# Asymptotic behaviour of random walks with long memory

PhD thesis

**Bálint Vető**

Institute of Mathematics
Budapest University of Technology and Economics

Supervisor:
prof. Bálint Tóth

2011

# Contents

# Chapter 1

# Introduction

The concept of random walks is a crucial subject in modern probability. The simple random walk can be defined as a sequence of i.i.d. (independent and identically distiributed) steps on the integer lattice $\mathbb{Z}^d$. The behaviour of this basic model is well understood, and there are natural ways to generalize it by relaxing the condition of independence of steps. The main difficulty here is the loss of Markov property, which makes standard methods not applicable. Hence the analysis of random walks with memory requires a completely new approach and different ideas.

More specifically, in this thesis, we consider self-repelling random walks which means that the random walker is pushed to areas which were less visited in the past. The definition can be made precise by introducing the local times which denote the amount of time spent on certain vertices (or edges) of the underlying lattice. In the main model of the present thesis, the transition rules of the random walker are given in terms of the discrete gradient of local times: the walker is driven locally by the *negative* discrete gradient of its own local time. This concept was originally called the *'true' self-avoiding random walk*, because it is a true walk, i.e. a path of length $n + 1$ can be sampled by performing one step according to some distribution after a path of length $n$. Note that it is not the case for the self-avoiding walk. In order to avoid confusion, throughout this thesis, we use the more intuitive name *myopic self-avoiding walk* (MSAW) instead of the 'true' self-avoiding walk.

The continuous space counterpart of the MSAW is also investigated in this thesis. The driving mechanism has essentially the same spirit: it is based on the local time profile, but its irregularity is smeared out by a convolution. The diffusion process defined in this way is the *self-repellent Brownian polymer* (SRBP) model.

In this introductory chapter, we describe the motivation and context of the models treated in the present thesis. We define the MSAW and SRBP models as they appeared first in the literature. To illustrate the context of our research, we describe related earlier results. Then, we give a summary of results of the thesis with rather informal statements. The precise formulation can be found in later chapters.

## 1.1 Motivation and background

The general motivation of the research which the present thesis is based on originates in the field of statistical physics. The central subject of our investigations, the MSAW which was already mentioned above was first introduced in the physics literature by Amit, Parisi and Peliti in 1983, see [APP83]. It was the first example for a non-trivial random walk with long memory which behaves qualitatively differently from the usual diffusive behaviour of random walks. The original definition of these authors is given in the following subsection.

This model belongs to the wider class of *self-interacting random walks* which attracted attention in recent times. Typical other examples are the *self-repellent Brownian polymer* (SRBP) model, the self-avoiding walk or the reinforced random walk. Some references are given later in this section. In all these cases, long memory of the random walk or diffusion is induced by a self-interaction mechanism defined locally in a natural way in terms of the local time (or occupation time) process. The asymptotic scaling behaviour of self-interacting random walks and processes has been a mathematical challenge since the early eighties. The two basic families of models considered in the physical and probabilistic literature are the MSAW and the SRBP model which, although having their origins in different cultures and having different motivations, are phenomenologically very similar.

### 1.1.1 The myopic self-avoiding random walk model

Let $X(n)$ be a nearest neighbour random walk on the integer lattice $\mathbb{Z}^d$ which starts from $X(0) = 0$. Denote its local time on the vertices $x \in \mathbb{Z}^d$ by

$$\ell(n,x) := \#\{0 < k \leq n : X(k) = x\} \tag{1.1}$$

where $\#\{\dots\}$ denotes the cardinality of the set. Let $X(n)$ be governed by the evolution rules

$$\begin{aligned}\mathbf{P}\left(X(n+1) = x+e \;\middle|\; \mathcal{F}_n, X(n) = x\right) &= \frac{\exp\{-\beta\ell(n,x+e)\}}{\sum_{|e'|=1}\exp\{-\beta\ell(n,x+e')\}} \\ &= \frac{\exp\{-\beta(\ell(n,x+e)-\ell(n,x))\}}{\sum_{|e'|=1}\exp\{-\beta(\ell(n,x+e')-\ell(n,x))\}}\end{aligned} \tag{1.2}$$

where $|e| = 1$, $\beta > 0$ is a fixed constant, and $\mathcal{F}_n$ contains all the information up to time $n$ including the local times. Then the random walk $X(n)$ is called the myopic self-avoiding walk (MSAW).

It has been already conjectured by the authors of [APP83] based on non-rigorous renormalization group arguments that the upper critical dimension of the MSAW is two. It means that in higher dimensions, the MSAW behaves diffusively similarly to the simple random walk, and logarithmic corrections appear in two dimensions. The one-dimensional behaviour was expected to be super-diffusive. In [PP87], Peliti and Pietronero used non-rigorous scaling arguments to show that the typical order of the displacement in one dimension is 2/3th power of time, but with no hint about the limiting distribution. For renormalization of the MSAW, see also [OP83].

