Asymptotic behaviour of random walks with long memory

outline of PhD thesis

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Introduction

The concept of random walks is a crucial subject in modern probability. The simple random walk can be defined as a sequence of i.i.d. (independent and identically distiributed) steps on the integer lattice $\mathbb{Z}^d$. The behaviour of this basic model is well understood, and there are natural ways to generalize it by relaxing the condition of independence of steps. The main difficulty here is the loss of Markov property, which makes standard methods not applicable. Hence the analysis of random walks with memory requires a completely new approach and different ideas.

More specifically, in the thesis, we consider self-repelling random walks which means that the random walker is pushed to areas which were less visited in the past. The definition can be made precise by introducing the local times which denote the amount of time spent on certain vertices (or edges) of the underlying lattice. In the main model of the present thesis, the transition rules of the random walker are given in terms of the discrete gradient of local times: the walker is driven locally by the *negative* discrete gradient of its own local time. This concept was originally called the *'true' self-avoiding random walk*, because it is a true walk, i.e. a path of length $n+1$ can be sampled by performing one step according to some distribution after a path of length n. Note that it is not the case for the self-avoiding walk. In order to avoid confusion, throughout the thesis, we use the more intuitive name *myopic self-avoiding walk* (MSAW) instead of the 'true' self-avoiding walk.

The continuous space counterpart of the MSAW is also investigated in the thesis. The driving mechanism has essentially the same spirit: it is based on the local time profile, but its irregularity is smeared out by a convolution. The diffusion process defined in this way is the *self-repellent Brownian polymer* (SRBP) model.

The general motivation of the research which the present thesis is based on originates in the field of statistical physics. The central subject of our investigations, the MSAW was first introduced in the physics literature by Amit, Parisi and Peliti in 1983, see [APP83]. It was the first example for a non-trivial random walk with long memory which behaves qualitatively differently from the usual diffusive behaviour of random walks. The original definition of these authors is given in the following subsection.

This model belongs to the wider class of *self-interacting random walks* which attracted attention in recent times. Typical other examples are the *self-repellent Brownian polymer* (SRBP) model, the self-avoiding walk or the reinforced random walk. In all these cases, long memory of the random walk or diffusion is induced by a self-interaction mechanism defined locally in a natural way in terms of the local time (or occupation time) process. The asymptotic scaling behaviour of self-interacting random walks and processes has been a mathematical challenge since the early eighties. The two basic families of models considered in the physical and probabilistic literature, the MSAW and the SRBP model, although having their origins in different cultures and having different motivations, are phenomenologically very similar.

The myopic self-avoiding random walk model

Let $X(n)$ be a nearest neighbour random walk on the integer lattice $\mathbb{Z}^d$ which starts from $X(0)=0$. Denote its local time on the vertices $x \in \mathbb{Z}^d$ by

$$\ell(n,x) := \#\{0 < k \le n : X(k) = x\} \tag{1}$$

where $\#\{\dots\}$ denotes the cardinality of the set. Let $X(n)$ be governed by the evolution rules

$$\begin{aligned}\mathbf{P}\left(X(n+1)=x+e \,\middle|\, \mathcal{F}_n, X(n)=x\right) &= \frac{\exp\{-\beta\ell(n,x+e)\}}{\sum_{|e'|=1}\exp\{-\beta\ell(n,x+e')\}} \\ &= \frac{\exp\{-\beta(\ell(n,x+e)-\ell(n,x))\}}{\sum_{|e'|=1}\exp\{-\beta(\ell(n,x+e')-\ell(n,x))\}}\end{aligned} \tag{2}$$

where $|e|=1$, $\beta>0$ is a fixed constant, and $\mathcal{F}_n$ contains all the information up to time n including the local times. Then the random walk $X(n)$ is called the myopic self-avoiding walk (MSAW).

It has been already conjectured by the authors of [APP83] based on non-rigorous renormalization group arguments that the upper critical dimension of the MSAW is two. It means that in higher dimensions, the MSAW behaves diffusively similarly to the simple random walk, and logarithmic corrections appear in two dimensions. The one-dimensional behaviour was expected to be super-diffusive. In [PP87], Peliti and Pietronero used non-rigorous scaling arguments to show that the typical order of the displacement in one dimension is 2/3th power of time, but with no hint about the limiting distribution. For renormalization of the MSAW, see also [OP83].

In many cases, it is more convenient to speak about MSAW in continuous time. One of the models of this thesis is also treated in the continuous time setting. The definition can be modified in a straightforward way, and it is given precisely later in Section 3.

The transition probabilities of the MSAW given by the last expression of (2) are indeed proportional to the exponential function of the negative discrete gradient of local times. We generalize the definition slightly by replacing the exponential function with an arbitrary non-decreasing function $w:\mathbb{R}\to\mathbb{R}_+$ with some mild technical assumptions imposed later. Formally, instead of (2), we use the definition

$$\mathbf{P}\left(X(n+1)=x+e \,\middle|\, \mathcal{F}_n, X(n)=x\right) = \frac{w(\ell(n,x)-\ell(n,x+e))}{\sum_{|e'|=1} w(\ell(n,x)-\ell(n,x+e'))} \tag{3}$$

which gives back (2) with the choice $w(u)=e^{\beta u}$. For this generalized model, our methods remain applicable as it will be seen in Section 2 and Section 3.

Self-repellent Brownian polymer model

We denote by $X(t)$ the d-dimensional self-repellent Brownian polymer which is a continuous time $\mathbb{R}^d$-valued stochastic process. (The same letter X is used as for the MSAW, but it will be always clear from the context which of them is under discussion.) The local time or occupation time measure here is given by

$$\ell(t,A) := |\{0<s\le t : X(s)\in A\}| \tag{4}$$

for any $A\subseteq\mathbb{R}^d$ measurable subset where $|\{\dots\}|$ is the Lebesgue measure of the set. Let $V:\mathbb{R}^d\to\mathbb{R}_+$ be a smooth spherically symmetric approximate identity, for instance, we may choose $V(x):=\exp(-|x|^2)$. Then, the SRBP is governed by the equation

$$\mathrm{d}X(t) = \mathrm{d}B(t) - \operatorname{grad}(V*\ell(t,\cdot))(X(t))\,\mathrm{d}t \tag{5}$$

where $*$ stands for convolution in $\mathbb{R}^d$ and $B(t)$ is a standard Brownian motion. The drift term in (5) is the negative gradient of the local time smeared out by convolution with V and taken at the current position.

The SRBP was first introduced by Norris, Rogers and Williams in 1987 in [NRW87]. Later, it has also been analysed mainly in one dimension in [DR92], [CL95] and [CM96]. The work [MT08] contains a survey of earlier results as well. In the recent paper [TTV11], super-diffusive bounds are given for the one-dimensional Brownian polymer.

The form (5), compared with (3), shows explicitly the phenomenological similarity with the MSAW. The non-rigorous scaling and renormalization group arguments originally formulated for the MSAW are equally well applicable to the SRBP. They suggest that the the dimension-dependent asymptotic scaling behaviour given for the MSAW above is also valid for the SRBP.

