

GROUP REPRESENTATIONS IN ENTANGLEMENT THEORY

PÉTER VRANA

Supervisor: DR. PÉTER LÉVAY

BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS
DEPARTMENT OF THEORETICAL PHYSICS
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Background

Entanglement is one of the distinguishing features of quantum mechanics. Entangled states show correlations which cannot be explained with any local probabilistic theory, and allow us to outperform any classical information processing protocol. One of the most important unsolved problems in quantum information theory is the classification of inequivalent types of entangled states. Quantification of entanglement would enable us to understand it better as a resource.

Nonlocal properties are studied via functions of the state which are invariant under local invertible transformations which describe interaction-free evolutions of a composite quantum mechanical system. Such functions are called entanglement measures in this context. There exist several approaches, in one of them one considers products of unitary transformations (called the local unitary group) acting separately on the Hilbert spaces of the subsystems. An advantage of this group is that it is enough to look for real polynomial invariants to find a set separating the equivalence classes. In some cases, however, considering products of the general linear groups (known as the group of stochastic local operations and classical communication) leads to a simpler structure, and in some of the simplest cases a full classification has already been obtained. Notable examples are the bipartite systems and the case of three and four qubits.

Although quantum information theory traditionally dealt with systems consisting of distinguishable parts, it is meaningful to consider bosonic and fermionic subsystems as well. In this case interaction-free evolutions are those which are generated by Hamiltonians built purely from one-particle operators meaning that the same linear transformation acts on every particle of the same kind. Apart from its importance as a fundamental question, entanglement of identical particles is of fundamental importance in the context of many body systems.

Compared to the case of distinguishable subsystems, much less is known in this area. One of the few results is the entanglement classification of the states of a pair of bosons with two dimensional single particle state space and that of a pair of fermions with four dimensional single particle state space. It is also interesting that these two cases are similar to each other and to the case of two distinguishable qubits. The roots of this similarity and its possible applications were not studied in detail so far.

Aims

In the first part of the thesis I study entanglement in quantum systems containing distinguishable subsystems. My aim was to study the algebras of polynomials which are invariant under the action of the local unitary groups and the product of the special linear groups of the subsystems. The former leads to a finer classification of entangled states, while the latter still contains physically meaningful transformations and leads to a coarser classification possibly resulting in a simpler orbit structure. In both cases, two main directions are possible, leading to different kinds of data. Firstly, one would like to find new invariants with explicit and easily computable formulae. Secondly, one would like to find a presentation of the algebra of invariants in terms of generators and relations or find the primary and secondary invariants. This allows us to extract all the nonlocal information from a state using a minimal set of invariant functions.

In the second part I study entanglement in quantum mechanical systems which contain identical particles. In this area a limited number of special cases have been studied, and in some cases the classification was completed. The invariants which one encounters in this area are often similar to the ones found in the case of distinguishable subsystems. My aim was to explore the connection between entanglement measures of quantum systems with identical and distinguishable subsystems. On the one hand, one can hope to identify new invariants of systems with distinguishable constituents which are special cases of fermionic or bosonic entanglement measures and can be expressed in a simpler way in terms of the system of identical particles. On the other hand, one may find fermionic or bosonic generalizations of invariants which are already known in the case of distinguishable subsystems.

New scientific results

- I. I have found every degree four local unitary invariant real polynomial of a multipartite quantum system with arbitrary finite dimensional single particle state spaces. Using Schur-Weyl duality, which is a close connection between the representation theory of the symmetric and general linear (and hence also the unitary) groups, I have managed to find analogous invariants in every even degree homogenous part of the algebra of polynomials. Among the degree four invariants I have identified those that have a permutation symmetry. [4]
- II. I have introduced an algebra to grasp key properties of the algebras of local unitary invariants of quantum systems with state spaces of var-

ious dimension. This algebra can be constructed as the inverse limit of all the graded algebras of local unitary invariant real polynomials of k -partite composite quantum systems with distinguishable constituents together with the algebra homomorphisms induced by the (local unitary-equivariant) inclusions of smaller Hilbert spaces into larger ones. This sequence of algebras shows a certain stabilization property: every element in the inverse limit is already represented in some of the algebras which the inverse limit is constructed from. [4]

- III. I have rederived a formula for the stable dimension of the homogenous subspaces of the algebras of real polynomial invariants, and interpreted it as the dimension of the corresponding homogenous subspace of the inverse limit. I have rewritten the Hilbert series of the inverse limit (that is, the formal power series with the stable dimensions above as coefficients) into the form of an infinite product, which allowed me to formulate and prove the following conjecture: The inverse limit of the algebras of real polynomial invariants of k -partite quantum systems is free with the number of degree $2m$ homogenous elements in an algebraically independent set of generators being equal to the number of conjugacy classes of index m subgroups in the free group on $k - 1$ generators. [4]
- IV. From a modification of my derivation of the stable dimension formula I have obtained an exact formula for the dimension of the space of degree m polynomial invariants of mixed states under the action of the local unitary group. There is a similar stabilization property in this case as well, and the inverse limit of the algebras in the mixed case is isomorphic to the inverse limit of the algebras of pure state invariants with one more subsystems. An isomorphism is induced by the partial trace over the extra subsystem. [4]
- V. Using Freudenthal's construction I have managed to find the equivalence classes of entangled states under the groups of stochastic local operations and classical communication in the following tripartite quantum systems: three fermions with six dimensional single particle state space, one qubit and two fermions with four dimensional single particle state space, three qubits (this one was already known), one qubit and two bosonic qubits and finally, three bosonic qubits. The construction results in a degree four polynomial invariant, which is a generalization of Cayley's hyperdeterminant of a $2 \times 2 \times 2$ hypermatrix. The orbit structure in each case parallels that of the already known three qubit case. [1, 2]

- VI. I have argued that this similarity between entangled states of different types of quantum systems with the same number of particles is a general phenomenon. Using the relationship between various groups of local transformations and between their representations, I have formulated rigorously this correspondence relating the equivalence classes and entanglement measures of different quantum systems. [2]
- VII. I have presented a method to find local unitary invariants for multi-fermion systems and express them in a form which is independent of the dimension of the single particle state space. Among the simplest invariants obtained this way one can already find generalizations (related by the correspondence above) of some known invariants of states of quantum systems with distinguishable subsystems. I have also found that every invariant under the action of the group of stochastic local operations and classical communication gives rise to a local unitary invariant constructed this way, which is then automatically generalized to larger single particle state spaces. [3]

References

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