-like variable in the deviatoric planes. I have obtained the analytical solution for this angle-like variable providing the explicit expression.

■ Related publications: [5], [10], [12].

Applications

The exact stress and strain solutions presented in the dissertation can be efficiently used to compute reference solutions for elastoplastic problems. The new stress update algorithms can be easily implemented into finite element codes yielding higher accuracy. The affirmation of the new methods is based on the citations made by other authors.

Citations related to publication [2]:

- Brannon, R. M., Leelavanichkul, S.: 2010. A multi-stage return algorithm for solving the classical damage component of constitutive models for rocks, ceramics, and other rock-like media. *International Journal of Fracture* **163**, 133–149. (IF = 1.043).
- Kaliszky S., Lógó J., Merczel D. B.: 2011. Softening and Hardening Constitutive Models and Their Application to the Analysis of Bar Structures. *Mechanics Based Design of Structures and Machines: An International Journal* **39**, 334–345. (IF = 0.472).
- Narvaez-Tovar, C. A., Lopez-Vaca, O. R., Garzon-Alvarado, D. A.: 2011. Modelo de endurecimiento isotrópico con esquema de integración explícita para biomateriales y su aplicación a la expansión de stents. *Revista Cubana de Investigaciones Biomédicas* **30**, 104–123.
- Rezaiee-Pajand, M., Sharifian, M.: 2011. A novel formulation for integrating nonlinear kinematic hardening Drucker–Prager’s yield condition. *European Journal of Mechanics / A Solids* **31**, 163–178. (IF = 1.414).

Publications

Journal papers:

- [1] Kossa, A.: 2007. Integration method for constitutive equation of von Mises elastoplasticity with linear hardening. *Periodica Polytechnica - Mechanical Engineering* **51**, 71–75.
- [2] Kossa, A., Szabó, L.: 2009. Exact integration of the von Mises elastoplasticity model with combined linear isotropic-kinematic hardening. *International Journal of Plasticity* **25**, 1083–1106. (*IF* = 4.791).
- [3] Kossa, A.: 2009. Rugalmas-képlékeny testek anyagegyenleteinek egzakt integrálási módszerei. (Exact integration methods of elastic-plastic materials) *Építés-Építésztudomány* **37**, 241–264.
- [4] Kossa, A., Szabó, L.: 2010. Numerical implementation of a novel accurate stress integration scheme of the von Mises elastoplasticity model with combined linear hardening. *Finite Elements in Analysis and Design* **46**, 391–400. (*IF* = 1.030).
- [5] Szabó, L., Kossa, A.: 2012. A new exact integration method for the Drucker–Prager elastoplastic model with linear isotropic hardening. *International Journal of Solids and Structures* **49**, 170–190. (*IF* = 1.677).

Conference papers:

- [6] Kossa, A., Szabó, L.: 2009. Computational aspects of the integration of von Mises elastoplasticity model with combined hardening. *Computational Plasticity X: Fundamentals and Applications*, 2-4 September 2009, Barcelona, Spain.

Conference presentations:

- [7] Kossa, A., Szabó, L.: 2006. Logarithmic stress rate in single crystal elastoplasticity. *6th European Solid Mechanics Conference*, 28 August - 1 September 2006, Budapest, Hungary.
- [8] Kossa, A., Szabó, L.: 2006. Using logarithmic stress rate in computational single crystal elastoplasticity. *35th Solid Mechanics Conference*, 4-8 September 2006, Cracow, Poland.
- [9] Kossa, A., Szabó, L.: 2007. Rugalmas-képlékeny testek konstitutív egyenleteinek integrálása lineáris izotróp és kinematikai keményedés esetén. (Stress integration of elastic-plastic constitutive equations with linear isotropic and kinematic hardenings.) *X. Magyar Mechanikai Konferencia*, 27-29 August 2007, Miskolc, Hungary.
- [10] Kossa, A.: 2011. Egzakt feszültségszámító eljárás a Drucker-Prager-féle rugalmas-képlékeny anyagmodellre izotrop keményedés esetén. (Exact stress update procedure for the Drucker–Prager elastoplastic model governed by linear isotropic hardening.) *XI. Magyar Mechanikai Konferencia*, 29-31 August 2011, Miskolc, Hungary.

Other presentations:

- [11] Kossa A.: 2007. Végés rugalmas-képlékeny anyagmodellek numerikus vizsgálata. (Computational aspects of elastic-plastic material models in finite strain description.) *Mechanika Szeminárium (Seminar on Mechanics)*, *BME Műszaki Mechanikai Tanszék (BME Department of Applied Mechanics)*. 10 May 2007.
- [12] Kossa A., Szabó L.: 2010. Exact stress update procedure for the hardening Drucker–Prager material model. *Seminar at Department of Aerospace and Mechanical Engineering, University of Arizona*, USA. 9 September 2010.