Overview of results on MSAW and SRBP

The first mathematically rigorous result for the MSAW was given by Tóth in 1995 in [T95]. He considered a modified model of the MSAW where the local times are defined on the *edges* of $\mathbb{Z}$. He proved a limit theorem for the sequence of local times of the random walk. The limit can be given in terms of a Brownian motion reflected in the Skorohod sense. He also gave a local limit theorem for the properly rescaled displacement in a late random time, which justified the conjectured scaling exponent 2/3.

A key point in the proof of [T95] is a kind of Ray – Knight type argument which works for the MSAW with edge repulsion but not for the original MSAW with site repulsion $X(n)$ given by (2). For the original idea of Ray – Knight theory, see [K63] and [R63]. It turns out that the sequence of local times of the MSAW with edge repulsion stopped at an inverse local time is a Markov chain and it can be thoroughly analysed.

It is a fact that the similar reduction does not hold for the original MSAW with site repulsion. The corresponding process is not Markovian in this case and thus the Ray – Knight type of approach fails. Hence no limit theorem is known for the original MSAW model with site repulsion in discrete time.

Later, in [TW98], Tóth and Werner constructed the *true self-repelling motion* which is believed to be the scaling limit of the one-dimensional MSAW. The construction of the process is intricate. It is based on an uncountable collection of coalescing Brownian motions starting from each point of the two-dimensional space-time. This system of trajectories was later called the Brownian web, see [FINR04]. The local time profile of the true self-repelling motion is constructed using the Brownian web, and the process itself can be recovered from that. The true self-repelling motion possesses all the analytic and stochastic properties of an assumed scaling limit of $A^{-2/3}X(\lfloor At \rfloor)$ as $A \to \infty$. The invariance principle for the MSAW model with edge repulsion has been clarified in [NR06].

The proofs of limit theorems in [T95] (and also in subsequent papers) have some built-in combinatorial elements which make it difficult (if possible at all) to extend these proofs robustly to a full class of 1d models of random motions pushed by the negative gradient of their occupation time measure. However, more recently, a robust proof was given for the super-diffusive behaviour of the 1d models: in [TTV11], inter alia, it is proved that for the $1d$ SRBP models $\underline{\lim}_{t\to\infty} t^{-5/4}\mathbf{E}\left(X(t)^2\right) > 0$ and $\overline{\lim}_{t\to\infty} t^{-3/2}\mathbf{E}\left(X(t)^2\right) < \infty$. These are robust super-diffusive bounds (not depending on microscopic details) but still far from the expected $t^{2/3}$ scaling.

Next, we present an overview about the most important results on the MSAW and

SRBP which have appeared in the literature so far.

One dimension: $X(n) \sim n^{2/3}$ with non-Gaussian scaling limit conjectured;

- Conjectures, renormalization group arguments in [APP83], [PP87];
- MSAW with edge repulsion: Tóth in [T95] and explained above;
- Construction of the true self-repelling motion as a scaling limit of one-dimensional MSAW: Tóth and Werner in [TW98] and above;
- MSAW with *oriented* edge repulsion: Tóth and Vető in [TV08] and Section 2;
- MSAW in continuous time with site repulsion: Tóth and Vető in [TV11] and Section 3;
- On-line demonstration of the one-dimensional MSAW models: Vető in [V09];
- Super-diffusive bounds on the one-dimensional SRBP: Tarrès, Tóth and Valkó in [TTV11].

Two dimensions: $X(n) \sim n^{1/2} \log^{1/4} n$ with Gaussian scaling limit conjectured;

- Conjectures, renormalization group arguments in [APP83], [OP83];
- Super-diffusive bounds on the SRBP: Tóth and Valkó in [TV10].

Three or higher dimensions: $X(n) \sim n^{1/2}$ with Gaussian scaling limit conjectured;

- Conjectures, renormalization group arguments in [APP83], [OP83];
- Diffusive bounds and central limit theorem for the SRBP: Horváth, Tóth and Vető in [HTV11] and Section 4;
- Diffusive bounds and central limit theorem for the MSAW: Horváth, Tóth and Vető in [HTV11]. This result is *not part of the present thesis.*

1 Reflected Brownian trajectories

The study of 1d Brownian trajectories pushed up or down by Skorohod reflection on some other Borwnian trajectories (running backwards in time) was initiated in [STW00] and motivated in [TW98] by the construction of the object what is today called the Brownian web, see [FINR04]. It turns out that these Brownian paths, reflected on one another, have very interesting, sometimes surprising properties. For further studies of Skorohod reflection of Brownian paths on one another, see also [SW02], [BN02], [W07] etc. In particular, in [W07], Warren considers two interlaced families of Brownian trajectories with paths belonging to the second family reflected off by paths belonging to the first (in Skorohod's sense) and derives a determinantal formula for the distribution of coalescing Brownian motions.

A particular case of Warren's formula is the following: fix a Brownian path and let two other Brownian paths be pushed upwards and respectively downwards by Skorohod reflection on the trajectory of the first one. The difference of the last two will be a three-dimensional Bessel process. Based on [TV07], in the thesis, we give an alternative, elementary proof of this fact which uses discrete approximation of the Brownian motion and Donsker's invariance principle.

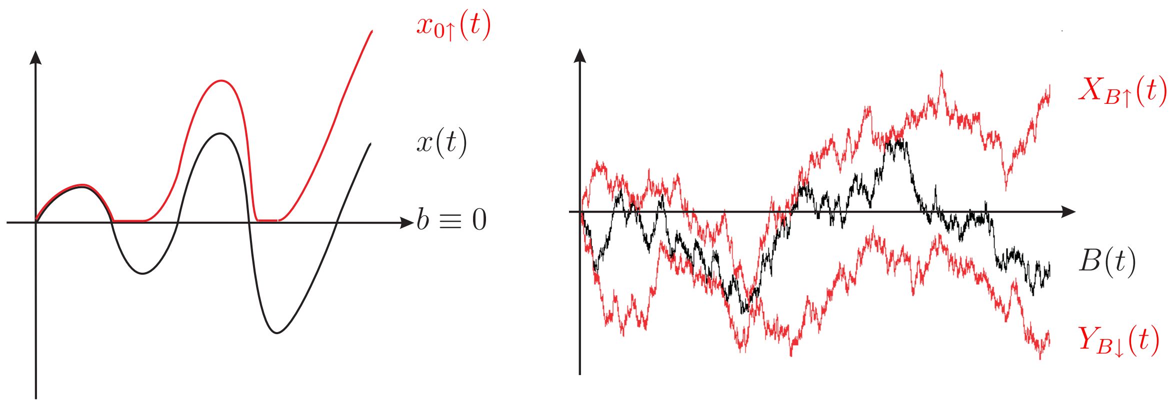


Figure 1: Example for Skorohod reflection and the reflected Brownian paths of Theorem 1.2

