PhD Dissertation

DYNAMICS OF TOWED WHEELS
– Nonlinear Theory and Experiments –

Author: Dénes Takács

Supervisor: Gábor Stépán, DSc

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<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
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<tbody>
<tr>
<td>$\beta$</td>
<td>The pseudo velocity</td>
<td>[rad/s]</td>
</tr>
<tr>
<td>$\chi$</td>
<td>The dimensionless mass moment of inertia parameter</td>
<td>[-]</td>
</tr>
<tr>
<td>$\delta$</td>
<td>The notation of virtual quantities</td>
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<td>$\Gamma$</td>
<td>The pseudo force</td>
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<td>$\kappa$</td>
<td>The dimensionless mass moment of inertia parameter</td>
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<tr>
<td>$\lambda$</td>
<td>The characteristic exponent</td>
<td></td>
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<tr>
<td>$\mathcal{A}$</td>
<td>The energy of acceleration</td>
<td>[Nm/s²]</td>
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<tr>
<td>$\mu_d$</td>
<td>The dynamic coefficient of friction</td>
<td>[-]</td>
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<tr>
<td>$\mu_s$</td>
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<tr>
<td>$\Omega$</td>
<td>The dimensionless angular velocity of the caster</td>
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</tr>
<tr>
<td>$\omega$</td>
<td>The self-excited angular frequency</td>
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<td>$\omega_n$</td>
<td>The natural angular frequency of the caster-wheel system at zero towing speed</td>
<td>[rad/s]</td>
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<td>$\Delta$</td>
<td>The Poincaré-Lyapunov parameter</td>
<td></td>
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<td>$\psi$</td>
<td>The caster angle</td>
<td>[rad]</td>
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<tr>
<td>$\sigma$</td>
<td>The relaxation length of the stretched string</td>
<td>[m]</td>
</tr>
<tr>
<td>$\tau$</td>
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<td>[s]</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>The wheel rotation angle</td>
<td>[rad]</td>
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<tr>
<td>$\Sigma$</td>
<td>The dimensionless relaxation length</td>
<td>[-]</td>
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<tr>
<td>$\vartheta$</td>
<td>The integration variable</td>
<td>[-]</td>
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<td>$\zeta$</td>
<td>The damping ratio of the caster-wheel system at zero towing speed</td>
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<tr>
<td>$a$</td>
<td>The half length of the contact patch</td>
<td>[m]</td>
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<tr>
<td>$B$</td>
<td>The dimensionless torsional damping</td>
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**NOMENCLATURE**

<table>
<thead>
<tr>
<th>Symbol</th>
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<tr>
<td>( b )</td>
<td>The specific lateral damping factor of the tyre ([\text{Ns/m}^2])</td>
</tr>
<tr>
<td>( b_l )</td>
<td>The lateral damping factor of the king pin ([\text{Ns/m}])</td>
</tr>
<tr>
<td>( b_t )</td>
<td>The torsional damping factor at the king pin ([\text{Nms/rad}])</td>
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<tr>
<td>( F_z )</td>
<td>The overall normal force between the tyre and ground ([\text{N}])</td>
</tr>
<tr>
<td>( f_n )</td>
<td>The natural frequency of the caster-wheel system at zero towing speed ([\text{Hz}])</td>
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<tr>
<td>( J_A )</td>
<td>The mass moment of inertia of the overall system with respect to the ( z ) axis at the king pin ([\text{kgm}^2])</td>
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<td>( J_A^+ )</td>
<td>The additional mass moment of inertia with respect to the ( z ) axis at the king pin ([\text{kgm}^2])</td>
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<td>( J_{cz} )</td>
<td>The mass moment of inertia of the caster with respect to the ( z ) axis at the centre of gravity of the caster ([\text{kgm}^2])</td>
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<td>( J_{wy} )</td>
<td>The mass moment of inertia of the wheel with respect to the ( y ) axis at the centre of gravity of the wheel ([\text{kgm}^2])</td>
</tr>
<tr>
<td>( J_{wz} )</td>
<td>The mass moment of inertia of the wheel with respect to the ( z ) axis at the centre of gravity of the wheel ([\text{kgm}^2])</td>
</tr>
<tr>
<td>( K )</td>
<td>The dimensionless torsional stiffness ([-\text{]})</td>
</tr>
<tr>
<td>( k )</td>
<td>The specific lateral stiffness parameter of the tyre ([\text{N/m}^2])</td>
</tr>
<tr>
<td>( k_l )</td>
<td>The lateral stiffness of the king pin ([\text{N/m}])</td>
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<tr>
<td>( k_t )</td>
<td>The torsional stiffness at the king pin ([\text{Nm/rad}])</td>
</tr>
<tr>
<td>( L )</td>
<td>The dimensionless caster length ([-\text{]})</td>
</tr>
<tr>
<td>( l )</td>
<td>The caster length ([\text{m}])</td>
</tr>
<tr>
<td>( l_c )</td>
<td>The distance between the king pin and the centre of gravity of the caster ([\text{m}])</td>
</tr>
<tr>
<td>( m_c )</td>
<td>The mass of the caster ([\text{kg}])</td>
</tr>
<tr>
<td>( m_w )</td>
<td>The mass of the wheel ([\text{kg}])</td>
</tr>
<tr>
<td>( p_0 )</td>
<td>The tyre inflation pressure ([\text{bar}])</td>
</tr>
<tr>
<td>( Q )</td>
<td>The dimensionless leading point lateral deformation ([-\text{]})</td>
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<tr>
<td>( q )</td>
<td>The lateral displacement ([\text{m}])</td>
</tr>
<tr>
<td>( R )</td>
<td>The radius of the wheel ([\text{m}])</td>
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<td>( R_e )</td>
<td>The effective rolling radius ([\text{m}])</td>
</tr>
<tr>
<td>( t )</td>
<td>The time ([\text{s}])</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
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<tr>
<td>--------</td>
<td>------------------------------------</td>
</tr>
<tr>
<td>$V$</td>
<td>The dimensionless towing speed</td>
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<tr>
<td>$v$</td>
<td>The towing speed</td>
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<tr>
<td>$Y_L$</td>
<td>The absolute position of the leading point</td>
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</table>
Chapter 1
Introduction

The wheel is one of the oldest and most important inventions of mankind. It has been developing during thousands of years reaching its actual form we can see today. The most relevant innovation was carried out by John Boyd Dunlop in 1887, who made the first pneumatic tyre to avoid the headache caused by the noise of his son’s bicycle on the paved road surface.¹ At that time, Dunlop did not suspected how much ”headache” will be generated by his invention in the future for engineers...

Almost all road vehicles roll on pneumatic tyres and their motion is primarily determined by the forces transferred to the vehicle from the ground by the wheels. Since the properties of the tyre strongly influences the manoeuvrability and stability of vehicles, the investigation of tyre and tyre-ground contact models is an essential task in vehicle dynamics. Precise mechanical models of tyres and tyre-ground contacts are required for numerical simulations in vehicle dynamics in order to explain experienced motions or to predict possible vehicle behaviour. The mechanical models of tyre also play a key role in the development of vehicle control systems such as the ABS, ESP, and so on.

Some of the vibrations in vehicles are related to tyre dynamics. In this study, the lateral vibration of towed wheels is under investigation, which is one of the most fascinating dynamic phenomena. Two essentially different low degree-of-freedom (DoF) mechanical models are considered and analysed by analytical, numerical and experimental methods.

In Chapter 2, a short review of the lateral vibration of towed wheels is presented, in which the most important results of the corresponding specialist literature are summarized and their critical analysis is presented.

In Chapter 3, the mechanical model is constructed from a rigid wheel towed by an elastic suspension system. The model considers a single contact point between the wheel and the ground and it neglects the elasticity of the tyre. The linear stability boundary of the model and the sense of the corresponding Hopf bifurcation is calculated analytically. The full geometric non-linearity of the system is analysed by a numerical bifurcation software and a combined stability chart and bifurcation diagram is constructed for the towed wheel.

A different mechanical model of the towed elastic tyre with one rigid-body DoF is

¹Despite of the fact that Dunlop is often said to be the inventor of the pneumatic tyre, note that Robert William Thomson patented his ‘Aerial Wheels’ in 1846 more than forty years before Dunlop.
considered in Chapter 4. The linear stability charts of the system are composed for different approximate models of the tyre, and the effect of an elastic steering mechanism is also investigated. The behaviour of the towed tyre is studied in the nonlinear domain by numerical simulations, where possible partial sliding is also included in the contact model.

The experimental validation and investigation of the towed tyre model is carried out in Chapter 5. First, the necessary system parameters of the mechanical model are identified in different experimental set-ups. The most relevant theoretical results are confirmed on a carefully constructed experimental rig.

The theoretical and practical conclusions together with possible further research directions are summarized in a separate booklet of theses that also presents the thesis points of this work.
Chapter 2

Lateral vibration of towed wheels

The lateral vibration of towed wheels, called shimmy, is one of the most fascinating phenomena in vehicle dynamics, which may be very dangerous when it occurs – it occurs rarely, though. In the subsequent sections, we summarize the most important references on the topic.

2.1 Classical examples for shimmy

Anybody can experience shimmy on supermarket trolleys or on two-wheeled suitcases (Kauzlarich et al., 2000; O’Reilly and Varadi, 1999; Plaut, 1996). Shimmy on vehicles usually presents a safety hazard and its ultimate elimination is a problem for engineers at design stage. It may appear on airplane landing gears (see Figure 2.1), on front wheels of motorcycles (see Figure 2.2) and cars, on caravans, rear wheels of semi-trailers and articulated buses. The terminology ‘shimmy’ appeared in the 1920’s when it was the name of a popular dance. In spite of the fact that the problem has been known and studied for almost a century (Brouilhet, 1925; De Lavaud, 1927), the elimination of shimmy is one of the most critical part of the landing gear design (Besselink, 2000; Coetzee, 2006; Sura and Suryanarayan, 2007; Thota et al., 2008). It is also well-known as one of the most dangerous vibration problem (also called ‘wobble’) for motorcycles (Catani and Mancinelli, 2007; Cossalter, 2006; Sharp et al., 2004), while accidents involving trailers and caravans are often originated in the so-called ‘waving’ of the vehicles (Sharp and Fernáñdez, 2002; Troger and Zeman, 1984), which is also related to the shimmy phenomenon. Some references in the specialist literature categorize these vibrations based on the frequency domain of the vibration: shimmy is used for the higher frequencies and wobble for the lower ones. Still we use the terminology shimmy only throughout this study.

In practice, the experienced properties of shimmy make the phenomenon somewhat ‘mystic’. One of these properties is that shimmy seems to occur randomly in certain speed ranges. This has obstructed the detailed and deep experimental analysis of the problem in real systems. Consequently, the possibility of shimmy can be hidden and unexplored during the design and even during the testing processes, which means a severe safety hazard in the sense that large and dangerous damage can occur with relatively low probability. Even the recall of some motorcycles from the market has become necessary in some cases due to fatal accidents caused by shimmy (Duke, 1997; Limebeer et al., 2002). The random appearance of shimmy can now be explained by means of recent
results and methods of bifurcation theory. Another counter-intuitive property of shimmy is that the vibrations often exhibit quasi-periodicity. Quasi-periodic vibrations are usually originated in the complexity of the mechanical structure, where multi DoF and increased number of system parameters show up leading to limitations in analytical studies. On contrary, quasi-periodic shimmy can appear in case of simple mechanical structures, too.

The shimmy of towed wheels may be caused by the elasticity of the towing bar suspension and the attached vehicle structure, by the elasticity of the tyre on the wheel or by the combination of the two cases. On one hand, the analyses of shimmy is difficult due to the fact that the vehicle itself is a complex dynamical system serving several low-frequency vibration modes which may all be important components of the dynamical behaviour at different running speeds and conditions. On the other hand, difficulties in the analyses are also originated in the modelling of the wheel-ground contact.

The analytical handling of complex multi DoF models are almost impossible, while the numerical simulations of these systems are mainly resolved by the development of multi-body dynamics and commercial computer codes (Schiehlen, 2006). Simulations of such large systems can be carried out by these appropriate numerical methods, but engineers do not have enough information about the effects of varying parameters in the vehicle system without analytical investigations. For example, it is difficult to suspect which parameters should be varied in order to stabilize an unstable vehicle. This
Chapter 2. Lateral vibration of towed wheels

problem can be even greater if the non-linearities have relevant effects on the vehicle behaviour (see True, 1999). In this case, the complex multi-body system can be analysed by means of enhanced numerical bifurcation methods. For example, nonlinear behaviour of motorcycles is investigated by Meijiaard and Popov (2006) with the help of a simplified mechanical model that involves an improved nonlinear tyre model. A different, industrial problem is investigated by Coetzee et al. (2006), where the manoeuvrability of an aircraft is studied by means of the numerical bifurcation software AUTO (Doedel et al., 1997). Coetzee et al. connect different computer codes and generate the stability maps of turning in order to determine the parameter domains of safe ground manoeuvres of airplanes. Numerical bifurcation methods like these, usually require large computational effort and the interpretation of the results is difficult due to the large number of parameters. Moreover, such complex systems can show very different behaviours for slightly different parameter set-ups, which causes the difficulties in the experimental validation of the chosen mechanical models. Nevertheless, these methods have the advantage that all the calculations can be done without the simplification of the original structure – if the computational power is large enough.

A complex system can be simplified if the relevant vibration modes are properly selected. Namely, the system can be modelled by a simple, usually analytically manageable low DoF mechanical model, which can also capture and explain the practically observed relevant vibrations. Of course, it is not always clear what the relevant vibration modes are, i.e., important properties of the original system could be ignored by an incorrect mechanical model. Even if a well constructed mechanical model of the vehicle structure is under consideration, an over-simplified dynamical model of the wheel-ground contact can jeopardise the validity of the results.

2.2 Development of tyre models

For the above reasons, the modelling of tyres and tyre-ground contacts have essential importance in vehicle dynamics. The exact modelling of the contact region leads to multi-scale mechanical models, where advanced finite element calculations need long computation times even in stationary cases; see Kalker (1991) for railway wheels and Böhm (1989); Chang et al. (2004) for tyres. Even nowadays, the dynamic contact problems usually require special codes, large computational power, and still, there are no analytical results available to check these calculations. There are several commercial numerical codes available, which can be used for different industrial problems in order to model tyre behaviour. For example, advanced dynamical tyre models are implemented in FTire (Gipser, 2007), RMOD-K (Oertel and Fandre, 1999), SWIFT (Pacejka, 2002) and TMeasy (Hirschberg et al., 2007). The basic assumptions of these models are presented and compared to each other in the paper of Lugner et al. (2005). All the above mentioned models can be used in one or more multi-body simulation software like ADAMS, MATLAB/Simulink, SIMPACK. Each of the models is constructed by taking into account different empirical or physical-based approaches, and each of them considers different levels of tyre complexity. Consequently, the computational times required by the different models may be far from each other. For example, the tyre model RMOD-K uses a detailed finite element model of the tyre structure with different layers in the belt, where an additional sensor layer
is built in to detect the tyre-ground contact. This complex, physically based model requires high computational effort relative to the empirically oriented SWIFT and TMeasy models. The model FTire applies another approach, namely, the deformable belt is constructed from 50–80 discrete mass-spring segments with several DoF and it describes the tyre-ground contacts with thousands of specific friction elements. FTire is also able to examine the temperature of the tyre and its impact on tyre behaviour. The wear of the tyre can also be investigated. Nevertheless, the numerical code of FTire requires several hundreds of tyre parameters, and some of them can be identified by special and expensive measurements only.

In another context, the tyre-ground contact problem has a key role in vehicle noise and rolling resistance reduction. At medium and high speeds of vehicles, the dominant part of the vehicle noise is originated in the tyre (FEHRL, 2006; Sandberg and Ejsmont, 2002). In order to reduce the tyre noise, the tyre-ground contact and the related noise generation of tyres are deeply analysed in many studies of the specialist literature. For example, experimental studies of tyre noise generation are presented by Andersson and Kropp (2009); Gauterin and Ropers (2005). Regarding the rolling resistance, essential part of the energy is consumed by the tyres that may warm up heavily. This statement is especially valid in case of heavy vehicles (see Miege and Popov, 2005, as a starting point to the literature survey).

2.3 Shimmy models

One of the first scientific reports on shimmy was presented by Schlippe and Dietrich (1941), where they analysed a simple low DoF mechanical model. In this so-called stretched string model, the tyre-ground contact was considered to be a contact line that becomes deformed due to the lateral displacement of the wheel. Furthermore, it was considered that each contact point sticks to the ground. Note that the tyre becomes deformed not only along the contact line but also in front of the leading point and behind the rear point. It was assumed that outside the contact patch the deformation decays exponentially, which was also confirmed by measurements. Because the resulting equations were too complicated to be analysed with the available mathematical tools at that time, they introduced a severe simplification, namely, the contact line was straight between the leading edge and trailing edge. This way, the resultant lateral force and torque induced by the elastic tyre deformation were calculated, leading to a delay differential equation (DDE) with a discrete delay. Using this equation, the linear stability of the stationary rolling motion was analysed with some further simplifications from mathematical theory view point. The discrete delay was just equal to the time period of a tyre point while in contact with the ground between the leading and the trailing points. Since then, several versions of the stretched string model have been developed and analysed. However, the mathematical simplification used by Schlippe and Dietrich was actually incorrect and as a result, many unstable regions were disregarded. Later, frequency response functions were calculated by Segel (1966) for the stretched string model without any restriction to the shape of the contact line. This so-called exact stretched string model has become a basic reference. This approach results in a DDE with a distributed delay but exact stability analysis was not been carried out at that time due to the mathematical difficulties
An alternative approach was taken by Pacejka (1966) who introduced different straight and curved contact line approximations by using stationary shape functions for the lateral tyre deformation calculated at constant drift angles. The resultant lateral creep force and torque are calculated from these ‘quasi-stationary’ deformations by the semi-empirical ‘Magic Formula’ (Pacejka, 2002). This way, the delay effects are completely eliminated. In the resulting simplified models only the caster angle and the lateral deformation of the leading point were used as state variables, leading to an ordinary differential equation (ODE) that made it very popular and easy-to-analyse. In the middle range of the towing speeds, the quasi-stationary deformation idea gives reasonable agreement with experiments. Several research reports prove the success of this approach in engineering (see Troger and Zeman (1984) on tractor-semi-trailers systems, Sharp et al. (2004) on motorcycles or Fratila and Darling (1996) on caravans). Non-linearities related to Coulomb friction were also introduced and their importance were emphasized by the experimental observation of unstable periodic motions in Pacejka (1966). The model was also generalised by considering the effect of the width of the contact area, that of sliding at the rear part of the contact patch, and that of the gyroscopic effects appearing when the wheel is allowed to be tilted from its vertical plane. Pacejka’s wisdom about tyres has accumulated in his book (Pacejka, 2002) that also includes a separate chapter for shimmy with an extensive reference list in it.

Considering the exact stretched string model with simplified boundary conditions, Stépán (1998) has introduced non-linearities into the system. He also investigated the linear stability with mathematical rigour leading to the possibilities of many disjunct unstable parameter domains and to the possible existence of quasi-periodic oscillations, which recently has been confirmed experimentally in Takács (2005); Takács and Stépán (2009).

The elasticity of the tyre was also considered in a single point contact model by Moreland (1954). The contact line was shrunk into a point where the force, induced by the elastic tyre, acts. A relaxation time was also introduced for the force to model its ‘delayed action’ and a torque coefficient was defined to relate the force and the so-called self-aligning torque. It was proven by Collins (1971) that the point contact model is equivalent to the stretched string model when the latter is restricted to the case of straight contact line. In the point contact model, however, the relaxation time and the torque coefficient have to be estimated or measured, while the stretched string model provides the corresponding parameters partly from the geometry of the contact region.

If there is elasticity in the suspension system, even a point contact model with rigid tyre can exhibit shimmy and complicated nonlinear (sometimes even chaotic) behaviour (De Falco et al., 2010; Goodwine and Stépán, 2000; Le Saux et al., 2005; Schwab and Meijaard, 1999; Stépán, 1991, 2002; Takács et al., 2008). Feed-back linearisation based controllers can be constructed for these nonlinear systems to suppress the vibrations (Goodwine and Zefran, 2002). As it was already mentioned, shimmy is one of the most critical problems of landing gear design, and recent studies about landing gear shimmy can also be found in the publicly available specialist literature. Some of them, are reviewed here briefly.

Somieski (1997) uses different mathematical methods in order to analyse a simple mechanical model of the towed tyre with zero rake angle. His tyre model is based on
the \textit{stretched string model} of Schlippe and Dietrich (1941), which results a nonlinear set of ODEs. The linear stability chart of the system is composed analytically, while the nonlinear behaviour of this system is analysed by means of numerical simulations. It is shown that the oscillations are limited by a stable limit cycle in the linearly unstable domain and an additional unstable limit cycle can appear in the presence of dry friction.

Although, shimmy is often known as the oscillatory problem of airplane nose gears, it can also appear on main landing gears. A recent essential study on main landing gear shimmy is written by Besselink (2000). Besselink composes the mechanical model of landing gears by taking into account the complexity of the real structure. The numerical simulations of the developed mechanical model are compared to and validated by measured time histories of velocity, acceleration and force signals. In a separate chapter, the fundamentals of shimmy are presented, where different mechanical models of former studies are analysed and compared to each other, too. The investigations of tyre modelling and tyre parameter identification are also important parts of that study.

A low DoF mechanical model of a simplified nose landing gear is under consideration in the paper of Thota et al. (2009). Two different vibration modes (torsion and bending) of the landing gear is composed in their model and the \textit{stretched string model} of Schlippe and Dietrich is applied in order to realize the nonlinear dynamics of the tyre. The inclination of the nose gear relative to the vertical direction is taken into account and the effect of non-zero rake angle is also investigated. In order to do this, the nonlinear ODEs describing the system are analysed by the numerical bifurcation software AUTO. The interaction between the lateral and torsional vibration modes of the gear are detected by the representation of double Hopf points in the two-parameter bifurcation diagrams. The practically observed quasi-periodic vibrations of nose gears are explained by this type of suspected interaction.

Further academic and industrial studies on the landing gear dynamics are reviewed in the recent paper of Pritchard (2001) with particular reference to shimmy.

\subsection*{2.4 Models to be used}

In the subsequent chapters, we consider the case when the wheel is pulled by a caster connected to a cart of constant velocity. Two different types of low DoF mechanical models are in the focus of our investigations. Both models consider zero rake angle, namely, the wheel centre plane is held vertical, perpendicular to the horizontal rigid ground. The first model consists of a rigid wheel with elastic suspension. This model is analysed by analytical and numerical methods with special attention to the nonlinear dynamics of the system. The corresponding mathematical model is a strongly nonlinear three dimensional system of ODEs that may produce several stable and unstable limit cycles leading to bistability and isola in certain realistic parameter regions.

The second model is constructed from a rigid suspension and a towed wheel of elastic tyre. The lateral deformations of the tyre are described by the \textit{exact stretched string model}, which is based on the work of Schlippe and Dietrich; Segel. With the appropriate choice of the boundary conditions, the relaxation length of the tyre is taken into account among other conventional tyre parameters like the specific stiffness and damping. The Newtonian equation of motion becomes a second order integro differential equation (IDE).
The rolling condition is formulated as a kinematic constraint that leads to a partial differential equation (PDE). The travelling wave solutions of the deformation allow us to transform the PDE-IDE system into a DDE with distributed delay. The linear stability investigation of the DDE shows that the stationary rolling motion may lose its stability via co-dimension one or co-dimension two Hopf bifurcations as the parameters (like the towing speed and the caster length) are varied. Consequently, self-excited periodic and quasi-periodic oscillations may appear. The stability chart in the plane of the above parameters is determined analytically and checked by numerical simulations and extensive laboratory experiments.
Chapter 3

Single contact point model

In this chapter a single contact point model is used to investigate shimmy. The model consists of a rigid wheel towed by a rigid caster, while the king pin is elastic in the lateral direction only. The equations of motion are derived with the help of the Appell-Gibbs equation (Gantmacher, 1975). The linear stability of the system is analysed by the Routh-Hurwitz stability criterion, and the linear stability chart is presented in the plane of the towing speed and the caster length for different values of the damping ratio. The locally nonlinear behaviour of the system is calculated by means of the so-called centre manifold reduction and the theory of Hopf bifurcation. The nonlinear system is also analysed by the numerical bifurcation software AUTO97 (Doedel et al., 1997), and a bifurcation chart is plotted, which summarizes all the different kinds of possible dynamical behaviours.