The Skorohod reflection

Let $T \in (0, \infty)$ and $b, x : [0, T) \to \mathbb{R}$ be continuous functions. Assume $x(0) \geq b(0)$. The construction of the following proposition is due to Skorohod. Its proof can be found either in [RY99] (see Lemma 2.1 in Chapter VI) or in [STW00] (see Lemma 2 in Section 2.1)

Proposition 1.1. *(1) There exists a unique continuous function* $x_{b\uparrow} : [0, T) \to \mathbb{R}$ *with the following properties:*

- *The function* $x_{b\uparrow} - b$ *is non-negative.*
- *The function* $x_{b\uparrow} - x$ *is non-decreasing.*
- *The function* $x_{b\uparrow} - x$ *increases only when* $x_{b\uparrow} = b$. *That is*

$$\int_0^T \mathbb{1}(x_{b\uparrow}(t) \neq b(t))\, \mathrm{d}(x_{b\uparrow}(t) - x(t)) = 0.$$

(2) The function $t \mapsto x_{b\uparrow}(t)$ *is given by the construction*

$$x_{b\uparrow}(t) = x(t) + \sup_{0 \leq s \leq t} \big(x(s) - b(s)\big)_{-}.$$

(3) The map $C([0, T)) \times C([0, T)) \ni \big(b(\cdot), x(\cdot)\big) \mapsto \big(b(\cdot), x_{b\uparrow}(\cdot)\big) \in C([0, T)) \times C([0, T))$ *is contunuous in supremum distance.*

We call the function $t \mapsto x_{b\uparrow}(t)$ the *upwards Skorohod reflection* of $x(\cdot)$ on $b(\cdot)$. As it is remarked in [STW00], the term *Skorohod pushup* of $x(\cdot)$ by $b(\cdot)$ would be more adequate. Skorohod reflection on paths $b(t) = \text{const.}$ plays a fundamental role in the proper formulation and proof of Tanaka's formula, see Chapter VI of [RY99]. See also Figure 1.

The downwards Skorohod reflection or Skorohod pushdown is defined for continuous functions $b, y : [0, T) \mapsto \mathbb{R}$ with $y(0) \leq b(0)$ by

$$y_{b\downarrow} := -\big((-y)_{(-b)\uparrow}\big), \qquad y_{b\downarrow}(t) = y(t) - \sup_{0 \leq s \leq t} \big(y(s) - b(s)\big)_{+}.$$

Result on reflected Brownian paths

Let $B(t)$, $X(t)$ and $Y(t)$ be independent standard 1d Brownian motions starting from 0 and define

$$X^+(t) := X_{B\uparrow}(t), \qquad \widehat{X}(t) := \quad X^+(t) - B(t), \tag{6}$$

$$Y^-(t) := Y_{B\downarrow}(t), \qquad \widehat{Y}(t) := -Y^-(t) + B(t) \tag{7}$$

as seen on Figure 1. We are interested in the difference process

$$Z(t) := X^+(t) - Y^-(t) = \widehat{X}(t) + \widehat{Y}(t). \tag{8}$$

It is straightforward that $2^{-1/2}\widehat{X}(t)$ and $2^{-1/2}\widehat{Y}(t)$ are both standard reflected Brownian motions. They are, of course, strongly dependent.

The following fact is a particular consequence of the main results in [W07]:

Theorem 1.2. *The process $2^{-1/2}Z(t)$ is BES^3, that is, standard 3d Bessel process:*

$$\mathrm{d}Z(t) = 2\frac{1}{Z(t)}\mathrm{d}t + \sqrt{2}\mathrm{d}W(t), \qquad Z(0) = 0. \tag{9}$$

In the thesis, we present an elementary proof of this fact.

2 One-dimensional myopic self-avoiding walk with oriented edge repulsion

Based on [TV08], we consider a variant of self-repelling random walk on the integer lattice $\mathbb{Z}$ where the self-repellence is defined in terms of the local time on *oriented* edges. This model is similar to the one examined in [T95], but the walker here is pushed by the local differences of occupation time measures on *oriented* rather than unoriented edges.

The phenomenological behaviour is surprisingly different from the unoriented case. We prove limit theorems for the local time process and for the position of the random walker under square-root-of-time (rather than time-to-the-2/3) space-scaling, but the limit laws are not the usual diffusive ones. The main ingredient is a Ray – Knight type of approach.

We formulate a limit theorem about the convergence in sup-norm and in probability of the local time process stopped at inverse local times. As a consequence, we also prove convergence in probability of the inverse local times to *deterministic values*. Then, we convert the limit theorems for the inverse local times to local limit theorems for the position of the random walker at independent random stopping times of geometric distribution with large expectation. Finally, we present some numerical simulations of the position and local time processes with particular choices of the weight function. The figures show the strange scaling behaviour of the walk considered.

Definition of the model

Let w be a weight function which is non-decreasing and non-constant:

$$w : \mathbb{Z} \to \mathbb{R}_+, \qquad w(z+1) \geq w(z), \qquad \lim_{z\to\infty}\big(w(z) - w(-z)\big) > 0. \tag{10}$$

We will consider a nearest neighbour random walk $X(n)$, $n \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$, on the integer lattice $\mathbb{Z}$, starting from $X(0) = 0$, which is governed by its local time process through the function w in the following way. Denote by $\ell^{\pm}(n, k)$, $(n, k) \in \mathbb{Z}_+ \times \mathbb{Z}$, the local time (that is: its occupation time measure) on oriented edges:

$$\ell^{\pm}(n, k) := \#\{0 \le j \le n - 1 : X(j) = k, X(j + 1) = k \pm 1\}$$

where $\#\{\dots\}$ denotes cardinality of the set. We will also use the notation

$$\ell(n, k) := \ell^{+}(n, k) + \ell^{-}(n, k + 1) \tag{11}$$

for the local time spent on the unoriented edge $\langle k, k + 1\rangle$.

Our random walk is governed by the evolution rules

$$\begin{aligned}
&\mathbf{P}\left(X(n + 1) = X(n) \pm 1 \bigm| \mathcal{F}_n\right) \\
&\quad = \frac{w(\mp(\ell^{+}(n, X(n)) - \ell^{-}(n, X(n))))}{w(\ell^{+}(n, X(n)) - \ell^{-}(n, X(n))) + w(\ell^{-}(n, X(n)) - \ell^{+}(n, X(n)))}, \\
&\ell^{\pm}(n + 1, k) = \ell^{\pm}(n, x) + 1\!\!1(X(n) = k, X(n + 1) = k \pm 1).
\end{aligned} \tag{12}$$

That is: at each step, the walk prefers to choose that oriented edge pointing away from the actually occupied site which had been less visited in the past. In this way, it balances or smoothes out the roughness of the occupation time measure.

The main results

As in [T95], the key to the proof is a Ray – Knight approach. Let

$$T^{\pm}_{j,r} := \min\{n \ge 0 : \ell^{\pm}(n, j) \ge r\}, \qquad j \in \mathbb{Z}, \quad r \in \mathbb{Z}_+$$

be the so-called inverse local times and

$$\Lambda^{\pm}_{j,r}(k) := \ell(T^{\pm}_{j,r}, k) = \ell^{+}(T^{\pm}_{j,r}, k) + \ell^{-}(T^{\pm}_{j,r}, k + 1), \qquad j, k \in \mathbb{Z}, \quad r \in \mathbb{Z}_+ \tag{13}$$

the local time sequence (on unoriented edges) of the walk stopped at the inverse local times. See Figure 2. We denote by $\lambda^{\pm}_{j,r}$ and $\rho^{\pm}_{j,r}$ the leftmost, respectively, the rightmost edges visited by the walk before the stopping time $T^{\pm}_{j,r}$:

$$\begin{aligned}
\lambda^{\pm}_{j,r} &:= \inf\{k \in \mathbb{Z} : \Lambda^{\pm}_{j,r}(k) > 0\}, \\
\rho^{\pm}_{j,r} &:= \sup\{k \in \mathbb{Z} : \Lambda^{\pm}_{j,r}(k) > 0\}.
\end{aligned}$$

We remark that, for fixed $j \in \mathbb{Z}$ and $r \in \mathbb{Z}_+$, all the quantities $T^{\pm}_{j,r}$, $\rho^{\pm}_{j,r} - \lambda^{\pm}_{j,r}$, $\sup_k \Lambda^{\pm}_{j,r}(k)$ are finite. The main result concerning the local time process stopped at inverse local times is the following:

Theorem 2.1. *Let $x \in \mathbb{R}$ and $h \in \mathbb{R}_+$ be fixed. Then*

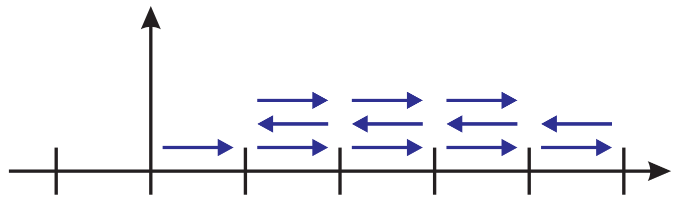
$$A^{-1}\lambda^{\pm}_{\lfloor Ax\rfloor,\lfloor Ah\rfloor} \overset{\mathbf{P}}{\longrightarrow} -|x| - 2h, \tag{14}$$

$$A^{-1}\rho^{\pm}_{\lfloor Ax\rfloor,\lfloor Ah\rfloor} \overset{\mathbf{P}}{\longrightarrow} \quad |x| + 2h, \tag{15}$$

and

$$\sup_{y\in\mathbb{R}} \left| A^{-1}\Lambda^{\pm}_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(\lfloor Ay\rfloor) - (|x| - |y| + 2h)_+ \right| \overset{\mathbf{P}}{\longrightarrow} 0 \tag{16}$$

as $A \to \infty$.



$$\Lambda^+_{3,2} = \quad 0 \quad 1 \quad 3 \quad 3 \quad 3 \quad 2 \quad 0$$

Figure 2: The local time sequence of the walk stopped at the inverse local time $T^+_{3,2} = 12$

Note that

$$T^{\pm}_{j,r} = \sum_{k=\lambda^{\pm}_{j,r}}^{\rho^{\pm}_{j,r}} \Lambda^{\pm}_{j,r}(k).$$

Hence, it follows immediately from Theorem 2.1 that

Corollary 2.2. *With the notations of Theorem 2.1,*

$$A^{-2}T^{\pm}_{\lfloor Ax\rfloor,\lfloor Ah\rfloor} \xrightarrow{\mathbf{P}} (|x|+2h)^2 \tag{17}$$

as $A \to \infty$.

Remark. Note that the local time process and the inverse local times converge in probability to deterministic objects rather than converging weakly in distribution to genuinely random variables. This makes the present case somewhat similar to the weakly reinforced random walks studied in [T97].

Let

$$\varphi(t,x) = \frac{1}{2\sqrt{t}}\mathbb{1}(|x| \le \sqrt{t})$$

be the density function of the UNI$(-\sqrt{t}, \sqrt{t})$ distribution, which is a natural candidate for being the marginal distribution of the scaling limit of the displacement. In order to prove a *local limit theorem* for the position of the random walker, some smoothening in time is needed, which is realized through the Laplace transform. Let

$$\widehat{\varphi}(s,x) := s\int_0^\infty e^{-st}\varphi(t,x)\mathrm{d}t = \sqrt{s\pi}(1-\Phi(\sqrt{2s}|x|))$$

where $\Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-y^2/2}\mathrm{d}y$ is the standard normal distribution function.

Theorem 2.3. *Let* $s \in \mathbb{R}_+$ *be fixed and* $\theta_{s/A}$ *a random variable with geometric distribution*

$$\mathbf{P}\left(\theta_{s/A} = n\right) = (1-e^{-s/A})e^{-sn/A} \tag{18}$$

which is independent of the random walk $X(n)$*. Then, for almost all* $x \in \mathbb{R}$*,*

$$A^{1/2}\mathbf{P}\left(X(\theta_{s/A}) = \lfloor A^{1/2}x\rfloor\right) \to \widehat{\varphi}(s,x)$$

as $A \to \infty$.

From (14) and (15), the tightness of the distributions $(A^{-1/2}X(\lfloor At\rfloor))_{A\ge 1}$ follows easily. Theorem 2.3 yields that if the random walk $X(\cdot)$ has any scaling limit, then

$$A^{-1/2}X(\lfloor At\rfloor) \Longrightarrow \mathrm{UNI}(-\sqrt{t}, \sqrt{t}). \tag{19}$$

Sketch proof of the limit theorems

The proof of Theorem 2.1 is organized as follows. We introduce independent auxiliary Markov chains η_k associated to the vertices $k \in \mathbb{Z}$ with the same distribution. These Markov chains are essentially the differences of local times on adjacent edges. The law of these Markov chains is analysed, the stationary distribution ρ is identified and an exponential bound is given on the rate of convergence to the stationary measure. It turns out that the stationary mean is $-1/2$.

The independence of η_k's enables us to represent the local time sequence

$$L_{j,r}(k) := \ell^+(T^+_{j,r}, k)$$

(which is essentially the "half" of $\Lambda^{\pm}_{j,r}$) as a random walk for some fixed $j \in \mathbb{Z}_-$ and $r \in \mathbb{Z}_+$, that is, we follow a Ray−Knight approach. We can deduce

$$\begin{aligned}
&L_{j,r}(j) = r, \\
&L_{j,r}(k+1) = L_{j,r}(k) + 1 + \eta_{k+1}(L_{j,r}(k)+1), \qquad && j \le k < 0, \\
&L_{j,r}(k+1) = L_{j,r}(k) + \eta_{k+1}(L_{j,r}(k)), && 0 \le k < \infty, \\
&L_{j,r}(k-1) = L_{j,r}(k) + \eta_{k}(L_{j,r}(k)), && -\infty < k \le j.
\end{aligned} \tag{20}$$

These formulas yield that starting from position $j \in \mathbb{Z}_-$, one might generate the sequence of local times $L_{j,r}(k)$ step by step by always adding a new independent random variable, that is, as a random walk. Note that the expected jump size is $1/2$ between j and 0, $-1/2$ above 0 and $-1/2$ below j backwards. It is in accordance with (16). At this point, we have to show that the two-sided random walk $L_{j,r}(k)$ does not differ too much from the expected behaviour.