3.1 Mechanical model

The mechanical model considered here is shown in Figure 3.1. The plane of the rigid wheel is always vertical to the ground, and it has a single contact point at P. The radius of the wheel is denoted by $R$, the mass of the wheel is $m_w$ and the mass moment of inertia of the wheel with respect to the $z$ axis at the centre of gravity $O$ is $J_{wz}$. The corresponding mass moment of inertia with respect to the $y$ axis of rotation is $J_{wy}$, where the subscripts w refer to wheel. The caster length is denoted by $l$, while $l_c$ stands for the distance of the centre of gravity $C$ of the caster and the king pin at $A$. The mass of the caster is $m_c$ and the mass moment of inertia with respect to the $z$ axis at $C$ is $J_{cz}$, with subscripts c referring to caster. The system is towed in the horizontal plane with constant velocity $v$. The king pin is supported by lateral springs and dampers of overall stiffness $k_l$ and viscous damping coefficient $b_l$, respectively.

Without rolling constraints, the system has 3 DoF, so one can choose the caster angle $\psi(t)$, the king pin lateral position $q(t)$, and the wheel rotation angle $\varphi(t)$ as general coordinates. The constraint of rolling (without sliding) means that the contact point $P$ has zero velocity $v_P = 0$. This rolling condition leads to two scalar kinematic constraint equations in the form of coupled first order nonlinear ODEs with respect to the general coordinates:

$$
l\dot{\psi}\sin\psi - R\dot{\varphi}\cos\psi + v = 0,
$$

$$
\dot{q} - l\dot{\psi}\cos\psi - R\dot{\varphi}\sin\psi = 0.
$$

(3.1)
The equations of motion of this rheonomic and non-holonomic system can be derived with the help of the Routh-Voss equations or the Appell-Gibbs equations (see Gantmacher, 1975). In the case of zero damping ($b_1 = 0$), they are given in the paper of Stépán (2002). Here we derive the equations for non-zero viscous damping with the help of the Appell-Gibbs equations.

In order to do this, the pseudo velocities have to be chosen. The number of the pseudo velocities is equal to the difference between the number of general coordinates and the number of the kinematic constraints, that is $3 - 2 = 1$. The pseudo velocity can be chosen freely till it describes the system uniquely, but the difficulty of the derivation of equations depends strongly on this choice, which is intuitive, anyway. In our case, for example, the angular velocity of the caster is a good candidate to be the single pseudo velocity:

$$\beta := \dot{\psi}. \quad (3.2)$$

The kinematic constraints (3.1) and the definition of the pseudo velocity can be arranged
into a system of linear algebraic equations:

\[
\begin{pmatrix}
0 & l \sin \psi & -R \cos \psi \\
1 & -l \cos \psi & -R \sin \psi \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\dot{q} \\
\psi \\
\dot{\phi}
\end{pmatrix} =
\begin{pmatrix}
-v \\
0 \\
0
\end{pmatrix}.
\]  

(3.3)

Its solution gives the 3 general velocities in terms of the 3 general coordinates and the single pseudo velocity:

\[
\begin{pmatrix}
\dot{q} \\
\psi \\
\dot{\phi}
\end{pmatrix} =
\begin{pmatrix}
v \tan \psi + \frac{l}{\cos \psi} \beta \\
\beta \\
\frac{v}{R} \beta \tan \psi
\end{pmatrix}.
\]  

(3.4)

Their differentiation with respect to time results the 3 general accelerations, which can be expressed as a function of the single pseudo acceleration, the single pseudo velocity and the 3 general coordinates:

\[
\begin{pmatrix}
\ddot{q} \\
\ddot{\psi} \\
\ddot{\phi}
\end{pmatrix} =
\begin{pmatrix}
v \tan \psi + \frac{l}{\cos \psi} \beta^2 + \frac{l}{\cos \psi} \dot{\beta} \\
\beta \\
\frac{v}{R} \beta \tan \psi + \frac{1}{R} \beta \dot{\beta}
\end{pmatrix}.
\]  

(3.5)

The Appell-Gibbs equations are derived from the so-called energy of acceleration \(A\) that can be calculated for a rigid body generally with the formula

\[
A = \frac{1}{2} m a_G \cdot a_G + \frac{1}{2} \epsilon^T J_G \epsilon + \epsilon^T (\omega \times (J_G \omega)) + \frac{1}{2} \omega^T (J_G \omega) \omega^2,
\]  

(3.6)

where \(a_G\) is the acceleration of the centre of gravity, \(\omega\) and \(\epsilon\) are the angular velocity and angular acceleration of the rigid body, respectively. In the above formula, \(J_G\) refers to the matrix of mass moment of inertia of the rigid body with respect to its centroid. In the Appell-Gibbs equations, the energy of acceleration is derived with respect to the pseudo accelerations. The last term of (3.6) cannot depend on the pseudo accelerations, since it does not include any acceleration term at all, consequently, it is not necessary to calculate it. In our case, the energy of acceleration of the system consists from the energy of acceleration of the caster and that of the wheel:

\[
A = \frac{1}{2} m_c a_C \cdot a_C + \frac{1}{2} \epsilon^T J_c \epsilon + \epsilon^T (\omega_c \times (J_c \omega_c)) + \ldots
\]  

(3.7)

\[
+ \frac{1}{2} m_w a_O \cdot a_O + \frac{1}{2} \epsilon^T J_w \epsilon + \epsilon^T (\omega_w \times (J_w \omega_w)) + \ldots.
\]

The accelerations of the centers of gravity of the caster and the wheel are given in the caster fixed \((x, y, z)\) coordinate system by means of the formulae

\[
a_C = \begin{pmatrix}
\dot{q} \sin \psi + l_c \beta^2 \\
\dot{q} \cos \psi - l_c \beta \\
0
\end{pmatrix} =
\begin{pmatrix}
v \tan \psi + (l \tan^2 \psi + l_c) \beta^2 + l \dot{\beta} \tan \psi \\
\beta \\
\frac{v}{\cos \psi} \beta + l \beta^2 \tan \psi
\end{pmatrix},
\]  

(3.8)

and

\[
a_O = \begin{pmatrix}
\dot{q} \sin \psi + l \beta^2 \\
\dot{q} \cos \psi - l \dot{\beta} \\
0
\end{pmatrix} =
\begin{pmatrix}
v \tan \psi + \frac{l}{\cos \psi} \beta^2 + l \dot{\beta} \tan \psi \\
\beta \\
\frac{v}{\cos \psi} \beta + l \beta^2 \tan \psi
\end{pmatrix},
\]  

(3.9)
respectively. The angular velocities of the caster and the wheel are
\[
\omega_c = \begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix} \quad \text{and} \quad \omega_w = \begin{pmatrix} \dot{\varphi} \\ \beta \end{pmatrix} = \left( \frac{v}{R \cos \psi} + \frac{l}{R} \beta \tan \psi \right),
\]
respectively. The angular accelerations can be calculated via differentiation of the angular velocities with respect to time:
\[
\varepsilon_c = \begin{pmatrix} 0 \\ 0 \\ \dot{\beta} \end{pmatrix} \quad \text{and} \quad \varepsilon_w = \begin{pmatrix} -\dot{\varphi} \beta \\ \dot{\beta} \end{pmatrix} = \left( \frac{-v \tan \psi}{R \cos \psi} \beta - \frac{l}{R} \beta^2 \tan \psi \right),
\]
which are all expressed in the moving caster fixed \((x, y, z)\) coordinate system.

After the substitution of these accelerations (3.8), (3.9), (3.11) and velocities (3.10) into the energy of acceleration (3.7), we obtain:
\[
A(\dot{\beta}, \beta, \psi) = \frac{1}{2} \left( m_w l^2 + \frac{I^2}{R^2} J_{wy} \right) \left( \dot{\beta} \tan \psi + 2 v \tan \psi \beta + 2 \frac{\beta^2}{\cos^2 \psi} \right) \dot{\beta} \tan \psi
+ \frac{1}{2} m_c l^2 \left( \left( \tan^2 \psi + \left( 1 - \frac{l_c}{l} \right)^2 \right) \dot{\beta} + 2 v \left( \frac{1}{\cos^4 \psi} - \frac{l_c}{l} \beta \right) \tan \psi \beta + 2 \frac{\beta^2}{\cos^2 \psi} \dot{\beta} \right)
+ \frac{1}{2} \left( J_{wz} + J_{cz} \right) \dot{\beta}^2 + \ldots .
\]
(3.12)

As already mentioned above, the Appell-Gibbs equation is constructed with the help of the derivative of the acceleration of energy with respect to the pseudo acceleration:
\[
\frac{\partial A}{\partial \dot{\beta}} = \Gamma,
\]
where the right hand side is the so-called pseudo force \(\Gamma\). The pseudo force can be calculated via the virtual power of the active forces, which are generated by the spring and the damper at the king pin only. The virtual power is given by
\[
\delta P = F_A^T \delta v_A = \left( \begin{matrix} 0 & -k_0 q - b \dot{q} & 0 \end{matrix} \right) \delta \begin{pmatrix} 0 \\ \dot{q} \\ 0 \end{pmatrix} = \Gamma \cdot \delta \beta,
\]
(3.14)
where \(\delta\) denotes virtual quantities. After the elimination of \(\dot{q}\) in the virtual power by means of (3.4), the pseudo force has the form
\[
\Gamma(q, \beta, \psi) = -k_0 q \frac{l}{\cos \psi} - b l v \tan \psi \frac{l}{\cos \psi} - b \dot{\beta} \frac{l^2}{\cos^2 \psi}. \quad (3.15)
\]

The Appell-Gibbs equation (3.13) then leads to
\[
\dot{\beta} = -\frac{N(\psi, \beta, q)}{D(\psi)}, \quad (3.16)
\]
where
\[
N(\psi, \beta, q) = \left( - (m_w l + m_c l_c) v + \frac{l v}{R^2} J_{wy} \tan^2 \psi + \frac{(m_w + m_c) l v}{\cos^2 \psi} + \frac{k_l l^2}{\cos \psi} \right) \beta \\
+ \left( (m_w + m_c) l^2 + \frac{l^2}{R^2} J_{wy} \right) \frac{\sin \psi}{\cos^2 \psi} \beta^2 + k_l l q + b_l l v \tan \psi
\]
(3.17)
and
\[
D(\psi) = \left( m_c l_c (l_c - 2l) - m_w l^2 + J_{wz} + J_{cz} + \frac{(m_w + m_c) l^2}{\cos^2 \psi} + \frac{l^2}{R^2} J_{wy} \tan^2 \psi \right) \cos \psi
\]
(3.18)
are odd and even functions of the general coordinates \( q \) and \( \psi \), respectively, while the third general coordinate \( \phi \) does not show up in the formulae.

The equations of motion of the system is governed by the Appell-Gibbs equation (3.16) and by the formulae of the general velocities (3.4):
\[
\dot{\psi} = \beta, \\
\dot{\beta} = - \frac{N(\psi, \beta, q)}{D(\psi)}, \\
\dot{q} = v \tan \psi + \frac{l}{\cos \psi} \beta, \\
\dot{\phi} = v + l \dot{\psi} \sin \psi \frac{1}{R \cos \psi},
\]
(3.19)
where the sequence of the equations is rearranged, as follows: the first equation is just the definition (3.2), the second is related to the derivative of the angular momentum and the last two are linked to the constraint of rolling. Only the last equation contains the wheel rotation angle \( \phi \), so \( \phi \) is a so-called cyclic coordinate and consequently, the first three equations can be decoupled from the last one. This also means that the system can be described uniquely in the three dimensional phase space of the caster angle \( \psi \), caster angular velocity \( \beta \) (that is also the single pseudo velocity) and the king pin lateral displacement \( q \).

### 3.2 Stability analysis

The trivial solution of the system corresponds to stationary rolling along a straight line defined by the vector of the towing velocity: \( \psi \equiv 0, \beta \equiv 0, q \equiv 0 \) and \( \phi \equiv v/R \).

When the towing speed is zero, the system forms a 1 DoF oscillator about the \( z \) axis. The corresponding angular natural frequency \( \omega_n \) of the undamped linear system and the damping ratio \( \zeta \) are given by
\[
\omega_n = \sqrt{\frac{k_l l^2}{J_{wz} + J_{cz} + m_c (l - l_c)^2}}, \quad \zeta = \frac{b_l}{2k_l} \omega_n.
\]
(3.20)
Let us introduce the new dimensionless parameters:
\[
L = \frac{l}{l_c}, \quad V = \frac{v}{\omega_n l_c},
\]
(3.21)
\( \kappa = \frac{m_c l_c (l - l_c)}{J_{wz} + J_{cz} + m_c (l - l_c)^2}, \quad \chi = \frac{(m_c + m_w)l^2 + J_{wy} l^2 / R^2}{J_{wz} + J_{cz} + m_c (l - l_c)^2}, \) (3.22)

where \( L \) and \( V \) are the dimensionless caster length and towing speed respectively, and \( \kappa \) and \( \chi \) are dimensionless mass moment of inertia parameters related to the inertias of the caster and wheel.

With these new parameters, the third order Taylor series expansion of the first three governing equations (3.19) about the trivial solution assumes the form:

\[
\begin{bmatrix}
\dot{\psi} \\
\dot{\beta} \\
\dot{q}/l_c
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
-2\zeta V \omega_n^2 / L & -(2\zeta + \kappa V) \omega_n & -\omega_n^2 / L \\
\omega_n V & L & 0
\end{bmatrix}
\begin{bmatrix}
\psi \\
\beta \\
q/l_c
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
0 \\
-\omega_n \left( (2\zeta + \kappa V)(1 - \chi) + V \left( \frac{2}{L} - \frac{\chi}{2} \right) \right) \psi^2 \beta \\
\frac{w_n V}{3} \psi^3 + \frac{L}{2} \psi^2 \beta \\
0
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
0 \\
-\chi \psi^2 \beta - \frac{w_n^2}{L} \left( \frac{1}{2} - \chi \right) \psi^2 q/l_c - \frac{2\zeta V \omega_n^2}{L} \left( \frac{5}{6} - \chi \right) \psi^3
\end{bmatrix}
\]

(3.23)

After the substitution of the exponential trial solution \( K e^{\lambda t} \) into (3.23), where \( K \in \mathbb{C}^3 \) and \( \lambda \in \mathbb{C} \), the characteristic equation is obtained from the linear coefficient matrix of (3.23):

\[
\lambda^3 + (2\zeta + \kappa V) \omega_n \lambda^2 + \left( 1 + \frac{2\zeta V}{L} \right) \omega_n^2 \lambda + \frac{V L}{\omega_n^3} = 0.
\]

(3.24)

According to the Routh-Hurwitz criterion (see Gantmacher, 1975), the asymptotic stability of the stationary rolling is equivalent to:

\[ V/L > 0, \]

(3.25)

and

\[ L > L_{cr}(V) = \frac{V(1 - 4\zeta^2 - 2\zeta \kappa V)}{2\zeta + \kappa V}, \quad \text{for} \quad \kappa > \kappa_{cr} = -\frac{2\zeta}{V} \]

(3.26)

considering positive parameter values only. At fixed damping ratios \( \zeta \), the stability boundary curves are characterized by

\[ V_{ext} = \frac{1 - 2\zeta}{\kappa} \quad \text{and} \quad V_{max} = \frac{1 - 4\zeta^2}{2\zeta \kappa}, \]

(3.27)

where \( L_{cr} \) has maximum at \( V = V_{ext} \) (i.e., \( L'_{cr}(V_{ext}) = 0 \)) and \( L_{cr} \) is zero at \( V = V_{max} \) (i.e., \( L_{cr}(V_{max}) = 0 \)). Stationary rolling is always stable for \( V > V_{max} \) or \( L > L_{cr}(V_{ext}) \).

Figure 3.2(a) shows the corresponding stability chart in the plane of the dimensionless towing speed \( V \) and caster length \( L \) for \( \kappa = 0.203 \) and \( \chi = 5.67 \). The stability region is shaded for \( \zeta = 0.1 \). The dash-dot line is the locus of \( V_{ext} \) as a function of \( \zeta \). In the figure, the values of \( \kappa \) and \( \chi \) come from the realistic towed wheel of a shopping trolley (see Table 3.1). We shall discuss the bifurcation diagram in Figure 3.2(b) in Section 3.3.

Note, that the formula of the critical caster length (3.26) is valid for \( \kappa > \kappa_{cr} \). If zero damping \( \zeta = 0 \) or \( V \to \infty \) is considered, the critical parameter is \( \kappa_{cr} = 0 \). This
means that linearly stable stationary rolling can exist for undamped case or for large values of the towing speed if $\kappa$ is positive. The dimensional form of this condition leads to the criterion $l_c < l$, which means that the centre C of gravity has to be closer to the king pin than the wheel is. This statement has essential importance, since it can be controlled easier in practice than the complicated stability condition (3.26). Although, the condition $l_c < l$ derived here is only a simple necessary (but not sufficient) condition of linear stability, it provides important instructions for loading trailers (see Figure 3.3) (see O’Reilly and Varadi, 1999). Similar, easy-to-check conditions were derived by Fratila and Darling (1996); Sharp and Fernández (2002) for caravans.

The stability chart in dimensional terms is of great practical importance for engineers, of course. However, the natural choice of scaling in (3.21) and (3.22) makes it difficult to immediately deduce the critical engineering design parameters from Figure 3.2(a). This is due to the fact that the values of $\kappa$ and $\chi$ also depend on the caster length (see (3.22)). Thus, we redraw the stability chart in dimensional terms in Figure 3.4, using the shopping trolley parameters.

**Table 3.1** The physical parameters of a realistic shopping trolley.

<table>
<thead>
<tr>
<th>Dimensional parameters</th>
<th>Dimensionless parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Caster</strong></td>
<td></td>
</tr>
<tr>
<td>$l$ [m]</td>
<td>0.02</td>
</tr>
<tr>
<td>$l_c$ [m]</td>
<td>0.012</td>
</tr>
<tr>
<td>$m_c$ [kg]</td>
<td>0.067</td>
</tr>
<tr>
<td>$J_{cz}$ [kgm$^2$]</td>
<td>$3.48 \times 10^{-6}$</td>
</tr>
<tr>
<td><strong>Wheel</strong></td>
<td></td>
</tr>
<tr>
<td>$R$ [m]</td>
<td>0.04</td>
</tr>
<tr>
<td>$m_w$ [kg]</td>
<td>0.312</td>
</tr>
<tr>
<td>$J_{wy}$ [kgm$^2$]</td>
<td>$4.63 \times 10^{-5}$</td>
</tr>
<tr>
<td>$J_{wz}$ [kgm$^2$]</td>
<td>$2.38 \times 10^{-5}$</td>
</tr>
<tr>
<td>$\kappa$ [-]</td>
<td>0.203</td>
</tr>
<tr>
<td>$\chi$ [-]</td>
<td>5.67</td>
</tr>
</tbody>
</table>
Figure 3.3 Effect of trailer loading on linear stability. Using the necessary condition $l_c < l$, some unstable set-ups of the trailer can easily be recognized in practice.

Figure 3.4 Stability boundaries of the linear system in dimensional form using the shopping trolley parameters.

3.3 Hopf bifurcation

Despite giving a more useful stability chart from engineering viewpoint, the dimensional form is less amenable to mathematical analysis. In dimensionless form the stability and bifurcation calculations are simpler and clearer.

The eigenvalues of the linear coefficient matrix of the dimensionless system can be determined on the stability boundary when $L = L_{cr}$ as given in (3.26). There are two complex conjugate eigenvalues with zero real part and one negative real eigenvalue:

$$\lambda_{1,2} = \pm i\omega, \quad \lambda_3 = -(2\zeta + \kappa V)\omega_n,$$  \hspace{1cm} (3.28)

where

$$\omega = \frac{\omega_n}{\sqrt{1 - 4\zeta^2 - 2\zeta \kappa V}}, \hspace{1cm} (3.29)$$
hence, there is Hopf bifurcation on the stability boundary. The corresponding eigenvectors \( \{s_1, s_2, s_3\} \) can also be determined:

\[
\begin{align*}
\mathbf{s}_1 = \mathbf{s}_2 &= \begin{pmatrix}
(2\zeta + \kappa V)
& \left(1 + i (2\zeta + \kappa V) \frac{\omega}{\omega_n} - i\right) \\
-(2\zeta + \kappa V) \omega & \left((2\zeta + \kappa V) \frac{\omega}{\omega_n} + i\right) \\
V \left((2\zeta + \kappa V)^2 + \frac{\omega^2}{\omega_n^2}\right) & -\frac{1}{2\kappa V}
\end{pmatrix}, \\
\mathbf{s}_3 &= \begin{pmatrix}
-\frac{1}{2\kappa V}
\end{pmatrix}.
\end{align*}
\]

The so-called Centre Manifold reduction (see Guckenheimer and Holmes, 1983) requires the construction of the transformation matrix \( \mathbf{T} \) from the eigenvectors in the following way:

\[
\mathbf{T} = \begin{pmatrix}
\text{Re} \mathbf{s}_1 & \text{Im} \mathbf{s}_1 & \mathbf{s}_3
\end{pmatrix}.
\]

(3.31)

Let us introduce new variables \( (x_1, x_2, x_3) \) such that:

\[
\begin{pmatrix}
\psi \\
\beta \\
q/l_c
\end{pmatrix} = \mathbf{T} 
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}. 
\]

(3.32)

If we substitute this into (3.23) and multiply the equation with the inverse of the transformation matrix, the Jordan normal form is calculated in the form

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{pmatrix} = \begin{pmatrix}
0 & \omega & 0 \\
-\omega & 0 & 0 \\
0 & 0 & -(2\zeta + \kappa V) \omega
\end{pmatrix} 
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} + \left(\sum_{j+k=3} a_{jk} x_j^1 x_k^1 + \ldots \right) 
\left(\sum_{j+k=3} b_{jk} x_j^2 x_k^2 + \ldots \right). 
\]

(3.33)

Since the nonlinearities are symmetric (i.e. there are no second degree terms in the nonlinear part of the Jordan normal form), the center manifold is approximated by a second degree surface. Thus, the transformation of the nonlinear part into the third degree Poincaré normal form needs only the terms in \( x_1 \) and \( x_2 \). The terms in which \( x_3 \) appear can be neglected, since the equation of the Centre Manifold is approximated by \( x_3 = 0 \) till the second degree. The sense of the Hopf bifurcation for a symmetric case comes from the reduced form of the Poincaré-Lyapunov parameter \( \Delta \) (see Stépán, 1989):

\[
\Delta = \frac{1}{8} \left(3a_{30} + a_{12} + b_{21} + 3b_{03}\right),
\]

(3.34)

so

\[
\Delta = \frac{(2\zeta + \kappa V)^2 \omega_n}{8 V^2} \cdot \frac{\zeta + \kappa V - (2\zeta + \kappa V) \frac{\omega^2}{\omega_n} \left(2 - \zeta \kappa V + \chi \left(\frac{\omega^2}{\omega_n^2} - 2\right)\right)}{\left(1 + (2\zeta + \kappa V)^2 + \frac{\omega^2}{\omega_n^2}\right) \left((2\zeta + \kappa V)^2 + \frac{\omega^2}{\omega_n^2}\right)}. 
\]

(3.35)

If \( \Delta \) is positive/negative then the periodic orbit emerging from the Hopf bifurcation point is unstable/stable, that is, the sense of the bifurcation is subcritical/supercritical. If we set \( V = 8 \) and take \( \kappa = 0.203 \) and \( \chi = 5.67 \) as before, \( \Delta \) can be plotted as a function of the damping ratio \( \zeta \) (see Figure 3.5). In our case, the sign of \( \Delta \) changes at a critical value of the damping \( \zeta_{cr} = 0.117 \), where the Hopf bifurcation is degenerate.
The general formula of the critical damping ratio could be determined by taking (3.35) equal to zero, i.e., $\Delta = 0$, which provides a fifth-degree polynomial equation for $\zeta_{cr}$. However, a useful upper estimation of the critical damping can be given if the condition $\kappa > 0$ is considered, which is also a necessary condition for the existence of the stable stationary rolling at large towing speeds. The estimation leads to

$$\zeta_{cr} < \zeta_{est}^{cr} = \sqrt{\frac{\kappa^2 V^2 + 2 - \kappa V}{4}},$$

(3.36)

which means that the sense of the Hopf bifurcation is supercritical for $\zeta > \zeta_{est}^{cr}$. In case of the shopping trolley parameters, this estimated value is $\zeta_{est}^{cr} = 0.132$, which results a 13% error relative to the numerically determined critical damping.