For taking the scaling limit, we set $j = \lfloor Ax \rfloor$ and $r = \lfloor Ah \rfloor$. We can prove that, as long as $L_{\lfloor Ax \rfloor, \lfloor Ah \rfloor}(k) > A^{1/2+\varepsilon}$, the Markov chains η_k which appear in the increments $L_{\lfloor Ax \rfloor, \lfloor Ah \rfloor}(k) - L_{\lfloor Ax \rfloor, \lfloor Ah \rfloor}(k-1)$ are exponentially close to the stationary distribution. This enables us to couple them efficiently to the i.i.d. copies $\widetilde{\eta}_k(m)$ of the *stationary* Markov chain, that is, for which $\mathbf{P}\left(\widetilde{\eta}_k(m) = x\right) = \rho(x)$ for all $k \in \mathbb{Z}$ and $m \in \mathbb{Z}$.

By replacing η_k's by $\widetilde{\eta}_k$'s in (20), we can define $\widetilde{L}_{j,r}(k)$ which has increments with distribution ρ. It is a standard large deviation estimate that the fluctuation of the random walk $\widetilde{L}_{j,r}$ around its mean value is uniformly smaller than $A^{1/2+\varepsilon}$ with high probability. By the construction of the coupling, it can be seen that $L_{\lfloor Ax \rfloor, \lfloor Ah \rfloor}(k) = \widetilde{L}_{\lfloor Ax \rfloor, \lfloor Ah \rfloor}(k)$ with high probability as long as both are above the threshold $A^{1/2+\varepsilon}$. This coupling breaks down when $L_{\lfloor Ax \rfloor, \lfloor Ah \rfloor}(k)$ approaches 0, but we can show that, after that it happens, L hits 0 in linear, i.e. $A^{1/2+\varepsilon}$ time with high probability. This completes the proof of Theorem 2.1.

The crucial identity for the proof of Theorem 2.3 is

$$\mathbf{P}\left(X(n) = k\right) = \sum_{m=0}^{\infty} \left(\mathbf{P}\left(T^+_{k-1,m} = n\right) + \mathbf{P}\left(T^-_{k+1,m} = n\right) \right) \tag{21}$$

which connects the distribution of the displacement $X(n)$ with that of the inverse local times $T^{\pm}_{k,m}$. Therefore, we can use the limit theorem stated in Corollary 2.2.

Computer simulations

We present computer simulation results on Figure 3. The first graph shows the sequence of local times of the random walk with $w(k) = 10^k$ after approximately 10^6 steps. More

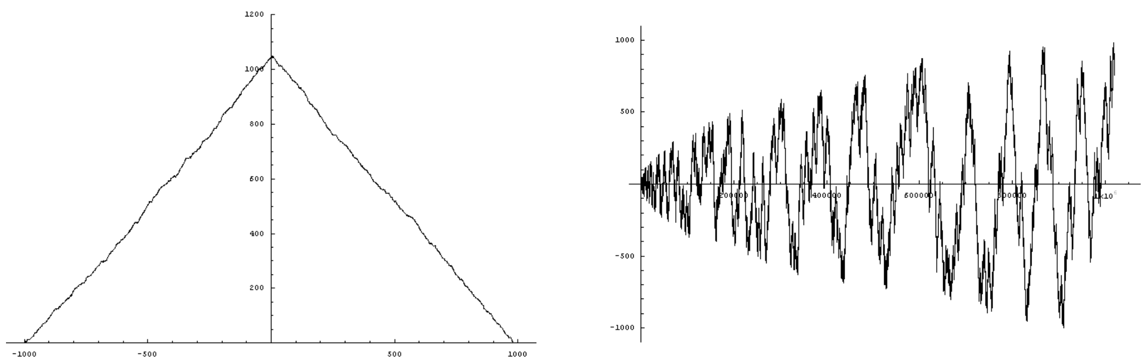


Figure 3: The local time process and the trajectories of the random walk with $w(k) = 10^k$

precisely, we have plotted the values of $\Lambda^+_{100,800}$. One the second graph, one can see the corresponding trajectory.

The sequence of local times fits apparently the theoretical value according to Theorem 2.1. The trajectory is more interesting. It draws a sharp upper and lower hull according to $\sqrt{t}$ and $-\sqrt{t}$. On the other hand, it oscillates very heavily between its extreme values. There are almost but not quite straight crossings from $\sqrt{t}$ to $-\sqrt{t}$ and back. It shows that there is no continuous scaling limit of the self-repelling random walk with directed edges.

If the random walk explores a new region, e.g. it exceeds its earlier maximum, then the probability of the reversal is $1/2$ independently of w, since both outgoing edges have local time 0. It can be a heuristic argument, why the upper and lower hulls $\sqrt{t}$ and $-\sqrt{t}$ are universal.

3 One-dimensional myopic self-avoiding walk with site repulsion in continuous time

In the present section and in [TV11], we consider the continuous time version of the 'true' or 'myopic' self-avoiding random walk (MSAW) with site repulsion in $1d$. The Ray−Knight type method which was applied first in [T95] and also in Section 2 to the discrete time and (unoriented and oriented) edge repulsion case is applicable to this model with some modifications. We present a limit theorem for the local time of the walk and a local limit theorem for the displacement. The main ideas of this section are similar to those of [T95], but there are essential differences, too. In particular, we use some new coupling arguments in the proofs.

The random walk considered and the main results

Now, we define a version of myopic self-avoiding random walk in continuous time which is a counterpart of the one given in (3) and for which the Ray−Knight type method is applicable. Let $X(t)$, $t \in \mathbb{R}_+$ be a *continuous time* random walk on $\mathbb{Z}$ starting from $X(0) = 0$ and having right continuous paths. Denote by $\ell(t, x)$, $(t, x) \in \mathbb{R}_+ \times \mathbb{Z}$ its local

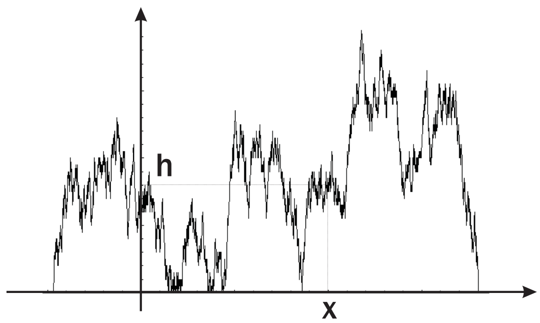


Figure 4: The two-sided reflected and absorbed Brownian motion $W_{x,h}$

time (occupation time measure) on sites:

$$\ell(t,x) := |\{s \in [0,t) \; : \; X(s) = x\}| \tag{22}$$

where $|\{\dots\}|$ denotes Lebesgue measure of the set indicated. Let $w : \mathbb{R} \to (0,\infty)$ be an almost arbitrary rate function. We assume that it is non-decreasing and not constant.

The law of the random walk is governed by the following jump rates and differential equations (for the local time increase):

$$\mathbf{P}\left(X(t+\mathrm{d}t) = x \pm 1 \;\middle|\; \mathcal{F}_t, \; X(t) = x\right) = w(\ell(t,x) - \ell(t,x \pm 1))\,\mathrm{d}t + o(\mathrm{d}t), \tag{23}$$

$$\dot{\ell}(t,x) = 1\!\!1(X(t) = x) \tag{24}$$

with initial conditions

$$X(0) = 0, \qquad \ell(0,x) = 0.$$

The dot in (24) denotes time derivative. Note that, for the choice of exponential weight function $w(u) = \exp\{\beta u\}$, this means exactly that conditionally on a jump occurring at the instant t, the random walker jumps to right or left from its actual position with probabilities $e^{-\beta\ell(t,x\pm1)}/(e^{-\beta\ell(t,x+1)} + e^{-\beta\ell(t,x-1)})$, just like in (2).

Fix $j \in \mathbb{Z}$ and $r \in \mathbb{R}_+$. We consider the random walk $X(t)$ running from $t = 0$ up to the stopping time

$$T_{j,r} = \inf\{t \geq 0 : \ell(t,j) \geq r\} \tag{25}$$

which is the inverse local time for our model. Define

$$\Lambda_{j,r}(k) := \ell(T_{j,r}, k) \qquad k \in \mathbb{Z} \tag{26}$$

the local time process of X stopped at the inverse local time.

Let

$$\lambda_{j,r} := \inf\{k \in \mathbb{Z} : \Lambda_{j,r}(k) > 0\},$$
$$\rho_{j,r} := \sup\{k \in \mathbb{Z} : \Lambda_{j,r}(k) > 0\}.$$

Fix $x \in \mathbb{R}$ and $h \in \mathbb{R}_+$. Consider the two-sided reflected Brownian motion $W_{x,h}(y)$, $y \in \mathbb{R}$ with starting point $W_{x,h}(x) = h$. See Figure 4. Define the times of the first hitting of 0 outside the interval $[0,x]$ or $[x,0]$ with

$$\mathfrak{l}_{x,h} := \sup\{y < 0 \wedge x : W_{x,h}(y) = 0\},$$
$$\mathfrak{r}_{x,h} := \inf\{y > 0 \vee x : W_{x,h}(y) = 0\}$$

where $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$, and let

$$\mathcal{T}_{x,h} := \int_{\mathfrak{l}_{x,h}}^{\mathfrak{r}_{x,h}} W_{x,h}(y)\,\mathrm{d}y. \tag{27}$$

The main result of this section is

Theorem 3.1. *Let $x \in \mathbb{R}$ and $h \in \mathbb{R}_+$ be fixed. Then*

$$A^{-1}\lambda_{\lfloor Ax\rfloor,\lfloor\sqrt{A}\sigma h\rfloor} \Longrightarrow \mathfrak{l}_{0\wedge x,h}, \tag{28}$$

$$A^{-1}\rho_{\lfloor Ax\rfloor,\lfloor\sqrt{A}\sigma h\rfloor} \Longrightarrow \mathfrak{r}_{0\vee x,h}, \tag{29}$$

and

$$\left(\frac{\Lambda_{\lfloor Ax\rfloor,\lfloor\sqrt{A}\sigma h\rfloor}(\lfloor Ay\rfloor)}{\sigma\sqrt{A}}, \frac{\lambda_{\lfloor Ax\rfloor,\lfloor\sqrt{A}\sigma h\rfloor}}{A} \le y \le \frac{\rho_{\lfloor Ax\rfloor,\lfloor\sqrt{A}\sigma h\rfloor}}{A}\right) \Longrightarrow \left(W_{x,h}(y), \mathfrak{l}_{0\wedge x,h} \le y \le \mathfrak{r}_{0\vee x,h}\right) \tag{30}$$

as $A \to \infty$ where $\sigma^2 = \int_{-\infty}^{\infty} u^2 \rho(\mathrm{d}u) \in (0, \infty)$ where ρ can be given explicitly.

Corollary 3.2. *For any $x \in \mathbb{R}$ and $h \ge 0$,*

$$\frac{T_{\lfloor Ax\rfloor,\lfloor\sqrt{A}\sigma h\rfloor}}{\sigma A^{3/2}} \Longrightarrow \mathcal{T}_{x,h}. \tag{31}$$

For stating Theorem 3.3, we need some more definitions. It follows from (27) that $\mathcal{T}_{x,h}$ has an absolutely continuous distribution. Define

$$\varphi(t, x) := \frac{\partial}{\partial t}\int_0^\infty \mathbf{P}\left(\mathcal{T}_{x,h} < t\right)\mathrm{d}h. \tag{32}$$

Theorem 2 of [T95] gives that, for fixed $t > 0$, $\varphi(t, \cdot)$ is a density function, that is, $\int_{-\infty}^{\infty} \varphi(t, x)\,\mathrm{d}x = 1$. One could expect that $\varphi(t, \cdot)$ is the density of the limit distribution of $X(At)/A^{2/3}$ as $A \to \infty$, but we prove a similar statement for their Laplace transform. We denote by $\widehat{\varphi}$ the Laplace transforms of φ:

$$\widehat{\varphi}(s, x) := s\int_0^\infty e^{-st}\varphi(t, x)\,\mathrm{d}t. \tag{33}$$

Theorem 3.3. *Let $s \in \mathbb{R}_+$ be fixed and $\theta_{s/A}$ a random variable of exponential distribution with mean A/s which is independent of the random walk $X(t)$. Then, for almost all $x \in \mathbb{R}$,*

$$A^{2/3}\mathbf{P}\left(X(\theta_{s/A}) = \lfloor A^{2/3}x\rfloor\right) \to \widehat{\varphi}(s, x) \tag{34}$$

as $A \to \infty$.

4 Central limit theorem for the self-repellent Brownian polymer in three or higher dimensions

As a continuous space-time counterpart of the myopic self-avoiding walk, we investigate the asymptotic behaviour of the self-repellent Brownian polymer (SRBP) in the non-recurrent dimensions. First, extending $1d$ results from [TTV11], we identify a natural time-stationary and ergodic distribution of the environment (essentially smeared-out occupation time measure of the process) as seen from the moving particle. As a main result, we prove that, in three and more dimensions, in this stationary (and ergodic) regime, the displacement of the moving particle scales diffusively and its finite dimensional distributions converge to those of a Wiener process.

The results of the present section which are based on [HTV11] settles parts of the conjectures in [APP83]. These conjectures were based on scaling and renormalization group arguments, but we present a rigorous proof which is a first mathematical result in the high dimensional regime. The main tool is the non-reversible version of the Kipnis – Varadhan type central limit theorem for additive functionals of ergodic Markov processes and the *graded sector condition* of Sethuraman, Varadhan and Yau, see [SVY00].