To compare our results with numerical continuation, the theoretical bifurcation branch has to be calculated. Accordingly, the real part of the implicit derivative of the characteristic equation (3.24) with respect to the bifurcation parameter $L$ has to be determined at the critical parameter value $L_{cr}$:

$$\text{Re} \frac{d\lambda}{dL} \bigg|_{L=L_{cr}} = \frac{(2\zeta + \kappa V)^2 \omega_n}{2V \left(1 + (2\zeta + \kappa V)^2 \frac{\omega_n^2}{\omega^2}\right)},$$

(3.37)

The amplitude of the periodic orbit is given by

$$r = \sqrt{-\text{Re} \frac{\lambda}{\Delta} \bigg|_{L=L_{cr}} (L - L_{cr})},$$

(3.38)

namely

$$r = \sqrt{\frac{4V \left((2\zeta + \kappa V)^2 + \frac{\omega_n^2}{\omega^2}\right)}{\zeta + \kappa V - (2\zeta + \kappa V) \frac{\omega_n^2}{\omega^2} \left(2 - \zeta \kappa V + \chi \left(\frac{\omega_n^2}{\omega^2} - 2\right)\right)} (L - L_{cr}).}$$

(3.39)

With the help of the transformation matrix $T$, the limit cycle can be given in the three dimensional phase space:

$$\begin{pmatrix}
\psi \\
\beta \\
q/l_c
\end{pmatrix} = T \begin{pmatrix}
r \cos(\omega t) \\
r \sin(\omega t) \\
0
\end{pmatrix}. $$

(3.40)
Thus, the amplitudes of the limit cycle with respect to the general coordinates can be given by

\[
A_\psi = r(2\zeta + \kappa V)\sqrt{1 + (2\zeta + \kappa V)^2 \frac{\omega_n^2}{\omega_n^2}},
\]
\[
A_\beta = \omega A_\psi,
\]
\[
A_q = rV\left((2\zeta + \kappa V)^2 + \frac{\omega_n^2}{\omega_n^2}\right).
\]

The theoretical branches of the amplitude of \(A_\psi\) are compared to the numerical results of the dimensionless system (3.23) in Figure 3.2(b). In this figure, as in the remainder of this chapter, dashed lines mean unstable branches and continuous lines mean stable solutions, and thick and thin lines show numerical and theoretical results, respectively. When the bifurcations are supercritical, the numerical results show the existence of folds for large values of the damping ratio \(\zeta\), leading to large amplitude unstable periodic motions.

As mentioned at the start of this section the analysis of the damped dimensional system is difficult. But in case of zero damping the sense of the Hopf bifurcation can be determined. This is calculated in the paper of Stépán (1991), where he uses simpler geometry with relationships between masses and mass moments of inertia of the wheel and the caster (i.e., \(l_c = l/2\), \(J_{cz} = \frac{1}{12}m_cl_c^2\), \(J_{wz} = \frac{1}{4}m_wR^2\) and \(J_{wy} = \frac{1}{2}m_wR^2\)). Without using any of these geometrical limitations, we can calculate the sense of the Hopf bifurcation for zero damping in closed form, leading to the Poincaré-Lyapunov parameter

\[
\Delta = \frac{v \omega_n^4 l}{8m_c(l - l_c)(v^2 + \omega_n^2 l^2)} \left( m_wl + m_cl_c + \frac{l}{R^2}J_{wy} \right) > 0,
\]

which is always positive for all realistic parameter values, namely the sense of the Hopf bifurcation is always subcritical, unstable periodic orbits exist always. The comparison of the theoretical and numerical results is shown in Figure 3.6 using the parameters in Table 3.1. The results show an excellent agreement till the vibration amplitude of the caster angle reaches 0.15 [rad]. After that, the vibration amplitudes of the unstable oscillation do not increase as much as predicted by the Hopf approximation, and they do not grow above 0.22 [rad].
3.4 Numerical Continuation

Returning now to the damped case (with $b = 0.2\, [\text{Ns/m}]$), the branch of unstable periodic solutions was followed using AUTO97 (Doedel et al., 1997) from the Hopf point and a saddle-node bifurcation (fold) was detected (see Figure 3.7(a)). This means that the damped system has stable periodic solutions, too. So if the system is perturbed enough then it will be attracted to these large amplitude stable periodic solutions. This is illustrated in Figure 3.7(b), where simulated motions with different initial conditions are shown in the three dimensional phase space, and the numerically determined unstable and stable periodic solutions are also plotted.

If the damping factor is increased, the notch of the stable branch and the peak of the unstable branch move closer and closer to each other. At a critical damping factor ($b_l = 0.269\, [\text{Ns/m}]$) the branches intersect each other and an isola is born, as shown in Figure 3.7(a). So a separated periodic solution branch occurs over the linearly stable parameter domain in the bifurcation diagram for $b_l > 0.269\, [\text{Ns/m}]$. There are now three folds in the figure, and by varying the damping factor $b$ or the towing velocity $v$, the locations of these folds also change. This means that the behaviour of the system can change significantly for different values of the system parameters.

For fixed damping factor, a bifurcation chart can be plotted in the $(v, l)$ plane, where the locations of the folds are also marked. With the help of different sections and projections of the chart, the nonlinear behaviour of the system is represented by bifurcation diagrams in Figure 3.8 for the damping factor $b_l = 0.5\, [\text{Ns/m}]$. The bistable parameter region is bordered by the corresponding curves of Fold 1, 2, 3 and by the linear stability boundary. The existence of this domain is very dangerous indeed as it is well presented by the bifurcation diagram over the bifurcation chart. It is constructed at $l = 0.06\, [\text{m}]$ with the bifurcation parameter $v$. As it is shown, the caster length is chosen from the linearly stable parameter domain, i.e., the stationary rolling is stable for any value of the towing speed. However, there is also a separated periodic solution branch, which means

![Bifurcation Chart](image_url)

**Figure 3.7** Panel (a) and (b) show the bifurcation diagrams of the damped system with the isola birth and the limit cycles with simulated trajectories in the 3D phase space, respectively.
that shimmy motion can still occur in a certain speed ranges. This region can not be
determined by linear analysis. Moreover, these large amplitude vibrations could not be
surely discovered either by extensive experimental investigation or by brute force simula-
tions of such systems. Therefore, without the detailed nonlinear investigation, a system
may be designed with parameter values that allows for the presence of these dangerous
oscillations. Note that the size of the bistable region can be reduced and bounded for
large enough damping ratio. This is shown in Figure 3.9.

Typical periodic motions of the spring-supported towed rigid wheel is demonstrated
in Figure 3.10. For the fixed parameter values of the parameter point B in Figure 3.8 ($v = 3 \, \text{m/s}$ and $l = 0.1 \, \text{m}$), the unstable and stable oscillations of the caster angle $\psi(t)$, the caster angular velocity $\beta(t)$ and the lateral displacement $q(t)$ are plotted versus the time $t \in [0, T]$. The vibration frequency of the detected stable periodic motion is $f = 1/T = 3.40 \, \text{Hz}$. For the stable periodic motion, the maximum value of the lateral displacement $q(t)$ is unrealistic for a shopping trolley wheel. In practice, such large amplitude stable periodic motions are expected to involve sliding of the wheel after a certain critical friction force is reached. This is the reason why the experienced violent stable oscillations of the shopping cart wheels are still smaller in amplitude than the ones calculated above in a model that considers rolling only with unlimited static coefficient of friction.
In spite of the quantitative discrepancy explained above, the qualitative behaviour of these wheels are captured quite well by the rolling wheel model as illustrated in Figure 3.10. The time history shows that the caster flaps widely and sharp turning points can be observed in the trace of the wheel. The same phenomenon is observed in case of an articulated bus that exhibits similar large amplitude oscillations after losing its stability in a subway (see Figure 3.11). The rear part of the articulated bus also presents sharp and sudden flaps during its periodic motion, although it involves certain sliding at these time instants. In the subsequent chapters, the switches between the dynamics of rolling and sliding wheels will be analysed in details in an improved mechanical model.

### 3.5 New results

**Thesis 1** A low degree-of-freedom mechanical model of a towed rigid wheel with elastic suspension was constructed and analysed. It was proven that the stationary rolling of the towed rigid wheel is asymptotically stable if and only if:

\[ L > L_{cr}(V) = \frac{V(1 - 4\zeta^2 - 2\zeta\kappa V)}{2\zeta + \kappa V}, \]

for the positive dimensionless parameter values of caster length \( L \), towing speed \( V \), damping ratio \( \zeta \) and inertia parameter \( \kappa \).

The Centre Manifold reduction of the damped system was carried out and the sense of the Hopf bifurcation related to the linear stability boundary was determined in closed form. It was shown that unstable self-excited vibrations exist around the stable stationary rolling if the damping ratio is smaller than a critical value. A useful upper estimation of the critical damping ratio was calculated, and it was shown that the loss of the stability of stationary rolling leads to small amplitude stable oscillations if

\[ \zeta > \frac{\sqrt{\kappa^2 V^2 + 2} - \kappa V}{4}, \]

considering positive parameters only. This estimation helps to guarantee that small amplitude oscillations will signal the danger of possible loss of stability. The analytical results were also confirmed by numerical continuation technique accomplished in the open source software AUTO97.

These results are published in Takács et al. (2008).
Thesis 2  The low degree-of-freedom model of the towed rigid wheel was investigated globally by numerical continuation technique outside the parameter region where the local bifurcation analysis was valid. Fold bifurcation of periodic orbits linked to the theoretical Hopf bifurcation point was detected, which verified the existence of stable large amplitude oscillations even around the unstable self-excited oscillations linked to the subcritical Hopf point. The separation of a periodic branch – a so-called isola – was discovered at certain critical parameter values. A bifurcation chart was constructed in the plane of the towing speed and the caster length, in which a so-called bistable parameter range of the towed wheel was identified where the stable stationary rolling and large amplitude stable oscillations coexist.

This provides essential new information for the design of such rolling systems, since neither linear stability analysis nor local bifurcation analysis, similarly, neither extensive numerical simulation nor experimental work can predict the existence of those violent oscillations, which appear with low probability at certain large perturbations only while the stationary rolling is quite robustly stable.

These results are published in Takács et al. (2008).
Chapter 4

Elastic tyre model

In this chapter, an elastic tyre model is used to investigate shimmy. The model consists of an elastic tyre towed by a rigid caster and king pin. The equations of motion of the system consisting of rigid-bodies and a massless stretched string, are derived by means of the Appell-Gibbs equation (Gantmacher, 1975). The linear stability of the system is analysed and stability charts are constructed in the plane of the towing speed and caster length for different values of other system parameters. The nonlinear system is also analysed by numerical simulations, where partial sliding is considered, too.

4.1 Mechanical model

The mechanical model in question is shown in Figure 4.1. The elastic tyre with rigid rim is towed by a rigid caster of length $l$ on the steady horizontal ground into the $X$ direction by the constant velocity $v$. The plane of the wheel is always vertical to the ground. The tyre is in contact with the ground along a final contact patch. This patch is modelled by a contact line of length $2a$. In this way, the lateral deformation of the tyre is described by the displacement function $q(x,t)$ of the centre line relative to the plane of the wheel rim. The coordinate $x$ describes the longitudinal position of the tyre centre points in the caster fixed $(x, y, z)$ coordinate system, and $t$ stands for the time. The longitudinal deformation of the tyre is not considered here, its effect on the wheel shimmy is assumed to be negligible. This approximation is not allowed in case of braking and accelerating wheels.

The contact forces acting between the tyre and the ground are calculated with the help of the elastic lateral deformation of the tyre by means of the specific lateral stiffness $k$ and the specific lateral damping factor $b$ both distributed along the contact length. In the mean time, the inertia of the deformable tyre is neglected and its mass is included in the rigid wheel only. This means that the mechanical model of our towed wheel is a very special combination of rigid bodies (rim, caster, king pin) and massless elastic continuum strings (tyre).

The caster is supported by torsional spring and damper at the king pin, which can model a steering mechanism and/or a shimmy damper. The corresponding torsional stiffness and the torsional damping factor are $k_t$ and $b_t$, respectively. We will refer to this system as torsionally supported at the king pin or simply as steered and towed wheel.

The rigid bodies of the system are characterized by the following parameters. The
mass of the wheel is $m_w$ and its mass moment of inertia with respect to the $z$ axis at the point O is $J_{wz}$. Its mass moment of inertia with respect to the $y$ axis of its rotation is $J_{wy}$, where subscripts w refers to wheel. The distance between the centre of gravity C of the caster and the point A of the king pin is $l_c$. The mass of the caster is $m_c$ and the mass moment of inertia with respect to the $z$ axis at C is $J_{cz}$, with subscripts c referring to caster.

Considering the geometrical constraints only, without the constraint of rolling, the system has 2 rigid-body DoF, so one can choose the caster angle $\psi$ and the wheel rotation angle $\varphi$ as general coordinates. The lateral deformation $q(x,\cdot)$ of the tyre is also a state variable of the system. When the time dependences of these discrete and distributed general coordinates are emphasized, we use the notation $\psi(t)$, $\varphi(t)$ and $q(x,t)$. During the derivation of the equation of motion, $q(x,t)$ will only emerge in the formula of the contact force, which acts on the wheel rim as an active distributed force. This is due to the fact that the tyre is modelled by a massless elastic string. The additional kinematic (in alternative terminology, non-holonomic) constraint of rolling (without any sliding) leads
to constraining equations for the elastic tyre and for the rigid-body system together.

### 4.1.1 Kinematic constraint for the lateral deformation

Considering the massless elastic stretched string model of the tyre, the kinematic constraint of rolling is formulated as follows. In the ground-fixed coordinate system \((X, Y, Z)\), the position vector of a contact point \(P\) is given as

\[
\begin{pmatrix}
X(x,t) \\
Y(x,t)
\end{pmatrix} = \begin{pmatrix}
vt - (l - x) \cos \psi(t) - q(x,t) \sin \psi(t) \\
-(l - x) \sin \psi(t) + q(x,t) \cos \psi(t)
\end{pmatrix}
\] (4.1)

for \(x \in [-a,a]\). The geometric constraint \(Z(x,t) \equiv 0\) is not spelled out here since it was already taken into account when the rigid-body general coordinates were chosen. The differentiation of this position vector with respect to time gives the velocities of the contact points taken to be zero to enforce the no-slip condition:

\[
\frac{d}{dt} \begin{pmatrix}
X(x,t) \\
Y(x,t)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\] (4.2)

for \(x \in [-a,a]\). Because the tyre points travel backwards relative to the caster, the element of the tyre located at a longitudinal position \(x\) changes in time similarly to the Eulerian description of flows. Consequently, \(q(x,t)\) describes the lateral displacement of the tyre particle located at \(x\) at the time instant \(t\) only. Then the total derivative of \(q(x,t)\) with respect to the time is

\[
\frac{d}{dt}q(x,t) = \dot{q}(x,t) + q'(x,t)x,
\] (4.3)

for \(x \in [-a,a]\). Here \(\frac{d}{dt}\) refers to total differentiation with respect to time \(t\) (also called material time derivative) while dot and prime refer to partial differentiation with respect to time \(t\) and space \(x\), respectively. Thus, the manipulation of (4.2) leads to

\[
\begin{cases}
\frac{d}{dt}q(x,t) = v \sin \psi(t) + (l - x) \dot{\psi}(t), \\
\dot{x} = -v \cos \psi(t) + q(x,t) \dot{\psi}(t).
\end{cases}
\] (4.4)

Using these formulas in (4.3), we obtain the constraining equation for the lateral elastic deformation in the form of a first-order scalar partial differential equation (PDE)

\[
\dot{q}(x,t) = v \sin \psi(t) + (l - x) \dot{\psi}(t) + q'(x,t)\{v \cos \psi(t) - q(x,t) \dot{\psi}(t)\},
\] (4.5)

where \(x \in [-a,a]\). A boundary condition (BC) is also needed, of course, which can be composed with the help of different approximate models of the tyre. Two different types of tyre models are considered in this study, the brush tyre model and the stretched string tyre model, which are explained in detail later in Section 4.3.

In several former studies of shimmy (see Pacejka, 1966; Schlippe and Dietrich, 1941), this PDE was eliminated and the deformation of the tyre was approximated by certain simple functions obtained from steady-state conditions. Now, we use this PDE to calculate the exact, dynamically varying deformations of the tyre during rolling.
4.1.2 Equations of motion of the rigid-body system

The kinematic constraint of rolling also leads to a condition in the longitudinal direction. Since we do not take into account the longitudinal deformation of the tyre, this condition can be formulated in the rigid-body system with the help of the effective rolling radius \( R_e \) (see Pacejka, 2002). The effective rolling radius produces connection between the longitudinal velocity of the centroid of the wheel and the wheel rotation angular velocity, namely, \( v_{Ox} = R_e \dot{\varphi} \). Thus, the rolling condition of the rigid-body system leads to a scalar kinematic constraint in the form of a first order nonlinear ODE with respect to the general coordinates:

\[
v \cos \psi - R_e \dot{\varphi} = 0 .
\]  

This also corresponds to the second constraining equation of (4.4), supposing that the distributed stiffness of the tyre in the longitudinal direction is zero.

The equation of motion of the rigid-body system is derived here with the help of the Appell-Gibbs equation (Gantmacher, 1975) that requires the (intuitive) definition of so-called pseudo velocities. The number of the pseudo velocities is equal to the difference between the number of rigid-body general coordinates and the number of the rigid-body kinematic constraints, which is \( 2 - 1 = 1 \). In our case, the angular velocity of the caster is chosen as the single pseudo velocity:

\[
\beta := \dot{\psi} .
\]

The kinematic constraint (4.6) and the definition of the pseudo velocity give the formulae of the two general velocities in terms of the two general coordinates and the pseudo velocity:

\[
\begin{pmatrix}
\dot{\psi} \\
\dot{\varphi}
\end{pmatrix} = \begin{pmatrix}
\frac{\beta}{R_e} \\
\cos \psi
\end{pmatrix} .
\]

The differentiation of the general velocities (4.8) with respect to time leads to the two general accelerations expressed by the two general coordinates and the pseudo acceleration:

\[
\begin{pmatrix}
\ddot{\psi} \\
\ddot{\varphi}
\end{pmatrix} = \begin{pmatrix}
-\frac{\dot{\beta}}{R_e} \\
\sin \psi
\end{pmatrix} .
\]

The energy of acceleration of the system is

\[
A = \frac{1}{2} m_c a_C \cdot a_C + \frac{1}{2} \varepsilon_c^T J_c \varepsilon_c + \varepsilon_c^T (\omega_c \times (J_c \omega_c)) + \ldots + \frac{1}{2} m_w a_O \cdot a_O + \frac{1}{2} \varepsilon_w^T J_w \varepsilon_w + \varepsilon_w^T (\omega_w \times (J_w \omega_w)) + \ldots ,
\]

where the accelerations of the centres of gravity of the caster and the wheel are given in the caster fixed \((x, y, z)\) coordinate system by the formulae

\[
a_C = \begin{pmatrix}
\varepsilon_c^T J_c \\
-l_c \dot{\beta}
\end{pmatrix} \quad \text{and} \quad a_O = \begin{pmatrix}
\varepsilon_w^T J_w \\
-l_w \dot{\beta}
\end{pmatrix} ,
\]

and

\[
\varepsilon_c = \begin{pmatrix}
l_c \beta^2 \\
-l_c \dot{\beta}
\end{pmatrix} \quad \text{and} \quad \varepsilon_w = \begin{pmatrix}
l_w \beta^2 \\
-l_w \dot{\beta}
\end{pmatrix} .
\]
respectively. The angular velocity of the caster and the wheel are
\[ \omega_c = \begin{pmatrix} 0 \\ 0 \\ \beta \cos \psi \end{pmatrix} \quad \text{and} \quad \omega_w = \begin{pmatrix} 0 \\ 0 \\ \frac{\dot{\varphi} \beta}{R_e} \cos \psi \end{pmatrix}, \tag{4.12} \]
respectively. The differentiation of the angular velocities with respect to time leads to the angular accelerations:
\[ \varepsilon_c = \begin{pmatrix} 0 \\ 0 \\ \ddot{\beta} \end{pmatrix} \quad \text{and} \quad \varepsilon_w = \begin{pmatrix} 0 \\ -\frac{\dot{\varphi} \beta}{R_e} \sin \psi \\ \frac{\ddot{\beta}}{R_e} \cos \psi \end{pmatrix}. \tag{4.13} \]
The energy of acceleration (4.10) with (4.11), (4.12) and (4.13) forms
\[ A(\dot{\beta}, \beta, \psi) = \frac{1}{2} \left( J_{cz} + m_c l_c^2 + J_{wz} + m_w l^2 \right) \dot{\beta}^2 + \ldots \tag{4.14} \]
The Appell-Gibbs equation can be written in the following form:
\[ \frac{\partial A}{\partial \dot{\beta}} = \Gamma, \tag{4.15} \]
where the right hand side is the so-called pseudo force \( \Gamma \), which has to be calculated via the virtual power of the active forces. In this case, the active forces are generated by the distributed forces related to the lateral tyre deformation and by the torsional spring and damper at the king pin. The distributed tyre forces can be integrated along the coordinate \( x \), where the limits of the integrals are \( -\infty \) to \( \infty \) (see Section 4.3.2), since the tyre may have lateral deformation outside the contact patch that depends on the approximating mechanical model of the tyre (brush model or stretched string model). The virtual power is given by
\[ \delta P = -(k_t \psi + b_t \beta) \cdot \delta \beta - \int_{-\infty}^{\infty} \left( k q(x,.) + b \frac{d}{dt} q(x,.) \right) \cdot \delta \left( \frac{d}{dt} q(x,.) \right) \, dx, \tag{4.16} \]
where \( \delta \) denotes virtual quantities. By means of (4.4), the virtual velocities of the tyre points can be described by the caster angle \( \psi \) and by the pseudo velocity \( \beta \):
\[ \delta \left( \frac{d}{dt} q(x,.) \right) = v \sin \psi + (l - x) \delta \beta. \tag{4.17} \]
After using (4.17) in (4.16), the pseudo force has the form
\[ \Gamma(\psi, \beta, q(x, .)) = -k_t \psi - b_t \beta - \int_{-\infty}^{\infty} (l - x) \left( k q(x,.) + b \frac{d}{dt} q(x,.) \right) \, dx, \tag{4.18} \]
where the third state variable of the system, namely, the lateral tyre deformation \( q(x, t) \) also appears.

The Appel-Gibbs equation (4.15) then leads to
\[ J_A \ddot{\beta} = -(k_t \psi - b_t \beta - \int_{-\infty}^{\infty} (l - x) \left( k q(x,.) + b \frac{d}{dt} q(x,.) \right) \, dx, \tag{4.19} \]
where \( J_A = J_{cz} + m_c l_c^2 + J_{wz} + m_w l^2 \) is the mass moment of inertia of the overall system with respect to the \( z \) axis at the king pin \( A \).
4.1.3 Governing equations of the towed elastic tyre

The overall mechanical model of the towed elastic tyre is governed by the Appel-Gibbs equation (4.19) and by the equations of the discrete general velocities (4.8) and the distributed general velocity (4.5) that relates to the lateral deformation of the tyre. After dropping the notation $\beta$ of the pseudo velocity and using the original general coordinates only, we obtain the governing equations in the rearranged sequence

$$J_A \ddot{\psi}(t) = -k_t \dot{\psi}(t) - b_t \dot{\psi}(t) - \int_{-\infty}^{\infty} (l - x) \left( k q(x, t) + b \frac{d}{dt} q(x, t) \right) dx,$$

$$\dot{q}(x, t) = v \sin \psi(t) + (l - x) \dot{\psi}(t) + q'(x, t) \left( v \cos \psi(t) - q(x, t) \dot{\psi}(t) \right),$$

$$\dot{\phi}(t) = \frac{v \cos \psi(t)}{R_e},$$

with the initial conditions:

$$\psi(0) = \psi_0, \quad \dot{\psi}(0) = \omega_0, \quad q(x, 0) = q_0(x), \quad \varphi(0) = \varphi_0.$$ (4.21)

Thus, the system is governed by an integro-differential equation (IDE), by a nonlinear PDE and by an ODE. Since the general coordinate $\varphi$ appears only in the third equation of motion, this coordinate is a so-called cyclic one and the system can be described uniquely with the essential coordinates $\psi$ and $q(x, \cdot)$ only, which are decoupled from the cyclic coordinate $\varphi$.