Definition of model and background

The self-repellent Brownian polymer (SRBP) model is defined as follows. We give a more precise definition here than the one in (4)–(5). Let $V : \mathbb{R}^d \to \mathbb{R}_+$ be an approximate identity, that is, a smooth (C^∞), spherically symmetric function with sufficiently fast decay at infinity, and

$$F : \mathbb{R}^d \to \mathbb{R}^d, \qquad F(x) := -\mathrm{grad}\, V(x). \tag{35}$$

For reasons which will be clarified later, we also impose the condition of *positive definiteness* of V:

$$\widehat{V}(p) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ip\cdot x} V(x)\, \mathrm{d}x \geq 0. \tag{36}$$

A particular choice could be $V(x) := \exp\{-|x|^2/2\}$.

Let $t \mapsto B(t) \in \mathbb{R}^d$ be standard d-dimensional Brownian motion and define the stochastic process $t \mapsto X(t) \in \mathbb{R}^d$ as the solution of the SDE

$$\mathrm{d}X(t) = \mathrm{d}B(t) + \left(\int_0^t F(X(t) - X(u))\, \mathrm{d}u \right) \mathrm{d}t. \tag{37}$$

Now, we introduce the occupation time measure

$$\ell(t, A) := \ell(0, A) + |\{0 < s \leq t : X(s) \in A\}| \tag{38}$$

where $A \subset \mathbb{R}^d$ is any measurable domain, and $\ell(0, A)$ is some signed initialization. The SDE (37) is equivalent to (5). In the present section, we address the three or higher dimensional case of the SRBP model.

Our general approach to the central limit theorem is that of martingale approximation for additive functionals of ergodic Markov processes initiated for reversible processes in the classic Kipnis – Varadhan paper [KV86] and extended to non-reversible cases in [T86], [V96] and [SVY00]. We shall refer to this approach as the *Kipnis – Varadhan theory*. In particular, validity of the efficient martingale approximation will rely on checking the *graded sector condition* of [SVY00].

Next, we give a state space along with a Gaussian probability measure. We also define the environment process on the space which turns out to be stationary for the given measure. Then, we state the main results and give a sketch of proof.

State space, Gaussian measure and environment profile

The proper state space of our basic processes will be the space of smooth scalar fields of slow increase at infinity, i.e. $\Omega := \{\omega \in C^\infty(\mathbb{R}^d \to \mathbb{R}) : \|\,\omega\,\|_{m,r} < \infty\}$ where $\|\,\omega\,\|_{m,r}$ are the seminorms $\|\,\omega\,\|_{m,r} := \sup_{x\in\mathbb{R}^d}(1+|x|)^{-1/r}\left|\partial^{|m|}_{m_1,\dots,m_d}\omega(x)\right|$ defined for the multiindices $m = (m_1, \dots, m_d)$, $m_j \geq 0$; and $r \geq 1$. The space Ω endowed with these seminorms is a Fréchet space.

Similarly to the 1d stationary distribution in [TTV11], our aim is to define a Gaussian random variables associated to the points of $\mathbb{R}^d$ with expectation and covariances

$$\mathbf{E}\left(\omega(x)\right) = 0, \qquad C(x-y) := \mathbf{E}\left(\omega(x)\omega(y)\right) = g * V(x-y) \tag{39}$$

where

$$g : \mathbb{R}^d \to \mathbb{R}, \qquad g(x) := |x|^{2-d} \tag{40}$$

is the Green function of the Laplacian in $\mathbb{R}^d$. From Minlos's theorem (see [S74]), it follows that such a Gaussian probability measure exists uniquely on Ω. Note that throughout this section, $d \geq 3$. This is the *massless free Gaussian field* whose ultraviolet singularity is smeared out by convolution with the smooth and rapidly decaying approximate identity V.

The group of spatial translations $\tau_z : \Omega \to \Omega$, $(\tau_z\omega)(x) := \omega(x+z)$ with $z \in \mathbb{R}^d$ acts naturally on Ω and preserves the probability measure $\pi(\mathrm{d}\omega)$. Actually, the dynamical system $(\Omega, \pi(\mathrm{d}\omega), \tau_z : z \in \mathbb{R}^d)$ is *ergodic.*

We note that the environment profile (the smeared-out local times) appearing on the right-hand side of (5) and (37) and as seen in a moving coordinate frame tied to the current position of the process $\eta(t) = (\eta(t,x))_{x\in\mathbb{Z}}$ with

$$\eta(t,x) := \eta(0, X(t)+x) + \int_0^t V(X(t)+x-X(u))\,\mathrm{d}u = \big(V * \ell(t,\cdot)\big)(X(t)+x) \tag{41}$$

is a Markov process in Ω. Rather surprisingly, it turns out that the Gaussian measure characterized by (39) is the natural *time-stationary and ergodic distribution* of the process given in (41). All further results will be meant for the process being in this stationary regime.

The main result of this section refers to the *diffusive limit* of the process $t \mapsto X(t)$. From (5) and (41), it arises that the displacement is written as

$$X(t) = B(t) + \int_0^t \varphi(\eta(s))\,\mathrm{d}s \tag{42}$$

where $\varphi : \Omega \to \mathbb{R}^d$ is a function of the state of the stationary and ergodic Markov process $t \mapsto \eta(t)$:

$$\varphi(\omega) = -\mathrm{grad}\,\omega(0). \tag{43}$$

Results

Theorem 4.1. *The Gaussian probability measure $\pi(\mathrm{d}\omega)$ on Ω with mean 0 and covariances (39) is time-invariant and ergodic for the Ω-valued Markov process $t \mapsto \eta(t)$.*

Corollary 4.2. *For π-almost all initial profiles $\eta(0,\cdot)$,*

$$\lim_{t\to\infty} \frac{X(t)}{t} = 0 \quad \text{a.s.} \tag{44}$$

Remark. It is clear that, in dimensions $d \geq 3$, other stationary distributions of the process $t \mapsto \eta(t)$ exist. In particular, due to transience of the process $t \mapsto X(t)$, the stationary measure (presumably) reached from starting with "empty" initial conditions $\eta(0, x) \equiv 0$ certainly differs from our $\mathrm{d}\pi$. Our methods and results are valid for the particular stationary distribution $\mathrm{d}\pi$.

The main result of the present section is the following theorem:

Theorem 4.3. *In dimensions $d \geq 3$, the following hold:*

1. *The limiting variance*

$$\sigma^2 := d^{-1} \lim_{t\to\infty} t^{-1} \mathbf{E}\left(|X(t)|^2\right) \tag{45}$$

exists and

$$1 \leq \sigma^2 \leq 1 + \rho^2 \tag{46}$$

where

$$\rho^2 := d^{-1} \int_{\mathbb{R}^d} |p|^{-2}\, \widehat{V}(p)\, \mathrm{d}p < \infty. \tag{47}$$

2. *The finite dimensional marginal distributions of the diffusively rescaled process*

$$X_N(t) := \frac{X(Nt)}{\sigma\sqrt{N}} \tag{48}$$

converge to those of a standard d-dimensional Brownian motion. The convergence is meant in probability with respect to the starting state $\eta(0)$ sampled according to $\mathrm{d}\pi$.