4.2 Memory effect of tyre – Travelling wave solution and time delay

The first two equations in (4.20) describe the system as a coupled IDE-PDE system. Since the contact points are sticking to the ground in $(X, Y, Z)$, they travel backwards relative to the caster fixed $(x, y, z)$. This gives the physical explanation for the introduction of the travelling wave solution

$$\left( \begin{array}{c} X(x, t) \\ Y(x, t) \end{array} \right) = \left( \begin{array}{c} X(a, t - \tau(x)) \\ Y(a, t - \tau(x)) \end{array} \right),$$ (4.22)

for $x \in [-a, a]$. The time $\tau(x)$ is needed for a tyre point touching the ground first at the leading edge L (where $x = a$) to travel backwards relative to the caster to the actual position P defined by the coordinate $x$. Accordingly, the recent history of the wheel motion is stored in the deformed contact area. Figure 4.2 represents this memory effect of the tyre: the lateral forces depend on the tyre contact point positions determined by the wheel leading edge positions in the past, when these points were put down to the ground. With the help of (4.1), the travelling wave (4.22) can be transformed into the form

$$\left\{ \begin{array}{l}
 l - x = v \tau \cos \psi(t) + (l - a) \cos(\psi(t) - \psi(t - \tau)) \\
 \quad -q(a, t - \tau) \sin(\psi(t) - \psi(t - \tau)), \\
 q(x, t) = v \tau \sin \psi(t) + (l - a) \sin(\psi(t) - \psi(t - \tau)) \\
 \quad + q(a, t - \tau) \cos(\psi(t) - \psi(t - \tau)), \\
 \end{array} \right.$$ (4.23)
which is still implicit with respect to the time delay due to the fact that $\tau(x)$ cannot be expressed in closed form. However, differentiating the first equation in (4.23) with respect to $\tau$, we obtain

$$\frac{d}{d\tau} = -v \cos \psi(t) + (l - a) \dot{\psi}(t - \tau) \sin(\psi(t) - \psi(t - \tau))$$

(4.24)

$$- \dot{q}(a,t - \tau) \sin(\psi(t) - \psi(t - \tau)) + q(a,t - \tau) \dot{\psi}(t - \tau) \cos(\psi(t) - \psi(t - \tau))$$

Now, one may substitute (4.23) into the first equation of (4.20) and use the change of variables $x$ and $\tau$ (integration by substitution) based on (4.24). This leads to a resulting retarded functional differential equation (RFDE), which contains the time dependent caster angle $\psi(t)$ and its delayed values $\psi(t - \tau)$, and the time dependent leading point lateral deformation $q(a,t)$ and its delayed values $q(a,t - \tau)$. The actual form of this RFDE will be presented in a linearised version later due to the high complexity of the nonlinear expressions (for a special case see Stépán (1998)). The PDE of the kinematic constraint of the contact line provides a one dimensional scalar differential equation with respect to the leading point lateral deformation:

$$\dot{q}(a,t) = v \sin \psi(t) + (l - a) \dot{\psi}(t) + q'(a,t) \left( v \cos \psi(t) - q(a,t) \dot{\psi}(t) \right),$$

(4.25)

which will be transformed into an ODE via the boundary condition of the PDE (see details later).

Pacejka’s creep force/moment model also uses the leading point lateral deformation as state variable, but instead of the travelling wave solution along the contact line, it uses the stationary lateral deformation obtained at constant drift angle $\psi$ (see Pacejka,
1966). This way, one obtains a three-dimensional nonlinear ODE instead of the infinite dimensional nonlinear RFDE-ODE, but loses the dynamics within the contact region, which may be important in certain parameter domains as identified later.

4.3 Tyre models

By the use of the travelling wave solution of the PDE (4.5) originated in the kinematic constraint, the system (4.20) of governing equations can be transformed into an RFDE-ODE in the following way. The boundary condition of the kinematic constraint (4.5) plays a key role in this transformation of the system equations. Both the brush tyre model and the stretched string tyre model are considered in this study, which are well-known and widely used in vehicle system dynamics (see Pacejka, 2002). In this section, we present the basic mechanical modelling and properties of the two different tyre models.

4.3.1 Brush tyre model

The brush tyre model in Figure 4.3(a) considers separate massless elastic thread elements, which do not effect each other, i.e., there is no elastic coupling between them. Consequently, the deformation outside the contact patch can be assumed to be zero and only the contact forces cause lateral deformation in the tyre. This leads to the Dirichlet boundary condition

\[ q(a, t) = 0 \]  \hspace{1cm} (4.26)

for \( t \in [0, \infty) \). With the help of this boundary condition, the travelling wave solution (4.23) depends on the caster angle \( \psi(t) \) and its delayed values \( \psi(t - \tau) \) only and the dynamics of the leading point lateral deformation (4.25) vanishes. This means that the equation of motion in (4.20) can be separated from the kinematic constraint (4.5), and the system is uniquely described by the RFDE with respect to the general coordinate \( \psi \).

4.3.2 Stretched string tyre model

The stretched string tyre model in Figure 4.3(b) was developed in Segel (1966). Segel neglected the bending stiffness of the tyre and modelled the tyre outside the contact patch by means of a massless stretched string with distributed elastic support. Here, we summarize the basic equations and assumptions of the stretched string tyre model.

The static deformation of the stretched string with elastic support is described by the ODE (see Kármán and Biot, 1940):

\[ \sigma^2 q''(s) - q(s) = -p(s)/k, \]  \hspace{1cm} (4.27)

where \( q(s) \) is the deformation of the string, \( \sigma \) is the so-called relaxation length, \( k \) is the distributed stiffness of the elastic support and \( p(s) \) is the lateral load on the string. The relaxation length is originated in the formula \( \sigma = \sqrt{F/k} \), where \( F \) is the tension force in the string. The points of the string are identified by the circumferential arc length coordinate \( s \).

In case of the tyre, the stretched string is used to model the deformation outside the contact patch, which means that the load on the string is zero there. This leads to a
homogeneous ODE with the general (hyperbolic) solution $q(s) = C_1 e^{-s/\sigma} + C_2 e^{s/\sigma}$. If the circumference of the tyre is considered to be much longer than the relaxation length, then it can be assumed that the deformation at the leading edge has negligible effect on deformation at the trailing edge and vice versa. This means that the deformation before the leading point and behind the rear point can be described separately. Moreover, the deformation of the equatorial line of the tyre can be projected to the ground and the circumferential coordinate $s$ can be approximated by the coordinate $x$. The theoretical extension $x \in (-\infty, \infty)$ is an acceptable standard approximation (see Pacejka, 2002; Schlippe and Dietrich, 1941; Segel, 1966) when the relaxation length $\sigma$ is small enough relative to the wheel radius $R$. For example, Segel (1966) experimentally identified the
condition $\sigma < (R\pi - a)/6$, but Besselink (2000) emphasized essential effects of tyre structure, too. Hence, the deformation outside the contact patch, before the leading point $L$ and behind the rear point $R$, is defined by exponentially decaying functions:

$$q(x,t) = \begin{cases} 
q(-a,t) e^{(x+a)/\sigma}, & \text{if } x \in (-\infty, -a), \\
q(x,t), & \text{if } x \in [-a, a], \\
q(a,t) e^{-(x-a)/\sigma}, & \text{if } x \in (a, \infty).
\end{cases} \tag{4.28}$$

These decaying functions are calculated from the ODE (4.27) of the undamped stretched string with vanishing boundary condition at infinity. If distributed damping is also considered, the stretched string is governed by a PDE, and the deformation outside the contact patch can not be determined in closed form. In that case, the dynamics of the stretched string can not be separated from the dynamics of the contact patch even in the massless case. Still, we will apply the above exponential decaying functions in order to estimate the distributed damping forces outside the contact patch, which approximation eliminates the non-relevant (higher order) dynamics of the travelling waves in the stretched string outside the contact region, but it enables the analytical handling of the equations.

If no sliding is considered in the contact patch, kinks may arise in the deformation of the tyre at the leading and trailing edges. However, due to the rolling process of the tyre, the kink at the leading edge $L$ disappears and $q(x,t)$ is continuously differentiable at $x = a$. The relaxation length $\sigma$ prescribes a line that is tangent to the lateral deformation function at the leading edge $L$ (see Figure 4.3(b)), which leads to the mixed boundary condition

$$q'(a,t) = -\frac{q(a,t)}{\sigma} \tag{4.29}$$

for $t \in [0, \infty)$. This condition is often called as ‘no kink at L’ condition (Pacejka, 1966).

With the help of the boundary condition (4.29), the differential equation (4.25) of the leading point lateral deformation results an ODE

$$\dot{q}(a,t) = v \sin \psi(t) + (l - a) \dot{\psi}(t) - \frac{v}{\sigma} q(a,t) \cos \psi(t) + \frac{1}{\sigma} q^2(a,t) \dot{\psi}(t). \tag{4.30}$$

This means, the system is governed by a nonlinear RFDE-ODE system in case of the stretched string tyre model, where the RFDE will be presented in Section 4.5.

### 4.4 Linear stability analysis of brush tyre model

In this section, the stability analysis of the system is presented for the brush tyre model without considering the elasticity of the steering mechanism and the shimmy damper at the king pin (i.e., $k_t = 0$ and $b_t = 0$). In case of the brush tyre model, the deformation outside the contact patch is assumed to be zero, which needs integration in the IDE (4.20) above the contact patch only. The boundary condition $q(a,t) = 0$ simplifies the form of the travelling wave solution (4.23), too.
4.4.1 Small oscillation around stationary rolling

The stationary rolling of the wheel corresponds to the trivial solution

\[ \psi(t) \equiv 0, \quad q(x, t) \equiv 0, \quad x \in [-a, +a]. \]  

Small shimmy oscillations around this stationary rolling can be described by the linearisation of the governing equations (4.20).

Substitute \( k_i = 0, b_i = 0 \) and the brush tyre model parameters in the IDE of (4.20). Then the equation of motion can be written as

\[ J_A \ddot{\psi}(t) = -\int_{-a}^{a} (l - x) \left( kq(x, t) + b \frac{d}{dt} q(x, t) \right) dx. \]  

In Section 4.2, the travelling wave solution (4.23) of the kinematic constraint (4.5) was calculated. If small oscillations are considered only around the stationary rolling, the linearised form of the travelling wave solution (4.23) can be used. Taking into account the boundary condition \( q(a, t) = 0 \), the travelling wave solution gives:

\[ \begin{cases} x = a - v\tau, \\ q(x, t) = (l - a + v\tau)\psi(t) - (l - a)\psi(t - \tau), \end{cases} \]  

for \( x \in [-a, a] \), which satisfies the linearised form of the kinematic constraint (4.5), of course. The time delay can be determined from the first equation explicitly, and we obtain it in the physically obvious linear approximation

\[ \tau(x) = \frac{a - x}{v}. \]  

The differentiation of \( x \) with respect to the time delay \( \tau \) leads to

\[ \frac{dx}{d\tau} = -v. \]  

After substituting (4.33) into the IDE (4.32) and using integration by substitution, we obtain a RFDE with respect to the present and delayed values of the caster angle \( \psi \):

\[ \ddot{\psi}(t) + 2\zeta\omega_n \dot{\psi}(t) + \omega_n^2 \psi(t) = \frac{kv}{J_A} \int_{0}^{2a/v} (l - a + v\tau)\psi(t - \tau)d\tau - \frac{bv}{J_A} 2la \psi(t), \]  

where the modal parameters

\[ \omega_n = \sqrt{\frac{2ka}{J_A} \left( l^2 + \frac{a^2}{3} \right)} \quad \text{and} \quad \zeta = \frac{b}{2k} \omega_n \]  

are the undamped natural angular frequency and the damping ratio of the steady wheel \( (v = 0) \), respectively.
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4.4.2 Rescaling and dimensionless parameters

Let us rescale the time as
\[ T := \frac{v}{2a} t, \] \hspace{1cm} (4.38)
define the new integration variable by
\[ \vartheta := -\frac{v}{2a} \tau. \] \hspace{1cm} (4.39)

Now, using dot as partial differentiation with respect to the rescaled time \( T \), too, we can define the dimensionless angular velocity as
\[ \Omega(T) := \dot{\psi}(T). \] \hspace{1cm} (4.40)

Furthermore, the dimensionless towing speed \( V \), the dimensionless caster length \( L \) are defined by
\[ V := \frac{1}{\omega_n} \frac{v}{2a}, \quad L := \frac{l}{a}, \] \hspace{1cm} (4.41)
respectively.

Using definitions (4.38)–(4.41), the RFDE (4.36) provides a 2-dimensional system of first-order DDEs:
\[
\begin{pmatrix}
\dot{\psi}(T) \\
\dot{\Omega}(T)
\end{pmatrix} = 
\begin{pmatrix}
-\frac{1}{V^2} & -\frac{4\zeta L}{V(L^2 + 1/3)} \\
-\frac{2\zeta}{V} & 0
\end{pmatrix} 
\begin{pmatrix}
\psi(T) \\
\Omega(T)
\end{pmatrix} + 
\frac{L - 1}{V^2(L^2 + 1/3)} \int_{-1}^{0} (L - 1 - 2\vartheta) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix}
\dot{\psi}(T + \vartheta) \\
\dot{\Omega}(T + \vartheta)
\end{pmatrix} d\vartheta. \] \hspace{1cm} (4.42)

4.4.3 Stability investigation

The stationary rolling motion (4.31) is now represented by the trivial solution
\[
\begin{pmatrix}
\psi(T) \\
\Omega(T)
\end{pmatrix} = 0 \] \hspace{1cm} (4.43)
of the linearised equation of motion (4.42). The Laplace transformation of (4.42) or the substitution of the trial solution
\[
\begin{pmatrix}
\psi(T) \\
\Omega(T)
\end{pmatrix} = Ke^{\lambda T}, \quad K \in \mathbb{C}^2, \quad \lambda \in \mathbb{C} \] \hspace{1cm} (4.44)
leads to the characteristic function
\[ D(\lambda; \mu) = V^2 \lambda^2 + 2\zeta V \lambda + 1 + \frac{4\zeta VL}{L^2 + 1/3} - \frac{L - 1}{L^2 + 1/3} \left( L \frac{1 - e^{-\lambda}}{\lambda} - \frac{1 + e^{-\lambda}}{\lambda} + 2 \frac{1 - e^{-\lambda}}{\lambda^2} \right), \] \hspace{1cm} (4.45)
where \( \mu \in \mathbb{R}^3 \) represents the dimensionless parameters ordered in a vector
\[ \mu = \text{col} [V \quad L \quad \zeta]. \] \hspace{1cm} (4.46)
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Generally, equation (4.45) has infinitely many complex zeros for the characteristic exponents (characteristic roots) \( \lambda \), but only a finite number of these may be situated in the right-half complex plane. The stationary rolling (4.31, 4.43) is asymptotically stable if and only if all the infinitely many characteristic exponents are situated in the left-half complex plane (see Stépán, 1989). At the limit of stability, bifurcation can take place in the corresponding nonlinear system when characteristic roots are located at the imaginary axis for some critical values \( \mu_{cr} \) of the parameter vector.

Saddle-node (SN) bifurcation (in other words, static loss of stability) of the stationary rolling can take place in the special case when the characteristic function is satisfied by the characteristic exponent \( \lambda = 0 \). In our case, the characteristic function is defined by its continuous extension for \( \lambda \to 0 \):

\[
D(0; \mu) = \frac{L + 4\zeta VL + 1/3}{L^2 + 1/3}.
\]

The critical dimensionless caster length of the static bifurcation can be calculated from \( D(0; \mu_{cr}) = 0 \), which leads to the negative value

\[
L_{SN} = -\frac{1}{3(1 + 4\zeta V)}.
\]

which refers to the case of pushing the wheel. Also, this critical dimensionless caster length of the saddle-node bifurcation depends on the dimensionless towing speed if the system is damped.

If the characteristic exponents located at the imaginary axis have non-zero imaginary parts, Hopf bifurcation may occur. In this case, a pair of pure imaginary complex conjugate characteristic exponents

\[
\lambda_{1,2}(\mu_{cr}) = \pm i\omega, \quad \omega \in \mathbb{R}^+)
\]

satisfies \( D(\lambda_{1,2}; \mu_{cr}) = 0 \) with the dimensionless angular frequency \( \omega \). Due to this bifurcation, self-excited vibrations may appear in the corresponding nonlinear system around the stationary rolling motion with dimensional frequency \( f = \omega v/(4a\pi) = \omega V_{\omega n}/(2\pi) \) in Hertz. Consequently, travelling waves propagate backward along the contact line with dimensional wave length \( v/f = 4a\pi/\omega \) in meter.

The stability boundaries are determined in the parameter space by the D-subdivision method. The characteristic exponent \( \lambda_1 = i\omega \) is substituted into (4.45) and \( D(i\omega; \mu_{cr}) \) is separated to real and imaginary parts:

\[
\text{Re} D(i\omega; \mu_{cr}) = 0, \quad \text{Im} D(i\omega; \mu_{cr}) = 0,
\]

which leads to:

\[
1 - V^2\omega^2 + \frac{6(2\zeta VL\omega^2 + L - 1)}{(3L^2 + 1)\omega^2} - \frac{6(L - 1)}{(3L^2 + 1)\omega^2} \cos \omega - \frac{3(L^2 - 1)}{(3L^2 + 1)\omega} \sin \omega = 0,
\]

\[
2\zeta V\omega + \frac{3(L^2 - 2L + 1)}{(3L^2 + 1)\omega} + \frac{6(L - 1)}{(3L^2 + 1)\omega^2} \sin \omega - \frac{3(L^2 - 1)}{(3L^2 + 1)\omega} \cos \omega = 0.
\]

In the 3-dimensional parameter space \( \mu \in \mathbb{R}^3 \), these formulae describe stability boundaries by means of the dimensionless angular frequency \( \omega \in \mathbb{R}^+ \). Fixing 1 of the 3 parameters, we obtain stability boundary curves in a parameter plane. In particular, we fix the
damping ratio $\zeta$ for different values, and construct stability boundary curves in the plane of the dimensionless towing speed $V$ and the dimensionless caster length $L$ parametrized by the dimensionless angular frequency $\omega$. To decide whether a certain region bounded by the intricate structure of stability curves is stable or not, we use the stability criteria derived in Stépán (1989).

The stability chart of the undamped system for the brush tyre model ($\zeta = 0$) is shown in Figure 4.4, where $\omega \in [0, 8\pi]$ and the stable parameter domains are shaded. The encircled numbers show the number of the unstable characteristic roots. Notice that there exist a stability boundary at $L = 1$, and for large $V$ the system is stable above this boundary and unstable below. For $L \to \infty$, (4.51) simplifies to

$$1 - V^2 \omega^2 - \frac{\sin \omega}{\omega} = 0,$$
$$1 - \frac{\cos \omega}{\omega} = 0,$$

which can be satisfied by $\omega = 2j\pi$ only, where $j = 1, 2, 3, \ldots$. Thus, the asymptotes of

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{stability_chart}
\caption{Stability chart in the $(V, L)$ parameter plane and the vibration frequencies of the Hopf stability boundaries for $\zeta = 0$.}
\end{figure}
the Hopf bifurcation boundaries are characterized by

\[ V_{cr} = \frac{1}{2j\pi} \quad \text{where} \quad j = 1, 2, 3, \ldots , \quad (4.53) \]

which are also marked in Figure 4.4. As it can be observed in the corresponding stability chart, there are large unstable islands around these critical values of the towing speeds even for very long caster lengths.

In case of zero damping, the critical dimensionless caster length of the SN bifurcation is \( L_{SN} = -1/3 \), which is also marked in the chart with double-dot-and-dashed line. Negative value of the dimensionless towing length means pushed tyre, namely, the centre point of the tyre \( O \) is before the king pin \( A \). Below the SN bifurcation boundary \( L < L_{SN} \), i.e., \( l < -a/3 \), the tyre loses its stability with exponentially diverging from the stationary rolling leading to the turn-over of the wheel to the other side of the king pin approaching a towed position. Above the SN bifurcation boundary \( L > L_{SN} \), vibrations may exist around the linearly stable or unstable stationary rolling.

Most of the stability boundaries are of Hopf type providing the intricate structure of the stability chart. The dimensionless self-excited angular frequencies \( \omega \) at the Hopf stability boundaries are also plotted relative to the natural angular frequency \( \omega_n \) (see the upper panel of Figure 4.4). At the parameter points, where the Hopf type stability boundary curves intersect each other, two pairs of pure imaginary characteristic exponents \( \pm i\omega_1 \) and \( \pm i\omega_2 \) co-exist corresponding to two dimensionless angular frequencies \( \omega_1 \) and \( \omega_2 \). This is a co-dimension two (or double) Hopf (HH) bifurcation which typically arises in delayed oscillators like the robot dynamics problem in Stépán and Haller (1995) or the machine tool vibration problem in Stépán (2001). Due to this HH bifurcation, quasi-periodic self-excited oscillations appear around the stationary rolling motion with dimensional frequencies \( f_1 = \omega_1 v/(4a\pi) = \omega_1 V\omega_n/(2\pi) \) and \( f_2 = \omega_2 v/(4a\pi) = \omega_2 V\omega_n/(2\pi) \). The corresponding ‘quasi-periodic travelling waves’ in the contact region possess the dimensional wave lengths \( v/f_1 = 4a\pi/\omega_1 \) and \( v/f_2 = 4a\pi/\omega_2 \).

It can be proved by means of the asymptotes (4.53) of the stability boundaries that the quasi-periodic vibrations exist in the parameter region \( 0 < L < 1 \) and \( V < 1/(2\pi) \), i.e., for short caster length and low towing speed:

\[ 0 < l < a \quad \text{and} \quad v < a \omega_n/\pi , \quad (4.54) \]

respectively.

In case of non-zero damping, the stability chart can also be composed for the brush tyre model in the \((V, L, \zeta)\) parameter space. The Hopf stability boundary surfaces are shown in Figure 4.5. The effect of damping is represented also in Figure 4.6, where the stability charts are plotted in the \((V, L)\) parameter plane for different damping ratios. The effect of damping is clear: the greater the damping is the smaller the unstable domains are. This is well-known in practice, consequently, shimmy dampers are often applied on motorcycles and at airplane nose gears.

If it exists for small enough damping, the most relevant quasi-periodic oscillation is found at the rightmost double Hopf bifurcation point of the stability chart, and it is located in the region \( 0 < L < 1 \) and \( 1/(3\pi) < V < 1/(2\pi) \), i.e.,

\[ 0 < l < a \quad \text{and} \quad 2a \omega_n/(3\pi) < v < a \omega_n/\pi . \quad (4.55) \]
Figure 4.5 Stability boundary surfaces in the \((V, L, \zeta)\) parameter space.

Figure 4.6 Stability charts in the \((V, L)\) parameter plane for different damping ratios \(\zeta\). Stable domains are shaded.

The global effect of damping is also illustrated in Figure 4.7. The critical towing length (4.48) of the SN bifurcation is also marked in the chart by double-dot-and-dashed line. Notice, that the critical towing length \(L_{\text{cr}}\) tends to zero at high speeds in case of
non-zero damping. However, damping can stabilize pushed tyre ($L < 0$) in a certain narrow parameter range (see Figure 4.7 for $\zeta = 0.08$).

### 4.5 Linear stability analysis of the stretched string tyre model

In this section, we investigate the linear stability behaviour of shimmying wheels in case of the *stretched string tyre model* without steering (i.e., $k_t = 0$ and $b_t = 0$). Since this model considers the tyre deformation also outside the contact patch and gives a mixed boundary condition for the PDE (4.20), the derivation of the characteristic function needs some extra effort. Although, the derivation of the characteristic function follows the same process like that of the *brush tyre model*, neither of the models and the stability results can be obtained as a limit case of the other one. This is due to the essential difference in the type of the boundary conditions applied at the contact region.

#### 4.5.1 Linearization of the governing equations

The stationary rolling motion of the wheel is described by the trivial solution

$$
\psi(t) \equiv 0, \quad q(x,t) \equiv 0, \quad x \in (-\infty, +\infty).
$$

(4.56)
Small shimmy oscillations around the stationary rolling can be described by the linearisation of the governing equations (4.20).

The integration in the IDE in (4.20) can be separated into three parts with the help of the exponential decay functions (4.28) of the stretched string tyre model outside the contact region in the following way:

\[
J_A \ddot{\psi}(t) = - \int_{-\infty}^{-a} (l - x) \left( k q(-a, t) + b \left( \dot{q}(-a, t) + \frac{\dot{x}}{\sigma} q(-a, t) \right) \right) e^{\frac{(x+a)}{\sigma}} \, dx
- \int_{-a}^{a} (l - x) \left( k q(x, t) + b \frac{d}{dt} q(x, t) \right) \, dx
- \int_{a}^{\infty} (l - x) \left( k q(a, t) + b \left( \dot{q}(a, t) - \frac{\dot{x}}{\sigma} q(a, t) \right) \right) e^{-\frac{(x-a)}{\sigma}} \, dx.
\]

In (4.57), the travelling wave solution (4.23) of the kinematic constraint (4.5) was calculated. If small oscillations are considered only around the stationary rolling, the linearised form of the travelling wave solution (4.23) can be used:

\[
\begin{aligned}
&x = a - v \tau, \\
&q(x, t) = (l - a + v \tau) \psi(t) - (l - a) \psi(t - \tau) + q(a, t - \tau)
\end{aligned}
\]

for \( x \in [-a, a] \), which satisfies the linearised form of the kinematic constraint (4.5), of course. The time delay can be determined from the first equation explicitly, and gives the same formula as (4.34) for the brush tyre model. The differentiation of \( x \) with respect to the time delay \( \tau \) leads to (4.35), again.

The first and the third integrals of (4.57) can be calculated in closed form with the help of (4.35):

\[
J_A \ddot{\psi}(t) = - \sigma(l + a + \sigma) \left( k q(-a, t) + b \left( \dot{q}(-a, t) - \frac{v}{\sigma} q(-a, t) \right) \right)
- \int_{-a}^{a} (l - x) \left( k q(x, t) + b \frac{d}{dt} q(x, t) \right) \, dx
- \sigma(l - a - \sigma) \left( k q(a, t) + b \left( \dot{q}(a, t) + \frac{v}{\sigma} q(a, t) \right) \right),
\]

in which the rear point lateral deformation \( q(-a, t) \) can be substituted by taking the travelling wave solution (4.58) at \( x = -a \):

\[
q(-a, t) = (l + a) \psi(t) - (l - a) \psi\left(t - \frac{2a}{v}\right) + q\left(a, t - \frac{2a}{v}\right).
\]

After substituting (4.58), (4.60) into the IDE (4.59) and using integration by substitution, we obtain a RFDE with respect to the present and delayed values of the caster angle \( \psi \) and the leading point lateral deformation \( q(a, .) \). This means that the RFDE is coupled with the ODE of the leading point lateral deformation (4.30), which can be linearised for small oscillations. In order to shorten the mathematical expressions, we temporarily use the notation

\[
Y_L(t) = (l - a) \psi(t) - q(a, t)
\]
for the absolute position $Y$ of the leading point $L$. Then, the RFDE-ODE governing equations of the towed elastic wheel is given by

$$
\ddot{\psi}(t) + 2\zeta\omega_n\dot{\psi}(t) + \omega_n^2\psi(t) = \frac{k\nu}{J_A} \int_0^{2a} (l - a + \nu \tau) Y_L(t - \tau) d\tau \\
+ \frac{k\sigma}{J_A} (l - a - \sigma) \left( Y_L(t) + \frac{2\zeta}{\omega_n} \left( \dot{Y}_L(t) + \frac{\nu}{\sigma} Y_L(t) \right) \right) \\
+ \frac{k\sigma}{J_A} (l + a + \sigma) \left( Y_L(t - \frac{2a}{v}) + \frac{2\zeta}{\omega_n} \left( \dot{Y}_L(t - \frac{2a}{v}) - \frac{\nu}{\sigma} Y_L(t - \frac{2a}{v}) \right) \right) \\
+ \frac{b\nu}{J_A} 2l(a + \sigma) \psi(t),
$$

(4.62)

$$
\dot{q}(a, t) = v\psi(t) + (l - a)\dot{\psi}(t) - \frac{v}{\sigma} q(a, t),
$$

where the parameters

$$
\omega_n = \sqrt{\frac{2k}{J_A} a\left(l^2 + a^2/3\right) + \sigma\left(l^2 + a^2 + a\sigma\right)} \quad \text{and} \quad \zeta = \frac{b}{2k}\omega_n
$$

(4.63)

are the undamped natural angular frequency and the damping ratio of the steady wheel (at $v = 0$), respectively.

### 4.5.2 Dimensionless state variables and parameters

Let us use (4.38)–(4.40), and define the dimensionless leading point lateral deformation by

$$
Q(T) := \frac{1}{a} q(a, T).
$$

(4.64)

Furthermore, we use the dimensionless towing speed $V$, the dimensionless caster length $L$ and the dimensionless relaxation length $\Sigma$:

$$
V := \frac{1}{\omega_n} \frac{v}{2a}, \quad L := \frac{l}{a}, \quad \Sigma := \frac{\sigma}{a},
$$

(4.65)

respectively.

By means of the definitions (4.38)–(4.40), (4.64) and (4.65), the RFDE-ODE system (4.62) provides a 3-dimensional system of first-order equations:

$$
\begin{pmatrix}
\psi(T) \\
\Omega(T) \\
Q(T)
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & 0 \\
-\frac{1}{\nu^2} + c_1 c_2 & -\frac{2\zeta}{\omega_n} - c_2 \Sigma (L - 1 - \Sigma) & 0 \\
-c_2 \int (L - 1 - 2\vartheta) (L - 1) & L - 1 & -1 \\
-c_2 \int (L - 1 - 2\vartheta) (L - 1) & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\psi(T) \\
\Omega(T) \\
Q(T)
\end{pmatrix}
$$

(4.66)

$$
\begin{pmatrix}
\psi(T + \vartheta) \\
\Omega(T + \vartheta) \\
Q(T + \vartheta)
\end{pmatrix}
$$

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\psi(T - 1) \\
\Omega(T - 1) \\
Q(T - 1)
\end{pmatrix}.
$$
where the new parameters
\[
c_1 = \frac{\Sigma}{2} (L - 1 - \Sigma)(L - 1) + 2\zeta V(L^2 + (1 + \Sigma)^2),
\]
\[
c_2 = \frac{1}{V^2 \left( \frac{L^2}{L^2 + 1/3 + \Sigma(L^2 + 1 + \Sigma)} \right)}
\]
are introduced.

### 4.5.3 Stability charts

The stationary rolling motion (4.56) is now represented by the trivial solution
\[
\begin{pmatrix}
\dot{\psi}(T) \\
\dot{\Omega}(T) \\
\dot{Q}(T)
\end{pmatrix} \equiv 0
\]
(4.68)
of the linearised equation of motion (4.66). The Laplace transformation of (4.66) or the substitution of the trial solution
\[
\begin{pmatrix}
\psi(T) \\
\Omega(T) \\
Q(T)
\end{pmatrix} = Ke^{\lambda T}, \quad K \in \mathbb{C}^3, \quad \lambda \in \mathbb{C}
\]
(4.69)
leads to the characteristic function
\[
D(\lambda; \mu) = \Sigma V^2 \lambda^3 + 2V(V + \Sigma \zeta)\lambda^2 + (\Sigma + 4\zeta V)\lambda + 2
- \frac{L - 1 - \Sigma}{L^2 + 1/3 + \Sigma(L^2 + 1 + \Sigma)} \times \left\{ \frac{4\zeta VL(\Sigma + 1)(2 + \Sigma \lambda)}{L - 1 - \Sigma} \right. \\
+ \frac{2}{\lambda^2} \left( (L - 1)\lambda + 2 - ((L + 1)\lambda + 2)e^{-\lambda} \right) \\
+ (L - 1 - \Sigma)(2\Sigma \zeta V\lambda + \Sigma + 4\zeta V) \\
+ (L + 1 + \Sigma)(2\Sigma \zeta V\lambda + \Sigma - 4\zeta V)e^{-\lambda} \right\},
\]
(4.70)
where \( \mu \in \mathbb{R}^4 \) represents the dimensionless parameters ordered in a vector
\[
\mu = \text{col} [V \ L \ \Sigma \ \zeta].
\]
(4.71)
The characteristic function at \( \lambda = 0 \) can be calculated by its continuous extension as \( \lambda \to 0 \):
\[
D(0; \mu) = 2 - 2\frac{L^2 + (\Sigma + 1)(4\zeta V - L)}{L^2 + 1/3 + \Sigma(L^2 + \Sigma + 1)} (\Sigma + 1).
\]
(4.72)
The critical dimensionless caster length of the SN bifurcation is given by
\[
L_{SN} = 4\zeta V - \frac{\Sigma^2 + \Sigma + 1/3}{(\Sigma + 1)^2}.
\]
(4.73)
As in case of the brush tyre model, the critical towing length depends on the towing speed if the system is damped.
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Figure 4.8 Stability charts of the undamped system \((\zeta = 0)\) in the \((V, L)\) parameter plane for different relaxation lengths \(\Sigma\). Stable domains are shaded.

Figure 4.9 The critical towing speed \(V_{cr}\) against the relaxation length \(\Sigma\).

The Hopf stability boundaries are determined by the D-subdivision method, and they are 3-dimensional hypersurfaces in the 4-dimensional parameter space \(\mu \in \mathbb{R}^4\) parametrized by the dimensionless angular frequency \(\omega \in \mathbb{R}^+\). Fixing 2 of the 4 parameters, we obtain stability boundary curves in a parameter plane. If we fix the dimensionless relaxation length \(\Sigma\) and the damping ratio \(\zeta\) for different values, we can construct stability boundary curves in the usual \((V, L)\) parameter plane parametrized by \(\omega\).

The effect of the relaxation length \(\Sigma\) is investigated in Figure 4.8, where \(\omega \in [0, 8\pi]\). The relaxation of the tyre amplifies both the stable and unstable domains of the stability chart. For example, the critical stability boundary \(L = 1\) of the brush model is transmitted to the boundary at \(L = 1 + \Sigma\) for any \(\Sigma\): above this line, the stationary rolling is stable for large values of \(V\). The encircled numbers are the numbers of the unstable characteristic roots. For \(L \to \infty\), the characterizing critical towing speeds \(V_{cr}\) of the
asymptotes of the Hopf bifurcation boundaries cannot be determined analytically in this case. Still, it is easy to determine their numerical values as a function of the relaxation length $\Sigma$ as shown in Figure 4.9. This characterizes the asymptote of the most relevant linearly unstable island for large caster lengths, which is also marked by dashed line in Figure 4.8. The asymptote tends to $V_{cr} = 1/\pi \approx 0.318$ when $\Sigma \to \infty$. This means that the relevant unstable island can exist near to and around the parameter domain $L > 1 + \Sigma$ and $1/(2\pi) \leq V \leq 1/\pi$, i.e., for

$$l > a + \sigma \quad \text{and} \quad a\frac{\omega_n}{\pi} \leq v \leq 2a\frac{\omega_n}{\pi}.$$  

(4.74)

The quasi-periodic vibrations can exist in the parameter region $0 < L < 1 + \Sigma$ and $V < 1/\pi$, i.e., for $0 < l < a + \sigma$ and $v < 2a\omega_n/\pi$. The most relevant quasi-periodic oscillation is in the towing speed region $1/(4\pi) < V < 1/\pi$, namely, $a\frac{\omega_n}{(2\pi)} < v < 2a\omega_n/\pi$. The predicted parameter region of this most relevant double Hopf bifurcation point is improved by means of the numerical analysis of the stability boundaries, which estimates this critical parameter point in the region $0.2 < L < 0.5$ and $1/(4\pi) < V < 3/(4\pi)$, which can be written in dimensional form as

$$0.2a < l < 0.5a \quad \text{and} \quad a\frac{\omega_n}{(2\pi)} < v < 3a\frac{\omega_n}{(2\pi)}.$$  

(4.75)

Stability charts are shown in Figure 4.10 for non-zero damping ratio $\zeta$ and tyre relaxation $\Sigma$. Note that the unstable islands disappear for damping ratios smaller than
Figure 4.11 Stability charts in the $(V, L)$ parameter plane for $\Sigma = 2$, for different values of damping $\zeta$ and for high towing speeds. Stable domains are shaded.

Those of the brush model (see Figure 4.6 and 4.10). The (red) crosses in the figure refer to parameter points where numerical simulation results are presented in Section 4.8.

The effect of the damping ratio for large towing speeds is shown Figure 4.11, where the critical towing length (4.73) of the SN bifurcation is also marked by double-dot-and-dashed lines. Qualitative difference can be observed in the critical towing lengths $L_{cr}$ of the two tyre models. In case of the stretched string tyre model, the stationary rolling can lose its stability without vibration for positive towing length at high speeds. The larger the damping is, the smaller the critical towing speed of the SN bifurcation is. This means that damping has a kind of destabilizing effect, too. Consequently, these systems will lose stability for (very) large towing speeds in a surprising static way, which will be examined in details in Section 4.7.

The stationary rolling of the pushed tyre can be stable in a certain narrow speed range if the system is damped. This can be observed in Figure 4.11 at $\zeta = 0.08$ and $\Sigma = 2$. The qualitative behaviour of the pushed tyre is checked in Section 4.8 at the parameter points marked by (red) crosses in the figure.
4.6 Linear stability analysis of the stretched string tyre model with elastic steering mechanism and shimmy damper

In this section, we investigate the linear stability behaviour of the stretched string tyre model extended by an elastic steering mechanism and shimmy damper at the king pin. New dimensionless parameters are introduced and their physical effects are studied.

4.6.1 Linear equations of motion with steering

The stationary rolling motion of the wheel is described by the trivial solution (4.56) and small shimmy oscillations around the stationary rolling can be described by the linearisation of the governing equations (4.20). As before, the integration in the IDE in (4.20) can be separated into three parts with the help of the exponential decay functions (4.28) of the stretched string tyre model. In Section 4.2, the travelling wave solution (4.23) of the kinematic constraint (4.5) was calculated. For small oscillations, the linearised form of the travelling wave solution (4.23) can be used again (see (4.58)) and the formula (4.34) is valid for the time delay, too. Also, the IDE-ODE system (4.62) remains the same except the meaning of the steady (at \( v = 0 \)) modal parameters like the natural angular frequency and the damping ratio:

\[
\omega_n = \sqrt{\frac{2k}{J_A}}\left(a(l^2 + a^2/3) + \sigma(l^2 + a^2 + a\sigma)\right) + \frac{k_t}{J_A},
\]

and

\[
\zeta = \frac{1}{2\omega_n}\left(\frac{2b}{J_A}\left(a(l^2 + a^2/3) + \sigma(l^2 + a^2 + a\sigma)\right) + \frac{b_t}{J_A}\right).
\]

4.6.2 Dimensionless steering parameters

Apart of the already introduced dimensionless parameters \( (V, L, \Sigma, \zeta) \), we also define the dimensionless parameters of the steering mechanism and the shimmy damper, namely, the dimensionless torsional stiffness \( K \) and the dimensionless torsional damping \( B \):

\[
V := \frac{1}{\omega_n} \frac{v}{2a}, \quad L := \frac{l}{a}, \quad \Sigma := \frac{\sigma}{a}, \quad K := \frac{k_t}{2a^3k}, \quad B := \frac{b_t}{2a^3b},
\]

respectively.

The usual Cauchy transformation of the steered system leads to a 3-dimensional system of first-order equations

\[
\begin{pmatrix}
\dot{\psi}(T) \\
\dot{\Omega}(T) \\
\dot{Q}(T)
\end{pmatrix} = \begin{pmatrix}
0 & 1 & 0 \\
-\frac{1}{2} \zeta + c_1 & -\frac{3}{2} \zeta & -c_2 \Sigma (L - 1 - \Sigma) \\
L - 1 & -\frac{2}{L} & 0
\end{pmatrix} \begin{pmatrix}
\psi(T) \\
\Omega(T) \\
Q(T)
\end{pmatrix} + c_2 \int_{-1}^{0} \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
\psi(T + \vartheta) \\
\Omega(T + \vartheta) \\
Q(T + \vartheta)
\end{pmatrix} d\vartheta
+ \frac{1}{2} (L + 1 + \Sigma) \begin{pmatrix}
0 & 0 & 0 \\
(L - 1)(c_2 \Sigma - 2c_3) - 2c_3 \Sigma & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
\psi(T - 1) \\
\Omega(T - 1) \\
Q(T - 1)
\end{pmatrix},
\]
with the new and extended parameters

\[ c_1 = c_2 \frac{\Sigma}{2} (L - 1 - \Sigma)(L - 1) + c_3 \left( L^2 + (1 + \Sigma)^2 \right) , \]
\[ c_2 = \frac{1}{V^2} \frac{1}{L^2 + 1/3 + \Sigma(L^2 + 1 + \Sigma) + K} , \]
\[ c_3 = \frac{2\zeta}{V} \frac{1}{L^2 + 1/3 + \Sigma(L^2 + 1 + \Sigma) + B} . \]  

(4.80)

4.6.3 The effect of steering parameters on stability

In this case, the characteristic function of the system is given by

\[ D(\lambda; \mu) = V^2 \Sigma \lambda^3 + (2V^2 + 2\zeta V \Sigma) \lambda^2 + (\Sigma + 4V) \lambda - \frac{L - 1 - \Sigma}{L^2 + 1/3 + \Sigma(L^2 + 1 + \Sigma) + K} \times \]
\[ \times \left\{ \Sigma \left( e^{-\lambda} - 1 + L(e^{-\lambda} + 1) \right) + \frac{2}{\lambda} \left( \left( L - 1 + \frac{2}{\lambda} \right) - \left( L + 1 + \frac{2}{\lambda} \right) e^{-\lambda} \right) \right\} \]
\[ + 2 - \frac{4\zeta V \lambda \Sigma}{L^2 + 1/3 + \Sigma(L^2 + 1 + \Sigma) + B} - \frac{2\zeta V(L - 1 - \Sigma)}{L^2 + 1/3 + \Sigma(L^2 + 1 + \Sigma) + B} \times \]
\[ \times \left\{ \Sigma \lambda \left( e^{-\lambda} - 1 + L(e^{-\lambda} + 1) \right) + \Sigma(e^{-\lambda} - 1) \right\} - 2 \left( e^{-\lambda} + 1 + L(e^{-\lambda} - 1) + \Sigma(e^{-\lambda} + 1) \right) \right\} , \]  

where \( \mu \in \mathbb{R}^6 \) represents the dimensionless parameters arranged in the vector

\[ \mu = \text{col} [V \ L \ \Sigma \ \zeta \ \ K \ B] . \]  

(4.82)

The value of the characteristic function at \( \lambda = 0 \) can be calculated by the usual continuous extension \( \lambda \to 0 \):

\[ D(0; \mu) = 2 - \frac{2L(L - 1 - \Sigma)(\Sigma + 1)}{L^2 + 1/3 + \Sigma(L^2 + 1 + \Sigma) + K} - \frac{8\zeta V(\Sigma + 1)^2}{L^2 + 1/3 + \Sigma(L^2 + 1 + \Sigma) + B} . \]  

(4.83)

The critical dimensionless caster length of the SN bifurcation can hardly be calculated in this case, but the critical dimensionless towing speed is still expressed in closed form:

\[ V_{SN} = \frac{(L^2 + 1/3 + \Sigma(L^2 + \Sigma + 1) + B)(L(\Sigma + 1)^2 + \Sigma(\Sigma + 1) + K + 1/3)}{4\zeta(\Sigma + 1)^2(L^2 + 1/3 + \Sigma(L^2 + \Sigma + 1) + K)} . \]  

(4.84)

The effect of the elastic steering mechanism characterized by the dimensionless torsional stiffness \( K \) is investigated in Figure 4.12, where \( \omega \in [0, 8\pi] \). The torsional spring at the king pin modifies the most relevant stability boundary in an essential way. Even for a small value of the torsional stiffness, the intersections of the most relevant Hopf stability boundaries disappear, i.e., the co-dimension two Hopf bifurcations do not exist for these cases. Moreover, the stability of the towed wheel with zero caster length (\( L = 0 \)) turns to be stable in a certain speed range. The larger the dimensionless torsional stiffness is, the wider this speed range is. Clearly, the spring can easily stabilize pushed wheels, too.

Now, let us examine the effect of the shimmy damper. The dimensionless torsional damping \( B \) gives information about the ratio between the tyre damping and the torsional
damping, see (4.78). With the help of this parameter, we can investigate which parameter has more relevant effect on the stability of stationary rolling: the tyre damping or the shimmy damper. As it can be seen in Figure 4.13, large stable parameter regions are lost if most of the damping is concentrated in the shimmy damper and not in the tyre. This is expressed by the stability charts presented for increasing $B$ while the damping ratio $\zeta$ is kept constant. In other words, if the designer has the option to choose to share the overall system damping between the tyre and the shimmy damper, the tyre should be the preferred solution from stability viewpoint. On the other hand, we must realize that the additional shimmy damper improves the stability properties if it appears additionally to the damping of the tyre, i.e., when two dimensionless damping parameters $B$ and $\zeta$ are increased together via $b_t$.

The structure of the stability charts is similar to the results presented in Pacejka (2002); Somieski (1997), where one rigid-body degree-of-freedom mechanical models are investigated with creep-force based tyre models. Such stability boundaries were already experimentally detected in Schrode (1957) without comparison to any theoretical results. While our model presents the stability properties in the standard parameter domains already explored in the literature, it also presents many uncharted instabilities, predicts quasi-periodic oscillatory behaviour and provides reliable tyre dynamics description in the low speed parameter regions.
4.7 Saddle-node bifurcation caused by tyre damping

The loss of stability of a pushed tyre through SN bifurcation is a well known phenomenon in vehicle dynamics, see for example, Besselink (2000); Pacejka (2002). In these studies, the damping of the tyre was neglected, therefore, the stability boundary of the SN bifurcation was detected at towing speeds, which were independent of caster length. This is represented by a horizontal stability boundary line in the \((V, L)\) parameter plane as shown in Figure 4.4. In the preceding Section, however, it was shown that the distributed damping of the tyre leads to a non-trivial relationship between the critical values of the caster length and the towing speed.

In case of the brush tyre model (see Section 4.4) without tyre damping \((b = 0)\), the
critical caster length was identified at \( L_{cr} = -1/3 \), i.e., in dimensional form \( l_{cr} = -a/3 \). This value was also identified by Pacejka (2002). If the distributed tyre damping is taken into account \( (\delta > 0) \), this simple formula turns to (4.48), which depends also on the dimensionless towing speed \( V \). Still, only pushed tyre, i.e., a system with negative caster length, can lose its stability through SN bifurcation since the critical caster length tends to zero from below as the towing speed tends to infinity (see Figure 4.7).

If the stretched string tyre model with tyre damping is considered, however, the critical caster length, where the SN bifurcation of the straight stationary rolling can occur at positive caster lengths, too, if the towing speed is high enough (see formula (4.73) and Figure 4.13). Since it is not trivial physically how the tyre damping and the tyre relaxation together can lead to stationary rolling with non-zero caster angle, further investigation of the SN bifurcation is presented in this section.

The stationary solution with non-zero caster angle and non-trivial lateral tyre deformation is given by

\[
\psi(t) \equiv \psi_s, \quad q(x,t) \equiv q_s(x), \quad x \in [-a,a].
\]  

(4.85)

Since the caster angle is constant and the tyre points in the contact region are fixed to the ground during rolling, the lateral deformation of the tyre is given by a straight line between the leading point L and the rear point R. The deformation does not change in time in the caster fixed \((x,y,z)\) coordinate system, i.e., \( \dot{q}_s(x) = 0 \) while the total derivative with respect to time is non-zero due to the backward translation of the tyre elements, i.e., \( \frac{d}{dt} q_s(x) = v \sin \psi_s \) as it follows from the formulae (4.3) and (4.4)). Using the trial solution \( q_s(x) = \alpha x + \gamma \) in the PDE of (4.20) with the BC (4.29), we can obtain the stationary deformation in the contact patch in the form

\[
q_s(x) = (\sigma + a - x) \tan \psi_s
\]  

(4.86)

for \( x \in [-a,a] \). This tyre deformation is illustrated in Figure 4.14, where the deformation outside the contact patch is approximated by the exponential decaying functions (4.28).

After the substitution of the stationary solution (4.85) and (4.86) into the IDE of
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(4.20), the equation of motion reduces to

\[ k_t \psi_s = -k \int_{-\infty}^{-a} (l - x) q_s(-a) e^{\frac{(x+a)}{\sigma}} \, dx - k \int_{-a}^{a} (l - x) q_s(x) \, dx - b \int_{-\infty}^{-a} (l - x) \frac{\dot{x}}{\sigma} q_s(-a) e^{\frac{(x+a)}{\sigma}} \, dx - b \int_{-a}^{a} (l - x) v \sin \psi_s \, dx - b \int_{-\infty}^{\infty} (l - x) \frac{\dot{x}}{\sigma} q_s(a) e^{\frac{(x-a)}{\sigma}} \, dx. \]

(4.87)

Using \( \dot{x} = -v \cos \psi_s \) from (4.4), the integrals can be calculated in closed form. With respect to the vertical axis at the king pin A, the resultant torque of the tyre damping forces before the leading edge \((x \in (a, \infty))\) is positive for short casters only \((l < (\sigma + a))\), but the same resultant torque of the damping forces behind the rear point \((x \in (-\infty, -a))\) is always positive trying to increase the caster angle. Thus, the damping forces outside the contact patch can ‘balance’ the negative (restoring) torque of the rest of tyre forces even for non-zero elastic tyre deformation. This effect is shown in Figure 4.14, where the travelling of the damping elements along the coordinate \(x\) is also illustrated.