Remark. The results are meant *in probability with respect to the initial profile $\eta(0, x)$* sampled from the stationary (and ergodic) initial distribution hinted at above. Recent results by Cuny and Peligrad [CP10] raise the hope that the Kipnis – Varadhan theory could be enhanced to central limit theorem for *almost all* initial conditions sampled according to the stationary distribution.

Sketch proof of the central limit theorem

First, we first introduce the Gaussian Hilbert space $\mathcal{H} = \mathcal{L}^2(\Omega, \pi)$ which the infinitesimal generator of the Markov process $\eta(t)$ acts on. This is a graded Hilbert space, that is, a direct sum of infinitely many orthogonal subspaces

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n. \tag{49}$$

The infinitesimal generator G of the environment process $\eta(t)$ can be written as

$$G = \frac{1}{2}\Delta + \sum_{l=1}^{d} \left(a_l^* \nabla_l + \nabla_l a_l\right) \tag{50}$$

where $a_l^* \restriction_{\mathcal{H}_n} : \mathcal{H}_n \to \mathcal{H}_{n+1}$ is a creation and $a_l \restriction_{\mathcal{H}_n} : \mathcal{H}_n \to \mathcal{H}_{n-1}$ is an annihilation operator. These are adjoints of each other. ∇_l is the second quantized differentiation in the l'th direction, and $\Delta = \sum_{l=1}^{d} \nabla_l^2$. G can be decomposed as $G = -S + A$ where S is the self-adjoint and A is the skew self-adjoint part. It can be seen that the operators

$$S = -\frac{1}{2}\Delta, \qquad A = \sum_{l=1}^{d} (a_l^* \nabla_l + \nabla_l a_l)$$

exist as self-adjoint and skew self-adjoint operators.

A natural symmetry holds in our model, $\eta(t)$ has the *Yaglom reversibility* property, i.e. starting from the stationary distribution, the flipped and time-reversed process $\widetilde{\eta}(t) = -\eta(-t)$ is equal with $\eta(t)$ in law. This property will be useful later. Ergodicity of the process $\eta(t)$ follows almost by definition, hence Theorem 4.1.

The starting point for proving central limit theorem for the displacement $X(t)$ of SRBP is the equation (42). In this representation, $B(t)$ is a Brownian motion, the integral term is an additive functional of the Markov process $\eta(t)$. Hence our aim is to use a version of the Kipnis – Varadhan theorem.

The first version of the Kipnis – Varadhan theorem in [KV86] is valid for general reversible Markov processes, i.e. if the skew self-adjoint part of the generator $A = 0$ vanishes. Tóth gave sufficient condition for the non-reversible central limit theorem in [T86]. In the presence of grading (49), the *graded sector condition* is sufficient for the central limit theorem, that is,

$$\left\| S^{-1/2} A S^{-1/2} \restriction_{\mathcal{H}_n} \right\| \leq C n^{\gamma} \tag{51}$$

with $\gamma < 1$. The H_{-1}-bound

$$\left\| S^{-1/2} f \right\| < \infty \tag{52}$$

on the function is also needed. In [SVY00], it is proved that, from (51) and (52), the central limit theorem for

$$\int_0^t f(\eta(s))\,\mathrm{d}s \tag{53}$$

follows.

For the SRBP, we apply this theorem and verify conditions (51) and (52). It is not hard to see that $\left\| \nabla_l |\Delta|^{-1/2} \right\| \leq 1$, and

$$\left\| |\Delta|^{-1/2} a_l^* \restriction_{\mathcal{H}_n} \right\| \leq C n^{1/2}$$

can be also shown by computation. Hence (51) follows with $\gamma = 1/2$.

The H_{-1}-bound (52) can be proved using a lemma from [SVY00]. It says that the limiting variance of the integral term in (42) is bounded by that of the integral of the same function along the trajectory of a different Markov process $\xi(t)$ on the same state space. The process $\xi(t)$ is a reversible Markov process with infinitesimal generator $-S$, i.e. the symmetric part of the generator G of $\eta(t)$. In the case of the SRBP, the process $\xi(t)$ is a diffusion in random scenery, for which the limiting variance of the integral is computed and it is finite in three or higher dimensions.

For showing (46), diffusive lower bound is also needed. With the latter, we rule out the case when the diffusive limit of the integral term in (42) cancels out with the martingale (in this case Brownian) term. In principle, it could happen just like in the famous case of the one-dimensional nearest neighbour symmetric simple exclusion process, see [A83]. The diffusive lower bound follows from the Yaglom reversibility of the SRBP model.

Further developments and conclusion

In order to provide a complete picture for the reader, we remark that the MSAW in three or higher dimensions and the SRBP model in one and two dimensions has also been investigated recently.

In [HTV11], diffusive bounds and central limit theorem are given also for the MSAW in high dimensions. The exact formulation is similar to Theorem 4.3, but these results are *not part of the present thesis.*

In one dimension, Tarrès, Tóth and Valkó proved essentially the bounds

$$C_1 t^{5/4} \le \mathbf{E}\left(|X(t)|^2\right) \le C_2 t^{3/2} \tag{54}$$

for the displacement of the SRBP with some $0 < C_1 < C_2 < \infty$ under some conditions on the function V, see [TTV11].

On the other hand, Tóth and Valkó proved in [TV10] among others that, for the two-dimensional SRBP, the super-diffusive bounds

$$C_3 t \log\log t \le \mathbf{E}\left(|X(t)|^2\right) \le C_4 t \log t \tag{55}$$

hold with some $0 < C_3 < C_4 < \infty$.

These bounds (54) and (55) are still not of the order conjectured in the 1980's for the MSAW, but they give robust estimates which do not depend on the particularities of the model. The existence of these recent results also shows that the self-repelling random walks and processes are at the cutting edge of modern probability theory.

For possible directions of future research, there is a wide variety of open questions. The conjectures of [APP83] are still unknown in general. The results of the present thesis and those of the quoted references are valid under some restrictive assumptions. For example, the powerful tool of Ray – Knight approach cannot be extended beyond the models treated here for some combinatorial reasons. One may ask for some robust argument for proving super-diffusive behaviour of the MSAW in one (and two) dimension.

Concerning the case above the critical dimension, the functional analytic tools used in Section 4 can be applied for *stationary* initialization of local times. It is a natural question to consider different (most interestingly empty) initial conditions. The recent paper [CP10] proves central limit theorem for additive functionals of reversible Markov processes for almost all initial values, but those methods do not seem to be directly applicable here and they are not valid for the empty initial condition. Another direction of future research might be the extension of the functional analytic methods for other types of models, e.g. random walks in random environment which is already the starting point of a new research project.

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