The algebraic manipulation of (4.87) leads to a transcendent equation

\[ k_t \psi_s - 2bv(\sigma + a)^2 \sin \psi_s + \left( l(\sigma + a)^2 + \sigma a(\sigma + a) + \frac{a^3}{3} \right) \tan \psi_s = 0, \]

(4.88)

which characterizes the possible stationary solutions. Note that the term, in which the damping factor \(b\) appears, has negative sign, and its value is independent from the caster length \(l\).

Considering \(k_t = 0\), i.e., no steering mechanism is applied, the non-trivial stationary solution is given by

\[ \cos \psi_s = \frac{k}{bv} \left( l + \frac{\sigma a}{\sigma + a} + \frac{a^3}{3(\sigma + a)^2} \right). \]

(4.89)

The non-zero stationary caster angle exists if and only if the absolute value of the right hand side of this formula is less than one. This condition leads to the dimensional form of the critical caster length already determined in (4.73), of course.

The above calculated result may have importance in some towing systems but it has to be noticed that the existence of the non-zero stationary caster angle for positive caster length can be originated in the exponential decay functions (4.28), which approximates the real deformation of the tyre outside contact patch. As it was stated in Section 4.3.2, these decay functions (4.28) are valid for an undamped massless stretched string and their application with distributed damping may be a rough approximation of the real system. Nevertheless, Smiley (1957b) also used these approximating functions with distributed...
damping in order to determine the hysteresis forces and moments of the tyre, but he did not investigate the SN bifurcation of the towed tyre with this model and assumptions. The possible existence of SN bifurcation for positive caster length should be investigated further experimentally. This would either prove the existence of SN bifurcation or would help to improve our particular mechanical models. Experimental results in Schrode (1957) does not confirm the existence of SN bifurcation, in spite of the fact that the stability properties of towed aircraft tyres were investigated in wide speed ranges, but with limited caster deflection angles ($\pm 4^\circ$), only.

The possible directions of the development of mechanical modelling could be, for example, the consideration of distributed damping and mass in the equation of the stretched string outside the contact region. These will lead to stationary relaxation shapes slightly different from exponential. Such models, however, can hardly be investigated by analytical methods (see Wickert and Mote Jr., 1990).

## 4.8 Simulations

An alternative way to investigate the dynamics of the *stretched string tyre model* is the numerical simulation of the IDE-PDE system (4.20) with the boundary condition (4.29). With the help of the analytically determined stability diagrams, we can verify the applied numerical methods. On the other hand, numerical simulation will provide information also about the appearing nonlinear oscillations, which is of primary interest when the stationary rolling motion is linearly unstable.

The applied numerical method is essentially a third order Runge-Kutta method in time, while the space discretization with respect to $x$ is tuned to the time steps by implementing the Courant-Friedrichs-Lewy stability condition (see Lax and Wendroff, 1960) usually used for the simulation of travelling waves. The number of spatial mesh points is 100. The initial conditions for the numerical simulations are set to zero initial deformation $q(x, 0) = 0$ and zero caster angle $\psi(0) = 0$, but non-zero angular velocity $\dot{\psi}(0) = -2$ [rad/s] is applied. This is an acceptable model of an impact-like perturbation of stationary rolling.

The simulations are carried out at the parameter points A–H and I–K marked by red crosses in Figure 4.10 and Figure 4.11, respectively. We fix the realistic dimensional parameter values in Table 4.1, and vary $v$, $l$, $b$, $J_A$ and $\sigma$ according to the values of $V$, $L$, $\zeta$ and $\Sigma$ at the investigated parameter point, see Table 4.2.

In case of each pairs of the parameter points A–B and C–D, the points are separated by a Hopf stability boundary in a way that one point lies in the stable (shaded) regime.

<table>
<thead>
<tr>
<th>$a$ [m]</th>
<th>0.0395</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$ [N/m$^2$]</td>
<td>62600</td>
</tr>
<tr>
<td>$k_t$ [Nm/rad]</td>
<td>0</td>
</tr>
<tr>
<td>$b_t$ [Nms/rad]</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_n$ [rad/s]</td>
<td>$4\pi$</td>
</tr>
</tbody>
</table>
### Table 4.2 The values of the parameters at the parameter points.

<table>
<thead>
<tr>
<th>Point</th>
<th>( v ) [m/s]</th>
<th>( l ) [m]</th>
<th>( b ) [Ns/m²]</th>
<th>( J_A ) [kgm²]</th>
<th>( \sigma ) [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.2978</td>
<td>0.1975</td>
<td>99.63</td>
<td>3.9742</td>
<td>0.079</td>
</tr>
<tr>
<td>B</td>
<td>0.3971</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>0.4467</td>
<td>0.0988</td>
<td>99.63</td>
<td>1.2256</td>
<td>0.079</td>
</tr>
<tr>
<td>D</td>
<td>0.5460</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>0.1489</td>
<td>0.0237</td>
<td>99.63</td>
<td>0.3622</td>
<td>0.079</td>
</tr>
<tr>
<td>F</td>
<td>0.1985</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>G</td>
<td>0.1489</td>
<td>0.0099</td>
<td>99.63</td>
<td>0.3186</td>
<td>0.079</td>
</tr>
<tr>
<td>H</td>
<td>0.1985</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>0.9927</td>
<td>-0.0040</td>
<td>797.05</td>
<td>0.3109</td>
<td>0.079</td>
</tr>
<tr>
<td>J</td>
<td>1.7373</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>K</td>
<td>2.4819</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

while the other is in the unstable (white) regime. Consequently, qualitatively different behaviours are expected for each pair of parameter points. The points E–H help to explore the behaviour of the system close to one of the double Hopf bifurcation points. Each of the four disjunct domains separated by two Hopf stability boundaries are represented by one of the four parameter points. The points I–J–K characterize the case of a pushed tyre at dynamically unstable, stable and statically unstable stationary rolling, respectively.

Figure 4.15 shows the simulation results for the parameter points A–H. In each of the panels, the time history of the caster angle \( \psi(t) \) is shown together with the spatial distribution of the lateral deformation \( q^*(x) = q(x, t^*) \) at the optionally chosen time instants \( t^* \) denoted at the corresponding left panels. The location of the king pin A is also marked in the figures in order to illustrate the structure of the towed tyre having a contact centre at zero. Case A belongs to the upper linearly unstable island of the stability chart. Due to the geometric non-linearities in the system, the motion tends to a stable periodic solution having an amplitude less than \( \pi/4 \). The deformation of the tyre at the detected periodic solution is quite large at \( t^* \), it is almost 100 [mm]. In practice, large amplitude stable oscillations like this are expected to involve sliding in the contact region, which is a physical non-linearity we have not modelled yet. It is worth to mention that such stable periodic motion is not detected by simulation of the brush tyre model, i.e., the solutions seem to grow in time without limitation. Case B shows a linearly stable stationary rolling motion, that is, small perturbations around this motion decay in time.

The cases C and D in Figure 4.15 confirm the most relevant stability boundary of the towed tyre. The shapes of the contact region deformations are almost straight lines at these parameter points, which explains why this stability boundary can be determined by shimmy models in which the deformation is approximated by straight line between the leading point L and rear point R (see Pacejka, 2002; Schlippe and Dietrich, 1941). Note, that the small amplitude oscillations are very close to harmonic for the cases A–D.

The simulations E–H show interesting time histories and tyre deformations. Case E demonstrates how the stationary rolling loses its stability in one of the lower unstable islands. Due to the intricate shape of the contact region deformation with an almost
vertical spatial derivative (tending to infinity), the numerical simulation does not converge after the time instant denoted by (red) dashed line in the corresponding panel.

Case F represents a stable stationary rolling close to the theoretical double Hopf bifurcation point. In the time history, both of the self-excited vibration frequencies of
the calculated stability boundaries appear and the oscillation is not periodic any more, but it becomes quasi-periodic. The parameter point G is chosen from the domain, where the number of the unstable characteristic roots is two. This can be observed in the time history as well. First, both the oscillations of the smaller and the larger frequencies are growing in time, then a stable non-harmonic periodic solution develops in which the smaller frequency appears as basic frequency. At the parameter point H, a very similar large amplitude periodic solution can be observed. The small amplitude oscillation contains also two frequencies, but the larger one decays in time as it was predicted by the linear stability chart. The large amplitude oscillations/waves in the cases G and H are not realistic physically due to the large caster angles reaching $\pi/2$. For the caster angles over $\pi/2$, the leading point L and the rear point R of the contact patch should be exchanged, which is not considered in our model. Consequently, the solutions for such large caster angles are not valid physically.

Simulation results are shown for the parameter points I–K in Figure 4.16. Case I is chosen from the linearly unstable domain, where one pair of unstable complex characteristic roots exists with non-zero imaginary part. Thus, the pushed tyre is oscillating around the unstable stationary rolling. The amplitude of the detected stable periodic motion is even smaller than $\pi/2$. Case J confirms the existence of the linearly stable domain for negative caster length, since the oscillation decays in time. Finally, case K represents how the tyre loses its stability without vibrations around the straight stationary rolling. In this case, the caster angle grows exponentially until the stationary solution with non-zero caster angle and deformation is reached, which was investigated in Section 4.7 analytically. The numerically determined stationary caster angle perfectly matches with the value of the analytical formula (4.89), which is $\psi_s = 0.718$ [rad]. The deformation of the tyre is a straight line as expected analytically and it is so large that the tyre will surely start sliding in practice. Still, this simulation result verifies the location of the SN bifurcation boundary shown in Figure 4.11.
4.9 Partial sliding as nonlinearity

The simulations in Section 4.8 often show unrealistically large tyre deformations for stable periodic oscillations, which obviously leads to the sliding of certain tyre points in practice. This was also experienced in our measurements (see Chapter 5), when the experimentally detected signals and the simulations with geometric non-linearities only showed discrepancies in most of the parameter domains.

For the theoretical explanation of the experimentally detected nonlinear vibrations, a reliable nonlinear mathematical model was needed that includes additional physical non-linearities. Since the experimental results convinced us that the sliding of the contact patch has a key role in the large amplitude oscillations, we decided to use the equations of motion (4.20), together with checking the lateral forces in the contact region whether they stay within the condition of sticking, that is, whether they are smaller than the corresponding normal forces multiplied by the static coefficient of friction $\mu_s$. When this condition does not hold, the lateral contact forces decrease to the value of sliding friction forces, that is, to normal forces multiplied by the dynamic coefficient of friction $\mu_d$.

Clearly, this means the implementation of the so-called extended Coulomb friction model, which is the simplest approximation of frictional contact forces. In the meantime, we do not consider the longitudinal deformation of the tyre, and we neglect the corresponding longitudinal distributed forces, too, which would be essential if we investigated the dynamics of braking or accelerating. Even so, this simplified model can be solved only numerically.

In order to simulate shimmy with partial sliding in the contact region, we approximate the normal force distribution in the contact patch by a parabolic function, which was developed by Pacejka during a simplified derivation of the creep–force idea (see Pacejka, 1966, 2002). The critical lateral forces and the sliding friction forces are described by

\[
p_{\text{cr}}(x) = \frac{3}{4} \mu_s \frac{F_z}{a} \left(1 - \frac{x^2}{a^2}\right) \quad \text{and} \quad p_{\text{d}}(x) = \frac{3}{4} \mu_d \frac{F_z}{a} \left(1 - \frac{x^2}{a^2}\right),
\]

respectively, where $F_z$ is the overall normal force between the tyre and the ground.

To decide whether the tyre starts partially sliding or not, we should calculate the required lateral force distribution in the contact patch, which would be needed to hold the actual deformed shape of the stretched string. In order to do this, we use the differential equation (4.27) of the stretched string extended with the damping forces:

\[
\sigma^2 q''(x, t) - q(x, t) - \frac{b}{k} \frac{d}{dt} q(x, t) = -\frac{p(x, t)}{k},
\]

where $\frac{d}{dt}$ is given by (4.3). Thus, the required lateral force distribution is

\[
p_{\text{req}}(x, t) = k \left(q(x, t) - \sigma^2 q''(x, t) \right) + b \left(\dot{q}(x, t) + q'(x, t) \dot{x}\right),
\]

where $\dot{x} = -v \cos \psi(t) + q(x, t) \dot{\psi}(t)$ in accordance with (4.4), since we neglect the longitudinal deformations of the tyre.

If the required lateral force of a tyre point at position $x$ reaches the critical lateral force, i.e., if $p_{\text{req}}(x, t) > p_{\text{cr}}(x)$, the corresponding tyre element starts sliding, the kinematic
constraint of rolling is relaxed, and (4.5) is not fulfilled any more. The lateral velocity of this sliding tyre point P relative to the ground is

$$v_P = -v \sin \psi(t) - (l - x) \dot{\psi}(t) + \dot{q}(x, t) + q'(x, t) \dot{x}.$$  \hspace{1cm} (4.93)

The lateral displacement of these sliding tyre points, i.e., the lateral deformation $q(x, t)$ of the tyre is described by the differential equation of the damped stretched string:

$$\ddot{q}(x, t) = \frac{k}{b} \left( \sigma^2 q''(x, t) - q(x, t) \right) + \frac{p_{\text{slide}}(x, t)}{b} - q'(x, t) \dot{x},$$  \hspace{1cm} (4.94)

where

$$p_{\text{slide}}(x, t) = \begin{cases} 
\text{sign}(p_{\text{req}}(x, t)) \cdot p_d(x), & \text{if } v_P = 0, \\
-\text{sign}(v_P) \cdot p_d(x), & \text{if } v_P \neq 0.
\end{cases}$$  \hspace{1cm} (4.95)

For the rest of the sticking tyre points that satisfy the conditions: $p_{\text{req}}(x, t) \leq p_{\text{cr}}(x)$ and $v_P(x, t) = 0$, the PDE in (4.20) is still valid.

The boundary conditions of (4.94) depend on the configuration of the sliding zone within the contact patch. For example, if sliding appears inside the contact patch, the continuity of the deformation function can be used as boundary condition at the two ends of the sliding zone. If the sliding zone reaches the rear and/or the leading point of the contact patch, the boundary conditions at the ends of the contact zone lead to $q'(-a, t) = q(-a, t)/\sigma$ and/or $q'(a, t) = -q(a, t)/\sigma$, respectively. This is due to fact that no discontinuity can occur in the lateral force at R and L, namely, the lateral forces are zeros at these points. This is somewhat related to the so-called ‘no kink at L’ condition used by Pacejka (2002).

In the simulations we use the same numerical method as before. The number of spatial mesh points is 100, and the space discretization is tuned to the time steps by taking into account the Courant-Friedrichs-Lewy stability condition (Lax and Wendroff, 1960), again. Some test examples showed that in the relevant towing speed range, an adequate simulation time step is much smaller when sliding is taken into account than it is in case of pure rolling. Moreover, the dynamics of sliding in the contact patch is one range faster than the dynamics of the rigid-body system.

Thus, we applied the following approximation in the simulation code. We simulate the original equations (4.20) with the same time step as for rolling (‘slow’ dynamics), and we check the lateral forces in each time step. If sliding would occur in the contact patch, we determine the modified deformations by simulating the PDE (4.94) (‘fast’ dynamics) meantime the simulation time of the main dynamics is stopped, and the caster angle and the angular velocity are kept constant. This means that the deformation function seems to jump infinitely fast to the shape that is given by the simulation of the PDE (4.94) after the sliding is finished, i.e., when $v_P(x, t) = 0$ and $p_{\text{req}}(x, t) \leq p_{\text{cr}}(x)$ for all of the tyre points. Of course, $v_P(x, t)$ never reaches exactly the zero value, therefore, we use the criterion $|v_P(x, t)| < v_{\text{err}}$, where $v_{\text{err}}$ is a properly chosen small positive number.

Simulations will be run at different parameter points in Section 5.3.2, where the experimentally detected and simulated time signals are to be compared. In this section, we show only one example to illustrate the nonlinear effect of sliding. Figure 4.17 shows the simulation result at the parameter point A (see Figure 4.10 and Table 4.2). The
Case A ($\psi$):

<table>
<thead>
<tr>
<th>$\psi$ [rad]</th>
<th>$q$ [mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\pi/8$</td>
<td>4</td>
</tr>
<tr>
<td>$-\pi/8$</td>
<td>-118.5</td>
</tr>
</tbody>
</table>

$t$ [s]:

<table>
<thead>
<tr>
<th>0</th>
<th>8</th>
<th>16</th>
<th>24</th>
<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q$ [mm]</td>
<td>0</td>
<td>-39.5</td>
<td>0</td>
<td>39.5</td>
</tr>
</tbody>
</table>

Figure 4.17 Simulation result at parameter point A in Figure 4.10 if sliding is considered.

Numerical values of the parameters, which are also necessary to simulate sliding, are $F_z = 200 \, [N]$, $\mu_s = 1.0 \, [-]$ and $\mu_d = 0.7 \, [-]$.

Let us compare the simulation results of Figure 4.15 and Figure 4.17. Without considering the sliding effect, a large amplitude periodic motion was detected by the simulation with amplitude close to $\pi/4$. If sliding is considered, the amplitude of the stable periodic motion is much smaller, namely, it is less than $0.016 \, [\text{rad}] = 0.92 \, [\text{o}]$. The lateral deformations within the tyre contact patch show the same reduction of oscillations (see Figure 4.17). This is due to the fact that the kink at the rear point R can not appear since the lateral force distribution should always be zero at R, and the rear part of the contact line starts sliding if this condition is not fulfilled. Practically, the rear end of the tyre contact region always slides.

4.10 New results

**Thesis 3** A low degree-of-freedom mechanical model of a towed wheel of elastic tyre was constructed where the tyre was modelled as a stretched string of elastic support. The corresponding equations of motion were transformed into a delay-differential equation with the help of a travelling wave trial solution for the tyre-ground contact region. Linear stability charts were constructed in the plane of the towing speed and caster length parameters.

It was shown that the unstable parameter domains of the towed tyre can be reduced by increased damping of the tyre, while the increase of the tyre relaxation length magnify the unstable parameter regions.

For large towing speeds, an essential stability limit was identified for the critical caster length

$$l_{ct} = a + \sigma,$$

where $a$ is the half length of the contact patch and $\sigma$ is the relaxation length of the tyre.

The existence of quasi-periodic oscillations were proven, the most relevant one was found at the rightmost double Hopf bifurcation point of the stability chart. If this quasi-periodic oscillation exists, it is in the parameter region:

$$0.2a < l < 0.5a \quad \text{and} \quad af_n < v < 3af_n,$$

where $f_n$ is the natural frequency of the steady caster-wheel system in Hertz.

A relevant unstable island can exist for large caster length ($l > l_{ct}$) around the towing speed domain

$$2af_n \leq v \leq 4af_n.$$

These results are published in Takács et al. (2009); Takács and Stépán (2009).
Thesis 4 The additional effects of elastic steering mechanism and shimmy damper were investigated. It was shown that the application of a torsional spring at the king pin can eliminate the double Hopf bifurcations and quasi-periodic self-excited oscillations can not occur. It was also shown that a fixed amount of viscous damping has a better effect on stability at the tyre than it has at the king pin.

It was proven for the model in question that even a pushed wheel of elastic tyre can exhibit linearly stable stationary rolling in a wide speed range if appropriate torsional stiffness and damping are installed at the king pin. It was also shown that the distributed damping of the tyre leads to towing speed dependent static (saddle-node) stability boundary even for towed wheels.

These results are published in Takács and Stépán (2010).
Chapter 5

Experiments on elastic tyre

The results in Chapter 3 and 4 promise good agreement with some practical observations on towed wheels. Since the majority of road vehicles use pneumatic tyres, the analysis of these rolling wheels of elastic tyre is an essential task. To validate the theoretical results of Chapter 4, experiments on the dynamic elastic tyre model were carried out and these experiments are described in details in this chapter. First, the design and the construction of an experimental rig are presented together with the identification of the elastic tyre parameters. Then, the theoretical results of Chapter 4 are confirmed and validated by the construction of experimental stability charts and the spectral analysis of the arising nonlinear vibrations. Finally, the dissipation effects are studied for unstable parameter islands by power consumption and surface temperature measurements together with the analysis of thermal camera records.

5.1 Experimental rig

The wheel used in the experiments is originally part of a small bicycle, and three different types of tyres are installed as shown in Figure 5.1. The wheel is towed by a special suspension system fixed above a conveyor belt. The designed suspension system is just the same as the one used in the mechanical model in Chapter 4.

The complied experimental rig is shown in Figure 5.2. The wheel is fit in a rigid fork, its longitudinal position can be fixed at adjustable positions on the rigid caster.

Figure 5.1 Different types of tyres used in the experiments.
Figure 5.2 The experimental rig. In panel (a), the whole experimental set-up is shown: the conveyor belt with the suspension system and the vibration measurement system PULSE. Panel (b) shows how the length of the contact patch, the caster length and the mass moment of inertia of the system can be adjusted. Panel (c) shows the measurement system PULSE with the modal hammer. In panel (d) and (e), the piezo-electric accelerometer and the photoelectric tachometer are shown, respectively.

to set different caster lengths \( l \) in the wide range of \(-0.020 \, [\text{m}] \leq l \leq 0.360 \, [\text{m}]\) (see Figure 5.2(b)). Actually, the experimental rig is also able to investigate the tyre dynamics even at small negative caster lengths. The wheel centre plane is held normal to the conveyor belt by a properly designed rigid king pin, which is supported by relatively large ball bearings. The length of the contact patch \( 2a \) can be tuned by the vertical position of the wheel, which is set by the variation of the vertical distance between the conveyor belt and the king pin. This is served by the adjustable spindle and fixing screws, as it is shown in Figure 5.2(b). The mass moment of inertia of the whole towed structure \( J_A \) with respect to the vertical axis at the king pin A can be increased with additional masses attached at a variable position of the rear part of the caster. This way, the natural angular frequency \( \omega_n \) of the steady wheel can be tuned, too.

The conveyor belt is made of rubber, and its controlled running speed can be varied
incrementally with speed steps $0.1 \, [\text{km/h}]$ in the range of $0.5 \, [\text{km/h}] \leq v \leq 18.0 \, [\text{km/h}]$. The surface of the endless belt is quite homogeneous, the continuity error at the glued surfaces is negligible, too. In order to avoid the combination of belt and tyre oscillations, the belt is stiffened laterally by a steel frame and the possible lateral buckling of the belt is also under control (see Figure 5.2(a)).

The experimental rig is connected to the dynamic measuring system PULSE (see Figure 5.2(c)). The tangential acceleration of the caster end point and the speed of the conveyor belt are measured by a piezo-electric accelerometer and a photoelectric tachometer, respectively (see Figure 5.2(d) and (e)). A modal hammer is also connected to the measuring system in order to identify the natural angular frequency $\omega_n$ and the damping ratio $\zeta$ of the steady towed wheel via its frequency response function.

The system parameters are identified in different experimental set-ups, which are presented in the subsequent section.

## 5.2 Measuring of the system parameters

The elastic tyre model is determined by the dimensional parameters $l$, $2a$, $v$, $\sigma$, $k$, $b$ and $J_A$. On one hand, the caster length $l$, the towing speed $v$ and the length of the contact patch $2a$ can be adjusted in relatively wide ranges in the experimental rig. On the other hand, the tyre parameters of the stretched string model, like the tyre relaxation length $\sigma$, the distributed lateral stiffness $k$ and the distributed lateral damping $b$ are originated in the tyre characteristics and can be tuned in relatively small ranges via the variation of tyre inflation pressure only. These parameters depend also on the vertical tyre deformation, which has a direct connection to the length of the contact patch $2a$. The mass moment of inertia $J_A$ is determined by the structure of the experimental rig, and it changes by the variation of the caster length $l$. It can also be increased by additional masses attached to the caster as explained above (see Figure 5.2(b)).

In this section, first, the measurement of the mass moment of inertia $J_A$ and its dependence on the caster length are described, then the tyre parameters are determined for different experimental set-ups.

### 5.2.1 Measuring mass moment of inertia

The mass moments of inertia of the caster and the wheel were measured with the help of a simple pendulum experiment. The caster with the wheel was hanged in the vertical plane with different caster lengths, and the time periods of the pendulations were measured in the domain of linear ('small') oscillations. In order to calculate the mass moment of inertia of the above constructed pendulums, the masses of the components had to be measured: $m_w = 1.305 \, [\text{kg}]$ and $m_c = m_{\text{bar}} + m_{\text{fork}} = 2.697 + 1.234 = 3.931 \, [\text{kg}]$. The centres of gravity were also determined for each caster lengths. In the equations of motion of the corresponding mechanical model (see (4.20)), the mass moment of inertia $J_A$ of the overall system with respect to the $z$ axis at the king pin $A$ and its dependence on the caster length are needed. The result of the above described process led to the formula:

$$J_A = 0.1561 + 2.539 \times (0.036 + l)^2 + J_A^+ \, [\text{kgm}^2] ,$$  \hspace{1cm} (5.1)
where $J_A^+$ is the mass moment of inertia of the additional masses with respect to the $z$ axis at the king pin $A$.

## 5.2.2 Measuring tyre parameters

The tyre parameters, namely, the relaxation length $\sigma$, the distributed lateral stiffness $k$ and distributed lateral damping $b$ were measured in two different experimental set-ups. Both methods are presented and compared in this section.

As it was mentioned above, the tyre parameters depend on the inflation pressure in the pneumatic tyre and on the related length of the contact patch $2a$, so the measuring process was accomplished for various inflation pressures and contact lengths. The measuring results clearly helped to identify the parameter domains, where the tyre characteristics is still linear and the vertical and lateral loads of the tyre can be tolerated by the suspension system and the conveyor belt. For example, we have to avoid cases when the geometrical non-linearity of the tyre deformation is dominant at low tyre inflation pressures, or when the lateral forces may cause the buckling of the conveyor belt at large vertical loads of the tyre. Thus, the shimmy experiments on the conveyor belt were carried out in a properly selected parameter domain, namely, when the tyre inflation pressure and the contact length were $p_0 = 2$ [bar] and $2a = 0.079$ [m], respectively. Here, we present the identification of the tyre parameters for these parameter values for the case of the tyre Rubena 47 - 305 16 × 1.75 × 2 (see Figure 5.1).

(A) Experiments with rigid frame

First, the stiffness and damping parameters of the tyre were measured. After fixing a certain tyre inflation pressure, the wheel was placed into a rigid frame and the contact length was adjusted with help of the variable size of the frame (see Figure 5.3(a)). To measure the stiffness parameters of the tyre, the wheel was pulled at its centre point in lateral direction with different forces, and the equivalent lateral stiffness of the tyre $k_e$ was estimated from the measured displacements of the centre point of the wheel relative to the measured contact length $2a$ in those domains where the characteristics were still linear as shown in Figure 5.3(b).

To obtain the natural frequency $f_{n,w}$ and the damping ratio $\zeta_w$ of the wheel in the rigid frame, the wheel was slightly hit in lateral direction, and the time history of the lateral acceleration of the wheel centre point was recorded. Since the tyre is a continuum, the modal mass $m_{w,mod}$ of the wheel belonging to the lateral vibration mode was determined from the measured lateral stiffness and natural frequency, namely, $m_{w,mod} = k_e/(4\pi^2 f_{n,w}^2)$. This is, of course, smaller than the mass of the wheel $m_w = 1.305$ [kg]. After that, the damping ratio was identified by means of the logarithmic decrement of the vibration signal. This way, the equivalent lateral damping factor of the tyre was calculated: $b_e = 4\pi\zeta_w f_{n,w} m_{w,mod}$. The measured and calculated parameters are given in Table 5.1. To calculate the specific stiffness and damping of the tyre, the relaxation length $\sigma$ is needed, too.

In order to identify the numerical value of the relaxation length $\sigma$, a transparent plastic plate was placed at one side of the rigid frame and the tyre with marked central line was fixed in the same set-up (see Figure 5.4(a)). Then the tyre was pulled in lateral
Figure 5.3 Measuring the lateral stiffness of the tyre in the rigid frame. In panel (a), it is shown how the lateral load was applied and how the displacement of the wheel centre was measured in the rigid frame. Panel (b) shows the measuring results for $p_0 = 2$ [bar] and $2a = 0.079$ [m]. The lateral force $F$ is plotted with respect to the lateral displacement of the wheel centre point. Cross means measured point, while red solid line shows the determined equivalent stiffness characteristics.

Table 5.1 Tyre modal parameters in the rigid frame for inflation pressure $p_0 = 2$ [bar] and contact length $2a = 0.079$ [m].

| $f_{n,w}$ [Hz] | 35.9 |
| $\zeta_w$ [-] | 0.036 |
| $m_{w,mod}$ [kg] | 0.92 |
| $k_e$ [kN/m] | 46.8 |
| $b_e$ [Ns/m] | 15.2 |

direction again. Since the deformation of the tyre was visible inside and outside the contact patch through the transparent plate, the relaxation length of the tyre could be determined. Figure 5.4(b) shows the enlarged (and distorted) picture of the deformed tyre in and around the contact region. Since the distance between the transparent plate and the camera was just 1543 [mm], the picture has a considerable optical error. This is the reason why the wheel diameter seems to be circa 350 [mm] in the picture while the real diameter is 396 [mm]. The nonlinear relationship between the real and the measured horizontal distances could be determined because the geometry of the experimental set-up was known. Thus, the deformation of the marked centre line was measured in the picture and its optical error was compensated with an appropriate geometrical transformation. The corrected measuring points are plotted in Figure 5.4(c) with crosses.

The identification of the relaxation length $\sigma$ from the above measured and corrected deformation shape was carried out by two different methods. First, it was assumed, that the deformation shape outside the contact patch follows the theoretical exponential decay functions (4.28) perfectly, and the relaxation length was identified with the help of the
Figure 5.4 Measuring of the relaxation length $\sigma$ through the transparent plate. In panel (a), the experimental set-up is shown. In panel (b), the enlarged and distorted deformation shape is shown for $p_0 = 2$ [bar] and $2a = 0.079$ [m]. Panel (c) shows the measured deformation inside and outside the contact patch by black and blue crosses, respectively.

tangent line at the end points of the contact patch. This method led to $\sigma = 0.072$ [m] in Takács et al. (2009). Later, experimental investigation on the stability boundaries of the towed wheel indicated larger relaxation length of the tyre. Therefore, the above measuring method was improved, namely, a thorough analysis of the theoretical and experimental deformation shapes of the laterally pulled tyre was accomplished.

In the rigid frame, the steady wheel is pulled in lateral direction. The ‘no kink at the leading edge’ condition, however, is not fulfilled because the deformation inside the contact patch is constant theoretically, i.e., $q(x,.) \equiv q_0$ for $x \in [-a,a]$, which leads to $\lim_{x \to -a} q'(x,.) = 0$ and $\lim_{x \to a} q'(x,.) = -q(a,t)/\sigma$, see also Besselink (2000); Pacejka (2002). Since the normal force distribution and the corresponding maximal lateral force distribution can be approximated by a parabolic function in the contact patch (see Section 4.9 and Pacejka (2002)), the maximum lateral forces are zeros at the end points L...
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Figure 5.5 In panel (a), the deformation shape of the stretched string is represented. Panel (b) shows the measured (blue crosses) deformation shape of the tyre along its perimeter outside the contact patch. The fitted theoretical deformation function is plotted with red solid curve.

and R. This means that no kink can occur at these points, consequently, sliding occurs in narrow transition zones of the contact patch near to L and R. This can be seen in Figure 5.4(c), where the measuring points inside the contact patch are shown by black crosses. Thus, the ‘no kink at the leading edge’ condition (4.29) is fulfilled if the effect of sliding is considered despite of the fact that the tyre is not rolling. Furthermore, no kink at the trailing edge condition could also be established. All this means that the transition points are characterized by (4.29) and the relaxation length could be measured with the help of the tangent lines. Unfortunately, the decaying functions (4.28) still do not approximate the real deformation outside the contact patch well when the relaxation length is identified this way.

One can suspect that this is due to the radial deformation of the tyre. Namely, the radial deformation of the tyre has kinks at the end points of the contact patch. These discontinuities may cause also changes in the lateral deformation and the stretched string theory cannot approximate the lateral deformation close to the transition points well. Smiley (1957a) also established the existence of these transition regions in his study. Thus, the identification of the relaxation length at the transition points can provide a worse result than the fitting of the decaying functions to the measured characteristics.

The theoretical deformation of the tyre in the rigid frame is shown in Figure 5.5(a).
When the stretched string model was deduced, we supposed that the tyre is deformed at one side only, which assumption leads to the exponential decaying functions (4.28). In the rigid frame, the tyre is deformed at both sides, and the decaying functions (4.28) do not approximate the real deformation perfectly. In order to determine the theoretical deformation of the tyre outside the contact patch precisely, we use the special form of the governing equation (4.27) of the stretched string:

$$q''(s, \cdot) - \frac{1}{\sigma^2} q(s, \cdot) = 0,$$

where $s \in [0, l_f]$ is the arc length coordinate along the perimeter of the tyre and $l_f \approx R\pi - 2a$ (see Figure 5.5(a)). The boundary conditions are characterized by $q(0, \cdot) \equiv q(l_f, \cdot) = q_0$ in the rigid frame if the symmetry of the deformation is considered, too. The solution for the theoretical deformation is given in the form

$$q(s, \cdot) = \frac{q_0}{1 + e^{-\frac{s}{\sigma}}} \left( e^{-\frac{s-l_f}{\sigma}} + e^{-\frac{s}{\sigma}} \right).$$

To identify the real relaxation length, the above determined theoretical deformation function is fitted to the measured deformation by the method of least squares. The fitted curve is shown in Figure 5.5(b) by (red) solid line.

The specific lateral stiffness and damping parameters can be calculated by the formulas

$$k = \frac{k_e}{4(a + \sigma \tanh \left( \frac{l_f}{2\sigma} \right))} \approx \frac{k_e}{4(a + \sigma)} \quad \text{and} \quad b = \frac{b_e}{4(a + \sigma \tanh \left( \frac{l_f}{2\sigma} \right))} \approx \frac{b_e}{4(a + \sigma)},$$

respectively. Their calculated values are shown in Table 5.2. These values were used in (4.63) to estimate the natural angular frequency and the damping ratio of the experimental rig. This resulted intolerable differences between the obtained damping values and the ones measured on the real structure on the conveyor belt during dynamic experiments. At this point, we recognized that the mechanical model should be extended with the torsional damping at the king pin, which does not seem to be negligible compared to the tyre damping.

(B) Experiments on the conveyor belt

In order to improve the estimation of the tyre parameters, the tyre with the whole suspension system was fitted on the conveyor belt. The length of the contact patch was tuned to a certain value, and the frequency response function of the steady towed wheel was determined with the help of the measuring system PULSE. The system was excited by modal hammer and the response acceleration of the end point on the caster was recorded. Then the natural angular frequencies and the damping ratios of the system were measured at different caster lengths. In these experiments there was no torsional spring added at the king pin, that is, $k_t = 0 [Nm/rad]$, and there was no additional mass on the caster either, i.e., $J_c = 0$. Since the mass moments of inertia $J_A$, the contact length $2a$ and the caster length $l$ were known, the tyre parameters $\sigma$, $k$, $b$ and the suspension damping parameter $b_t$ could be determined by fitting the theoretical curves (4.76) and (4.77) to the measured data (see Figure 5.6). The relative errors between the theoretical curves...
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Figure 5.6 The natural frequency $f_n$ and the damping ratio $\zeta$ of the steady towed tyre at different caster lengths for $p_0 = 2$ [bar] and $2a = 0.079$ [m]. Crosses mean measuring points, solid (red) lines are the fitted theoretical curves.

Table 5.2 The numerical values of the identified parameters at $p_0 = 2$ [bar] and $2a = 0.079$ [m].

<table>
<thead>
<tr>
<th></th>
<th>(A) Rigid frame</th>
<th>(B) Conveyor belt</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tyre</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma$ [m]</td>
<td>0.134</td>
<td>0.130</td>
</tr>
<tr>
<td>$k$ [kN/m$^2$]</td>
<td>68.8</td>
<td>62.6</td>
</tr>
<tr>
<td>$b$ [Ns/m$^2$]</td>
<td>22</td>
<td>19</td>
</tr>
<tr>
<td>Suspension</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k_t$ [Nm/rad]</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>$b_t$ [Nms/rad]</td>
<td>-</td>
<td>0.61</td>
</tr>
</tbody>
</table>

and the measured data were minimized with the method of least squares. Without the suspension damping $b_t$ even the qualitative characteristics of the damping ratio against the caster length was incorrect.

Since the vibration frequencies can be measured more precisely than the damping ratio, first, the relaxation length $\sigma$ and the specific lateral stiffness $k$ were identified from the measured vibration frequencies. After that, the specific lateral damping $b$ and the torsional damping $b_t$ were determined from the measured damping ratios. This method leads to the parameters in Table 5.2.

(C) Comparison of the parameter identification measurement results

The experiments with the wheel in the rigid frame and on the conveyor belt give almost the same numerical values for the relaxation length, the specific stiffness and the specific damping factor of the tyre; the relative errors are less than 3, 10 and 16%, respectively. The largest deviation is in the specific damping factor $b$, but 16% error is reasonable in case of damping parameters in engineering systems. Since the torsional damping at the king pin could not be identified in the rigid frame, we trust the results of the experiments on the conveyor belt.

We give some further discussion on the measured relaxation length. As it was written in Section 4.3.2, Segel (1966) determined a maximal value for the relaxation length experimentally. In our case, this maximum value gives $(R\pi - a)/6 = 98$ [mm], which
corresponds to the relaxation length \( \sigma = 72 \text{[mm]} \) measured with the help of the tangent lines in the rigid frame (see Figure 5.4). However, the improved value \( \sigma = 134 \text{[mm]} \) of the relaxation length (see Figure 5.5) exceeds Segel’s maximal value. There is no available information about the type of tyres used by Segel. Later, different approximations of the relaxation lengths were compared by Besselink (2000), and it was emphasized that the maximal values of the relaxation length may strongly depend on the types of the tyre. For example, in case of a passenger car tyre, the relaxation length can be five times the contact length, which could be 395 [mm] in our case. This condition is fulfilled by our improved relaxation length value. Nevertheless, the linear stability boundary detected on the experimental rig also shows a better agreement for the improved (larger) value of the relaxation length \( \sigma = 134 \text{[mm]} \).

5.2.3 Experiments on static and dynamic coefficients of friction

The validation of our mechanical model also requires the identification of the static and dynamic coefficients of friction between the tyre and the conveyor belt. We applied a simple experiment to estimate the coefficients, which is shown in Figure 5.7(a). Panel (b) shows the slick surface of the test specimen (contact surface: 80 [mm] \( \times \) 80 [mm]), which was towed on the conveyor belt meantime the towing force was measured by a dynamometer. This simple experiment helps to estimate the static and the dynamic coefficients of friction only, and do not provides the exact values, of course. More precise experiments would have required large efforts (see, for example, Andersson and Kropp, 2009), which were not reasonable in our case. Actually, our mechanical model was constructed by means of a simple Coulomb friction model as described in Section 4.9 in details.

The experiments were carried out after the specimen was warmed up to the stationary temperature. The mean values of the measurement results using different normal forces are shown in Table 5.3 with two digits accuracy. The slick surface adheres to the conveyor

![Figure 5.7 Experiments on the coefficients of friction.](image)

**Table 5.3** Static and dynamic coefficients of friction between the specimen and the conveyor belt.

<table>
<thead>
<tr>
<th>Specimen</th>
<th>( \mu_s )</th>
<th>( \mu_d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slick</td>
<td>1.5</td>
<td>1.0</td>
</tr>
</tbody>
</table>
belt very well, namely, the static coefficient of friction is larger than 1, and the dynamic coefficient of friction is the two third of the static one. This ratio could be even less in case of rubber and asphalt surfaces (Pacejka, 2002).

5.3 Validation of the theoretical results

To simplify the validation of the theoretical results, the dimensionless stability chart is transformed into the space of the dimensional parameters used during the experiments. Namely, all of the relationships of the system parameters (4.76), (4.77) and (4.78) are taken into account to realize the surface in the dimensionless parameter space, on which the series of measurements can be accomplished. Its intersection with the hypersurface of the dimensionless stability boundaries leads to the stability chart in the dimensional plane \((v, l)\) of the towing speed and caster length. The dimensionless relaxation length \(\Sigma\), the dimensionless torsional stiffness \(K\) and damping \(B\) do not depend on \(v\) and \(l\), their values are kept constant during the experiments as shown in Table 5.4. The above described intersection of the parameter surfaces and the stability boundaries can be determined in the space \((V, L, \zeta)\) (see Figure 5.8). We have options which parameter surfaces are used during the measurement since we can still vary the additional mass moment of inertia \(J_k^+\). The corresponding surfaces of the measurement parameters are also shown in the space \((V, L, \zeta)\) (see Figure 5.8) for \(J_k^+ = 0 \ldots 3 [\text{kgm}^2]\). The thick solid lines represent the stability boundaries corresponding to the intersections of these surfaces and these can be transformed to the dimensional \((v, l)\) plane in Figure 5.9.

Despite of all our efforts to design the system for a wide range of parameters, we managed to identify experimentally only one of the most important stability boundaries due to the intricate connections of the relatively large number of system parameters. Namely, in order to amplify the unstable islands in the stability charts, we intended to decrease the damping ratio \(\zeta\) of our system by increasing the mass moment of inertia with an added mass on the caster. This way, we also decreased the natural angular frequency \(\omega_n\) of the system and the values of the dimensionless towing speeds \(V\) were increased (see (4.78)). Due to the minimum speed limit \((0.5 [\text{km/h}])\) of the conveyor belt, this procedure made it possible to reach that part of the stability chart where the first of the multiple unstable islands just appear. This way, we could identify experimentally the most relevant stability boundary, and we could steer the parameters close enough to one of the double Hopf points identified in the stability charts of Section 4. Also, the most

Table 5.4 The parameters of the experimental rig at \(p_0 = 2 [\text{bar}]\) and \(2a = 0.079 [\text{m}]\).

<table>
<thead>
<tr>
<th>Dimensional parameters</th>
<th>Dimensionless parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma [\text{m}])</td>
<td>0.130</td>
</tr>
<tr>
<td>(2a [\text{m}])</td>
<td>0.079</td>
</tr>
<tr>
<td>(k [\text{kN/m}^2])</td>
<td>62.6</td>
</tr>
<tr>
<td>(k_t [\text{Nm/\text{rad}}])</td>
<td>0</td>
</tr>
<tr>
<td>(b [\text{Ns/m}^2])</td>
<td>19</td>
</tr>
<tr>
<td>(b_t [\text{Nms/\text{rad}}])</td>
<td>0.61</td>
</tr>
</tbody>
</table>
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Figure 5.8 The intersection of the dimensionless stability boundaries and the surfaces of the measurement parameters for different values of the additional mass moment of inertia $J_A^+$. 

Figure 5.9 Thick curves show the theoretical stability boundaries in dimensional ($v, l$) plane for different values of the additional mass moment of inertia. The stable domain is shaded. Thin curves are the theoretical stability boundaries of the undamped system. Dashed (red) lines demonstrate the realizable minimum towing speed and maximum caster length.

relevant unstable island in the upper region can be detected by increasing the additional mass moment of inertia as shown in Figure 5.9.
5.3.1 Experimental stability analysis

During the experiments, the wheel was towed with different but constant speeds, while the vibrations of the caster and the stability of the system were observed at different but fixed values of the caster length. To identify the stability boundary, the caster was perturbed and the stability of the stationary rolling was examined. The experiments show subcritical Hopf bifurcation at the most relevant stability boundary, consequently, slight perturbations led to exponential decay of the vibrations, while somewhat larger perturbations could generate large amplitude vibrations close to the stability boundary. In these cases, the system was linearly stable but small amplitude unstable periodic motions bounded the domain of attraction. Unfortunately, the unbalance of the wheel excited the caster at larger towing speeds, and the identification of the stability boundary was slightly disturbed by the presence of this subcritical Hopf bifurcation.

The experimentally identified stability charts are shown in Figure 5.10. The measured stability boundaries are plotted by dashed lines through the blue crosses of the corresponding critical measurement points. Above the charts, the theoretical vibration frequencies are plotted by thick lines relative to the natural frequency of the steady wheel. The frequencies of the small amplitude oscillations were also measured and they are marked by blue crosses relative to the measured natural frequencies.

The experimental and theoretical stability boundaries are qualitatively similar and only some quantitative difference can be observed. As shown in panel (a), the towed wheel become unstable at somewhat lower towing speeds than it is predicted by the theory. In panel (b), the most relevant stability boundary has good agreement even quantitatively. The vibration frequencies follow the theoretical curves for small and medium towing speeds in case of both measurement configurations. For large towing speeds, however,
there are differences between the theoretical and measured frequencies but the tendency of the curves are qualitatively similar.

The unstable island in the upper parameter domain could not be detected in these experiments. For that reason, the nonlinear behaviour of the mechanical model was analysed by several numerical simulations. The simulations considering sticking in the whole contact region (‘pure rolling’) show large amplitude periodic oscillations at these unstable islands in accordance with the linear theory and its extension with geometric non-linearities (see Section 4.8). After implementing the sliding effect into the mechanical model, only small (even tiny) amplitude vibrations occurred at the same parameter set-up (see Section 4.9). Although, the stationary rolling is linearly unstable, the predicted small amplitude vibration is almost invisible and its detection required further detailed experimental investigation that is described later in Section 5.3.5.

5.3.2 Comparison of the measured and simulated nonlinear vibrations

The stability and the vibration frequencies of the towed tyre were analysed at many measurement parameter points in the stability charts of Figure 5.10. The time histories of the measured vibration signals were recorded during these experiments when the stationary rolling was always unstable and shimmy motion (self-excited vibration) appeared. In this section, some of these time histories are presented, analysed and compared to the results of numerical simulations described in Section 4.9.

The actual measurement points are marked by $M_1$, $M_2$, $M_3$ and $M_4$ and they are denoted by (red) crosses in Figure 5.10. Because some of these parameter points ($M_1$, $M_2$) are in the theoretically linearly stable domain, the numerical simulations show qualitatively different behaviour at these points. Therefore, each of the investigated parameter points was paired to the simulation parameter points $S_1$, $S_2$, $S_3$ and $S_4$ where the simulations were carried out. These parameter points are at the same caster lengths like their measurement counterparts, but they are in the theoretically linearly unstable domain, namely, their towing speeds are larger. Both the measurement and the simulation parameter points are characterized in Table 5.4 and 5.5. The reason for selecting somewhat altered parameter points is that we decided to disregard the difference between the measured and the theoretical stability maps in order to compare the large amplitude nonlinear vibrations close to the stability limits but definitely in the unstable side in both cases.

If the sliding effect was not considered, the simulations gave much larger vibration amplitudes than it was detected during the experiments. Thus, the partial sliding of the contact line had to be taken into account in the mechanical model used in the simulation code (see Section 4.9). In order to do this, the normal force on the tyre was measured, which was $F_z = 180 \, [N]$. The static and dynamics coefficients of friction were already known as a result of the measurements in Section 5.2.3, i.e., $\mu_s = 1.5 \, [-]$ and $\mu_d = 1.0 \, [-]$ were used in the simulations. The initial conditions for the numerical simulations were set to a lateral impact-like perturbation: $\psi(0) = 0$, $\dot{\psi}(0) = -0.1 \, [\text{rad/s}]$ and $q(x, 0) = 0$. Since the tangential acceleration $a_t$ of the caster end point was measured and recorded during experiments, this signal was also calculated from the simulation results by means of the formula $a_t(t) = l_p \, \ddot{\psi}(t)$, where $l_p = 0.5 \, [\text{m}]$ was the distance between the king pin
and the piezo-electric accelerometer.

**Table 5.5** The numerical values of the system parameters at the parameter points of Figure 5.10.

<table>
<thead>
<tr>
<th>Point</th>
<th>( v ) [m/s]</th>
<th>( l ) [m]</th>
<th>( J_A^S ) [kgm^2]</th>
</tr>
</thead>
<tbody>
<tr>
<td>M_1 S_1</td>
<td>0.484</td>
<td>0.020</td>
<td>0</td>
</tr>
<tr>
<td>M_2 S_2</td>
<td>1.750</td>
<td>0.140</td>
<td>0</td>
</tr>
<tr>
<td>M_3 S_3</td>
<td>0.186</td>
<td>0.040</td>
<td>3.17</td>
</tr>
<tr>
<td>M_4 S_4</td>
<td>0.692</td>
<td>0.140</td>
<td>3.17</td>
</tr>
</tbody>
</table>

The measured and simulated time histories are compared in Figure 5.11, 5.12, 5.13 and 5.14. In all the figures that follow, the experimental and the simulation results are shown in the left and the right panels, respectively. In panel (a), the time history of the tangential acceleration of the caster end point can be seen when the stationary rolling is unstable and shimmy motion occurs. Panel (b) is the enlarged part of the same signal that corresponds to the sampled signal in the black dashed box in panel (a). On the horizontal axis, the time \( t \) is scaled to the period \( T \) of the shimmy motion in order to show that the measured and the simulated shimmy motion is qualitatively very similar. Panel (c) shows the spectra of the vibrations, and the curves with different colours refer to the corresponding coloured sampling windows in panel (a). This way, the vibration frequencies of the small and the large amplitude vibrations can also be analysed and compared.

The measurement and simulation results at the parameter points \( M_1 \) and \( S_1 \) are shown in Figure 5.11. It can be observed in panel (a) and (b) that the time histories are very similar despite of the fact that the amplitude of the simulated shimmy motion is somewhat smaller than that of the measured one. A shift of the frequencies can also be observed in panel (c), namely, the measured vibration frequencies are slightly smaller both for the transient motion where the linear behaviour dominates (brown curve) and for the fully developed large amplitude motion where the nonlinear behaviour dominates (green curve). It is worth mentioning that the basic frequency in the measured spectrum shifts to smaller values as the large amplitude oscillation develops as the transient motion dies out, while the simulation result does not shown this phenomenon, the basic frequency remains the same as the vibration increases. Also, the measurement result contains many small frequency components between the basic frequency and its higher harmonics, while the basic frequency and its harmonics dominate the simulation result.

Vibrations at large caster length are investigated at the parameter points \( M_2 \) and \( S_2 \). The corresponding results are shown in Figure 5.12. The amplitudes and the frequencies of the detected shimmy motions have good agreement. It is interesting to observe that the basic frequency shifts to smaller values as the vibration fully develops during the simulation while it does not change at all for the measurement.
Figure 5.11 The measured and the simulated time signals at the parameter points M₁ and S₁, respectively.

Figure 5.12 The measured and the simulated time signals at the parameter points M₂ and S₂, respectively.

A low frequency shimmy motion can be observed in Figure 5.13, which corresponds to the parameter points M₃ and S₃. In panel (b), the measured and the simulated periodic shimmy motions have quite large differences, while the frequencies in panel (c) show excellent agreement.
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At last in this series, the measured and the simulated results at the parameter points \( M_4 \) and \( S_4 \) are plotted in Figure 5.14. The detected periodic motions and vibration frequencies have very good agreement (see panel (b) and (c)). The shifts of the basic frequencies can be detected in both spectra of the measured and the simulated signals,
and as in case of parameter point $M_1$ and $S_2$, the large amplitude oscillations have smaller frequencies. While it might be difficult to explain the somewhat contradictory observation in cases of $M_1 - S_1$ and $M_2 - S_2$, the consequent result of $M_4 - S_4$ is in good agreement with the general properties of Hopf bifurcations where the vibration frequencies of the self-excited oscillations can slightly change as the vibration amplitudes increase.

The further small quantitative differences between the measurement and the simulation results can be explained in many different ways. For example, the coefficients of friction could be larger than the ones measured in Section 5.2.3. If a larger static coefficient of friction (for example, $\mu_s = 1.8$) is used in the simulation code, the shapes of the simulated signals and the amplitudes of the vibrations have a better agreement with the measured results. Another explanation can be that we applied an approximation in the simulation code to hold the CPU time on an acceptable value, see Section 4.9. This approximation reduces the time of the sliding in the contact patch and may cause smaller amplitude vibrations since the energy stored by the stretched string dissipates faster.

5.3.3 Modal testing of the rolling wheel

In order to validate the theoretically determined self-excited vibration frequencies and their dependence on the towing speed, the modal testing (experimental modal analysis) of the towed wheel was carried out during its rolling. At a fixed caster length, the caster was hit with the modal hammer while the wheel was rolling with different speeds and all the other fixed parameters were set in a way that the stationary rolling was close to the stability boundary on the stable side. At each towing speed, the responses were measured by a piezo-electric accelerometer at the end of the caster and the corresponding frequency response functions were determined and recorded. The measured vibration frequencies should correlate with the imaginary parts of the numerically determined roots of the characteristic function, which was determined via the linear stability analysis of the mechanical model of the towed wheel in Chapter 4. Note that there are infinitely many characteristic roots, but only some of them are relevant in the sense that they are located close to the imaginary axis of the complex plane, i.e., their real parts are close to zero. This property is known from the theory of retarded functional differential equations (Stépán, 1989).

The results of the modal testing are shown in Figure 5.15, 5.16 and 5.17, where experiments were carried out at the most important parameter domain of the stability chart. In all the figures, panel (a) shows the frequency response functions of the system in the form of waterfall diagram in the parameter space $(f, A, v)$ of the frequency, the absolute value of the frequency response function and the towing speed. The relevant frequency components of the frequency response functions are denoted by (blue, green, red) dots and they are distinguished by colours according to the corresponding characteristic roots, the real and imaginary parts of which are shown in panel (b) by (blue, green, red) crosses. The imaginary parts are presented relative to the natural angular frequency $(\omega/\omega_n = f/f_n)$ of the steady wheel (when $v = 0$). In these diagrams, the theoretically calculated characteristic roots of the mechanical model of the towed wheel are also presented by means of their real and imaginary parts by continuous lines using the same colour code as in case of the experiments. Note that the real parts of the measured characteristic roots are not presented in the lower graphs of panel (b) partly because they have less importance.
Figure 5.15 The result of the modal testing for $l = 0.030$ [m] and $J_z^+ = 0$ [kgm$^2$]. Panel (a) shows the frequency response function of the towed wheel against the towing speed. In panel (b), the measured vibration frequencies (crosses) are compared to the imaginary parts of the theoretically calculated characteristic roots (continuous lines), while the relevant (near-to-zero) real parts of the corresponding characteristic roots are also presented by continuous lines.

During the validation process, partly because the identification of the slight damping of the signals caused difficulties close to the stability boundary.

Two of the presented experiments were carried out at parameter points with small caster lengths, namely, at $l = 0.030$ [m] and at $l = 0.060$ [m]. Close to these parameter points, two self-excited vibration frequencies are predicted in case of a certain towing speed and small damping (at the double-Hopf bifurcation point). In Figure 5.15(b), there are two pairs of theoretically predicted characteristic roots with small negative real parts at $0.3 - 0.4$ [m/s], which is an explanation why two relevant and rationally independent vibration frequencies appear in the frequency response function at this speed range. The most relevant characteristic root (blue dots, crosses, lines) corresponding to the most relevant stability boundary of the towed wheel has the largest real part for any value of the towing speed. This means that the corresponding frequency should appear in the frequency response function for any towing speed. Still, this frequency can not be observed in Figure 5.15(a) at small towing speed but this is related to the limitation of our measurement system only; the identification of small frequencies below 0.7 [Hz] can not be accomplished precisely by our piezo-electric accelerometer and the measurement system cuts this frequency domain. The most important conclusion of this series of measurements is that the theoretically predicted quasi-periodic oscillation frequencies are perfectly identified by modal testing.

There is no speed range in Figure 5.16(b), where more than one pair of characteristic roots has almost zero real part (compare to Figure 5.15(b)), consequently, there is no speed range, where more than one relevant frequency can be detected in the measured frequency response functions of Figure 5.16(a). Still, three different frequency ranges show up (marked with red, green, blue colour code) related to three disjunct towing speed domains, which is predicted again by the delay effects in our tyre model only.

In contrast to the above conclusion, there exists a fourth frequency range denoted by
Figure 5.16 The result of the modal testing for $l = 0.060$ [m] and $J_A^+ = 0$ [kgm$^2$]. Panel (a) shows the frequency response function of the towed wheel against the towing speed. In panel (b), the measured vibration frequencies (crosses) are compared to the imaginary parts of the theoretically calculated characteristic roots (continuous lines), while the relevant (near-to-zero) real parts of the corresponding characteristic roots are also presented by continuous lines.

The modal testing of the towed wheel is also presented in Figure 5.17, where the caster length is $l = 0.300$ [m] > $a + \sigma = 0.170$ [m] and the non-zero additional mass moment of inertia $J_A^+ = 3.17$ [kgm$^2$] is applied. This means that the actual parameter configuration is in the region of the linearly unstable island of large caster lengths $l > a + \sigma$ that was not found or detected during the experimental stability analysis (see Section 5.3.1). Despite of this, the modal testing confirms the existence of the theoretical characteristic roots with a vibration frequency close to the natural frequency. A slight quantitative difference can be observed between the theoretically predicted and experimentally detected frequencies. This statement also appeared in Section 5.3.1 (see also Figure 5.10) for large caster lengths. It is important to note that the above modal testing confirmed the validity of our time delayed tyre model even in the long caster region due to the appearance of the predicted frequencies, while the self-excited vibrations did not seem to show up even in those speed ranges where the real part of the relevant characteristic root was predicted to be positive in accordance to the theory (see the (cyan) continuous line in the lower diagram of panel (b) crossing $\text{Re} \lambda = 0$ for several towing speeds).

In Figure 5.17, the above described periodic excitation and their higher harmonics can be detected in the frequency response functions again. They are marked by (black) dots and circles in panel (a) and (b), respectively. As it can be seen in both panels, the higher harmonics of the periodic excitation can cause a kind of resonance in the wheel lateral...
vibration in certain speed ranges, which effect may perturb the modal testing results for the towed wheel. This explains why the experimentally detected vibration frequency has larger amplitude in the frequency response function at these speed ranges despite that the corresponding theoretically calculated characteristic root has smaller real part.

5.3.4 Quasi-periodic signal analysis

One of the most exciting properties of shimmy is that the detected vibrations usually show notable beating effect, namely, the vibrations are quasi-periodic. Such oscillations of the towed wheel of elastic tyre were observed in our experiments at small caster length and towing speed, close to that parameter domain, where two theoretical stability boundaries intersect each other, and two different self-excited vibration frequencies exist for a small damping ratio (see the stability charts in Chapter 4). The parameter domain, where the quasi-periodic vibrations were observed experimentally, are marked by a green spot and by the letter Q in Figure 5.10(a).

Some measurement results are shown in Figure 5.18, which were obtained for \( l = 0.015 \) [m] and \( v = 0.28 \) [m/s]. In this experiment, no additional mass moment of inertia was used, i.e., \( J^+ = 0 \) [kgm²]. In panel (a), the time history of the detected quasi-periodic vibration is presented and it shows an essential beating effect. In panel (b), the corresponding spectrum is shown. However, this intricate spectrum clearly refers to a complex nonlinear vibration, where the theoretical frequencies of the double Hopf bifurcation point can also be identified: \( f_1 \approx 0.23f_n \) and \( f_2 \approx f_n \) together with several higher harmonics and side peaks. For example, the third harmonic of \( f_1 \), marked by \( f_{1,3} \), is also a well isolated peak. The waterfall diagram of the detected signal is shown in panel (c). The transient vibration (the first 12 seconds) contains the two self-excited vibration frequencies \( f_1 \) and \( f_2 \) only, which are predicted by the theory and can clearly be identified.

**Figure 5.17** The result of the modal testing for \( l = 0.300 \) [m] and \( J^+ = 3.17 \) [kgm²]. Panel (a) shows the frequency response function of the towed wheel against the towing speed. In panel (b), the measured vibration frequencies (crosses) are compared to the imaginary parts of the theoretically calculated characteristic roots (continuous lines), while the relevant (near-to-zero) real parts of the corresponding characteristic roots are also presented by continuous lines.
Figure 5.18 The measured quasi-periodic vibration at $l = 0.015 \, [m]$, $v = 0.28 \, [m/s]$ and $J^+_A = 0 \, [kgm^2]$. Panel (a) and (b) show the time history of the tangential acceleration of the caster end point and its spectrum, respectively. In panel (c), the waterfall diagram of the measured signal is presented.

In order to show that such complex vibrations can be described by means of our mechanical model, numerical simulations were also carried out in the corresponding parameter domain using the same numerical method we described and tested in details in Section 4.9. The initial conditions for the numerical simulations were set to an impact-like lateral perturbation: $\psi(0) = 0$, $\dot{\psi}(0) = -0.1 \, [rad/s]$ and $q(x, 0) = 0$ just as we did in Section 5.3.2. The numerically simulated transient motion is plotted in Figure 5.19(a), where quasi-periodic oscillation can be observed, which corresponds to both of the linearly predicted frequencies. A typical shape of the contact line is shown in panel (b), which refers to the time instant $t^*$ in panel (a). Although the tyre has only small deformation at this time instant, the rear part of the contact patch already slides, and no kink occurs at the trailing edge R (at $x = -a$). The 2D phase portrait of the transient motion projected from the infinite dimensional phase space shows an interesting whirling of the system in panel (c), which corresponds to an intricate transient behaviour.

The simulation of the fully developed vibration of the caster angle is shown in Figure 5.20. In panel (a), the vibration signal of the towed wheel indicates a small periodic variation of the vibration amplitude that refers to quasi-periodicity, again. In panel (b), the presented deformation corresponds to $t^*$ that is just before the time instant when the whole contact patch starts sliding and the presented deformation shape is fully destroyed. The phase portrait in panel (c) confirms that the simulated motion contains many frequencies. The trace of the attractor in the 2D projection could be either the trace of a torus referring to quasi-periodic oscillation, or that of some higher dimensional objects referring to ‘slight’ chaotic oscillations.

To simplify the comparison of the measured and simulated oscillations, the simulated time history of the tangential acceleration of the caster end point, its spectrum, and its waterfall diagram are shown in Figure 5.21. The slight beating effect can still be
Figure 5.19 The simulated transient motion. Panel (a) shows the transient time history of the caster angle. Panel (b) shows the transient shape of the contact line at $t^\star$. The 2D projection of the phase portrait of the transient motion is plotted in panel (c).

Figure 5.20 The simulation of the fully developed vibration. Panel (a) shows the time history of the caster angle. Panel (b) shows the transient shape of the contact line at $t^\star$. The 2D projection of the phase portrait of the transient motion is plotted in panel (c).

Figure 5.21 The simulated quasi-periodic vibration at $l = 0.010$ [m], $v = 0.43$ [m/s] and $J_A^+ = 0$ [kgm$^2$]. Panel (a) and (b) show the time history of the tangential acceleration of the caster end point and its spectrum, respectively. In panel (c), the waterfall diagram of the simulated signal is presented.
observed in panel (a) after 48 seconds of the oscillation. The corresponding spectrum is plotted in panel (b), where the predicted frequencies of the beating can be detected with similar side peaks and higher harmonics as in case of the spectrum of the measured signal (compare to Figure 5.18(b)). The waterfall diagram of the simulated motion has an excellent qualitative and good quantitative agreement with the measurement results as it can be observed in panels (c) of Figure 5.18 and 5.21.

The only quantitative difference between the experimentally observed and numerically calculated beating effects is that the simulation shows smaller variation in the vibration amplitudes. However, this type of quasi-periodic oscillation was detected both numerically and experimentally only in this theoretically predicted parameter domain. This confirms that the practical experiences are in strict relationship with the theoretical double Hopf bifurcation point in the stability chart although the complicated attractor embedded in the infinite dimensional phase space has not been analysed in this study with the methods of bifurcation theory of non-smoothed delayed systems.

5.3.5 Heat generation of small amplitude shimmy with microslips

In Section 5.3.1, the experimental stability analysis of the towed wheel was described in details. As it was mentioned there, the linearly unstable parameter domain of large caster lengths was not observable experimentally. Since the relevance of the memory effect in tyres could unquestionably be confirmed by the validation of this unstable island, further experiments were carried out at the corresponding parameters in this section. The initiation of this additional series of experiments were also encouraged by the modal testing results presented in Figure 5.17, where the critical pair of complex characteristic roots in the unstable parameter island were clearly identified by the impact experiments carried out during rolling.

If the system parameters of the towed wheel are situated in the upper unstable parameter domain \( l > a + \sigma \) of the stability chart in Figure 5.10, self-excited vibrations should appear. Although the system is linearly unstable and shimmy motion is predicted by the theory, the amplitude of the vibrations might be limited by the nonlinear effect of sliding so much that the vibration remains unobservable within the noise of the measured signals. Still there exists a maximum lateral force distribution in the contact patch within which the vibrations can increase without sliding of the tyre points. When the lateral force distribution in the contact patch reaches this maximum value somewhere within the contact patch, the corresponding tyre points start sliding and the vibrations decay in time till the sliding is completed, the tyre points stick again, and the vibrations can start increasing again. In the unstable island, this effect leads to a small (practically invisible) self-excited vibration with several microslips in the contact patch. This was recognized during the detailed analysis of the numerical simulations described in Section 4.9 that also included the study of the slow-motion animation of the wheel oscillation and that of the tyre lateral deformations in the contact patch. Although the vibrations are small, the microslips dissipate most of the mechanical energy introduced by the self-excited shimmy vibration. This means that it is ‘more difficult’ to pull the wheel in these parameter regions, namely, the resistance to rolling has a higher level in these unstable parameter domains of shimmying wheels. As a consequence of the above described effect, the tyre may warm up to a somewhat higher temperature, and the wear of the tyre may speed
up in these parameter domains while the shimmy vibration itself can be observed neither visually nor with sophisticated dynamic measurement systems due to the presence of noise.

The above described power dissipation of wheel shimmy was verified experimentally in the unstable parameter island when the temperature of the tyre and the electric input current of the conveyor belt were measured. After fixing all of the system parameters, the towing speed was tuned to a desired value within and around the unstable island, and the tyre was run for some minutes at the same speed in order to reach the stationary temperature distribution. Then, the temperatures of the tyre was measured by a pyrometer. The resolution of the pyrometer was 1 [$^\circ$C]. The generation of heat was also observed by a sensible thermal camera in order to verify the locations of the power dissipation. Two pictures are shown in Figure 5.22 in order to demonstrate that the generation of heat has a measurable level and the thermal camera was able to visualize the wheel-track in real time (see panel (a)). The average of the input current of the conveyor belt was also recorded by an electric-supply meter with the resolution of 0.01 [A].

The normal load on the tyre generates normal forces and the related friction forces between the belt and its supporting plate. The power of the friction forces heats the belt, and heat transfer occurs between the belt and the tyre, too, which was confirmed by the thermal camera (see Figure 5.22(b)). Unfortunately, this effect make the validation of the microslips more difficult. Nevertheless, the power dissipation of the friction forces between the belt and the plate depends linearly on the towing speed while the power dissipation of the microslips may occur in a certain speed range only. If the towing speed is chosen from the prescribed speed range of the unstable island, higher temperatures and higher electric input current should be measured.

The experimental results are presented in Figure 5.23, where the caster length was $l = 0.300$ [m] and the additional mass moment of inertia $J_A^+ = 3.17$ [kgm$^2$] was applied. In panel (a), the theoretical stability chart is constructed for the experimental parameter configuration in question. In panel (b), the common real part of the theoretically calculated relevant characteristic roots is plotted for $l = 0.300$ [m] against the towing speed. The fully developed shimmy oscillations were simulated for different values of the towing

\[ \text{Figure 5.22} \] Pictures made by the thermal camera at different parameter points.
Figure 5.23 Heat generation of small amplitude shimmy at the caster length \( l = 0.300 \) [m] and at the additional mass moment of inertia \( J_A^+ = 3.17 \) [kgm\(^2\)]. Panel (a) shows the theoretical stability chart for the experimental parameter configuration. In panel (b), the real part of the theoretically calculated relevant characteristic root is plotted against the towing speed. Panel (c) shows the bifurcation diagram at the fixed caster length. Panel (d) and (e) present the actual temperature of the tyre at the tyre trailing edge and the measured increments of electric current, respectively.

speed, and the bifurcation diagram of the system was conjectured for the fixed caster length. This bifurcation diagram is shown in panel (c), where the numerically calculated stable branch is represented by (green) solid line, while the hypothetic unstable periodic branch is illustrated by (red) dashed line. It seems to be a non-conventional supercriti-
Chapter 5. Experiments on elastic tyre

cal Hopf bifurcation at \( v = 0.75 \m/s \) in the sense that it does not follow the standard parabola shape. This is common, however, for bifurcations in non-smooth systems with Coulomb friction as shown by Leine (2006) for stick-slip tasks. It is likely that the sub-critical Hopf bifurcation at \( v = 0.44 \m/s \) is also degenerate in the same way. The exact bifurcation analysis of the system at these parameter points is an exciting challenge for future mathematical research, since our mechanical model has an infinite dimensional phase space and the Centre Manifold reduction is a much more difficult task than it was in Chapter 3 where the phase space was only 3 dimensional.

Panel (d) shows the actual temperature of the tyre at the trailing edge, while panel (e) shows the increments of the electric current that were measured between the cases when the conveyor belt runs with the wheel on it and when it runs free of the wheel that was lifted up. The measurement points are presented by blue crosses in these panels. A higher level of the temperature can clearly be observed at the towing speeds \( 0.25 - 0.6 \m/s \), which perfectly coincides with the region where the stable periodic vibrations were predicted with microslips, while the linearly unstable speed range is predicted for the somewhat higher towing speed region \( 0.44 - 0.75 \m/s \). As explained above, the shift between the observed and the predicted critical speed ranges is a straightforward consequence of the bifurcation diagram in panel (c). The numerical bifurcation analysis of the system shows that the largest amplitude vibrations of the towed wheel and consequently the largest gradient in the temperature increase may appear somewhat ‘before’ the unstable island of the stability chart, where the linearly stable stationary rolling coexists with the tiny stable periodic motion with microslips.

The electric current in panel (e) also shows a definite local maximum in the same critical speed range. It must be mentioned that the characteristics of the electric motor driving the conveyor belt is not known and the dashed line in panel (e) illustrates only the estimated power requirement caused by the friction forces between the belt and the plastic plate under the belt. A certain percentage of the power increments in the range of the increased current may also be related to the change in the efficiency of the motor at different speeds, but still, a relevant part of this power increment must be related to the temperature increase of the tyre through the microslips caused by the tiny shimmy motion.

These measurement results confirm the existence of the linearly unstable island in the stability chart for long caster where the self-excited vibration has so small amplitude due to the microslips that it cannot be distinguished from the background noise in the system. This would explain the introduction of the idea of ‘micro-shimmy’. Although, this vibration can hardly be detected by means of dynamic measurements, its practical relevance is unquestionable since it causes increased resistance to rolling, increased thermal load and increased wear of the tyre. Since the rolling resistance consumes substantial energy (see Miege and Popov, 2005), simple adaptive control strategies may help to reduce tyre wear and fuel consumption by avoiding these critical towing speed ranges. It may be also important to know that efficiency of the ABS is reduced for these towing speed domains due to the many fractional microslips within the contact patch.
5.4 New results

**Thesis 5** By means of the modification of a conveyor belt, a test rig was designed and constructed to carry out a series of experiments on the shimmy of towed wheels of elastic tyres. The most relevant theoretically predicted stability boundary and the corresponding vibration frequencies of the arising self-excited vibrations were identified during the experiments in a wide range of towing speeds for caster lengths less than the sum of the half contact length and the tyre relaxation length \((l < a + \sigma)\).

The memory effect of the delayed tyre model was also validated by the modal testing of the towed wheel during rolling with various towing speeds. The impact experiments clearly identified three branches of frequencies belonging to three different pairs of complex characteristic roots with near-zero real parts for short caster \((l < a + \sigma)\), while a single branch of frequencies was found for long caster \((l > a + \sigma)\), which perfectly followed the predicted variation of a relevant pair of characteristic roots around the imaginary axis of the complex plane.

These results are published in Takács and Stépán (2010).

**Thesis 6** The nonlinear behaviour of the system with respect to the geometric nonlinearities and Coulomb friction force was analysed by numerical simulations. The partial sliding of the contact patch was modelled by the implementation of the governing equation of the damped stretched string tyre model into the simulation code as a phenomenon of fast time scale relative to the wheel oscillation of slow time scale. The experimentally detected transient and fully developed vibration signals were compared to these simulation results, and a good match was presented.

The memory effect of the delayed tyre model was confirmed also by the measured transient small-amplitude vibrations and the fully developed large-amplitude quasi-periodic ones. Waterfall diagrams were measured and simulated which explained the evolution of the peaks in the spectra as the nonlinear vibrations developed for parameters near to the theoretically predicted double Hopf bifurcation points of the stability charts. This validated our fast and slow time scale approximation in the improved mechanical model including partial sliding within the contact patch.

These results are published in Takács and Stépán (2009).

**Thesis 7** Experimental procedure was developed to measure the increased temperature of the tyre trailing point and the power increment of the conveyor belt due to the possible increased rolling resistance of the towed wheel at certain speeds. With the help of these measurements, we proved the physical existence of the linearly unstable islands in the stability charts for long caster \((l > a + \sigma)\) where the self-excited vibration has so small amplitude due to the microslips that it cannot be distinguished from the background noise in the system. This way we also proved that the memory effect in the delayed tyre model is relevant also for long caster at certain towing speed ranges where the shimmy motion itself is hidden but its effect is relevant with respect to increased tyre temperature, rolling resistance and wear, and possible decreased ABS efficiency if implemented.
Bibliography


Youtube: 2006, Bus in Tunnel!  
[URL](http://www.youtube.com/watch?v=0D4kEQLu5cA)

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