### 1.1.2 Related models

First, we give the definition of the self-repellent Brownian polymer (SRBP) model which is investigated in high dimensions in Chapter 5 in details. On the other hand, we present two other models of self-interacting random walks that are related to the topic of this thesis but not analysed here.

We denote by $X(t)$ the $d$-dimensional self-repellent Brownian polymer which is a continuous time $\mathbb{R}^d$-valued stochastic process. (The same letter $X$ is used as for the MSAW, but it will be always clear from the context which of them is under discussion.) The local time or occupation time measure here is given by

$$\ell(t, A) := |\{0 < s \leq t : X(s) \in A\}| \tag{1.3}$$

for any $A \subseteq \mathbb{R}^d$ measurable subset where $|\{\dots\}|$ is the Lebesgue measure of the set. Let $V : \mathbb{R}^d \to \mathbb{R}_+$ be a smooth spherically symmetric approximate identity, for instance, we may choose $V(x) := \exp(-|x|^2)$. Then, the SRBP is governed by the equation

$$\mathrm{d}X(t) = \mathrm{d}B(t) - \operatorname{grad}(V * \ell(t, \cdot))(X(t))\,\mathrm{d}t \tag{1.4}$$

where $*$ stands for convolution in $\mathbb{R}^d$ and $B(t)$ is a standard Brownian motion. The drift term in (1.4) is the negative gradient of the local time smeared out by convolution with $V$ and taken at the current position.

The SRBP was first introduced by Norris, Rogers and Williams in 1987 in [NRW87]. Later, it has also been analysed mainly in one dimension in [DR92], [CL95] and [CM96]. The work [MT08] contains a survey of earlier results as well. In the recent paper [TTV11], super-diffusive bounds are given for the one-dimensional Brownian polymer.

Another self-interacting random walk model is the *self-avoiding walk* where the steps to sites which were already visited is forbidden. Therefore, the walker might be trapped by itself, i.e. a finite self-avoiding trajectory cannot necessarily be extended to a longer self-avoiding trajectory. It shows that the self-avoiding walk is not a 'true' random process that could be sampled step by step according to some distribution as opposed to the MSAW. For a textbook on the topic, see [MS93], further references can be found therein.

A self-attracting rather than self-repelling random walk model is the reinforced random walk: it is governed by the discrete gradient of its own local time in such a way that it prefers to step to vertices that had been more visited before than others. See [T97] for a limit theorem for a family of reinforced random walks, [T04] for a recent result on the long time behaviour and [P07] for a survey.

For further surveys of self-interacting random motions, see also [T99] and [T01].

### 1.1.3 First one-dimensional results on the MSAW

The first mathematically rigorous result for the MSAW was given by Tóth in 1995 in [T95]. He considered a modified model of the MSAW (the random walk $\widetilde{X}(n)$ in our notation) where the local times are defined on the *edges* of $\mathbb{Z}$, that is,

$$\widetilde{\ell}(n,x) := \#\{0 \le k < n : \{\widetilde{X}(k), \widetilde{X}(k+1)\} = \{x, x+1\}\}. \tag{1.5}$$

The transition probabilities are

$$\begin{aligned} &\mathbf{P}\left(\widetilde{X}(n+1) = x+1 \,\middle|\, \mathcal{F}_n, \widetilde{X}(n) = x\right) \\ &\quad = 1 - \mathbf{P}\left(\widetilde{X}(n+1) = x-1 \,\middle|\, \mathcal{F}_n, \widetilde{X}(n) = x\right) \\ &\quad = \frac{\exp\{-\beta(\widetilde{\ell}(n,x) - \widetilde{\ell}(n,x-1))\}}{\exp\{-\beta(\widetilde{\ell}(n,x) - \widetilde{\ell}(n,x-1)) + \exp\{-\beta(\widetilde{\ell}(n,x-1) - \widetilde{\ell}(n,x))\}}. \end{aligned} \tag{1.6}$$

Tóth proved a limit theorem for the sequence of local times of the random walk defined in (1.6). The limit can be given in terms of a Brownian motion reflected in the Skorohod sense. He also gave a local limit theorem for the properly rescaled displacement of $\widetilde{X}$ in a late random time, which justified the conjectured scaling exponent 2/3.

A key point in the proof of [T95] is a kind of Ray – Knight type argument which works for the MSAW with edge repulsion $\widetilde{X}(n)$ defined in (1.6) but not for the original MSAW with site repulsion $X(n)$ given by (1.2). For the original idea of Ray – Knight theory, see [K63] and [R63]. Let

$$\widetilde{T}_{x,h} := \min\{n \ge 0 : \widetilde{\ell}(n,x) \ge h\}$$

be the so-called inverse local times for $x \in \mathbb{Z}$ and $h \in \mathbb{Z}_+$ and

$$\widetilde{\Lambda}_{x,h}(y) := \widetilde{\ell}(\widetilde{T}_{x,h}, y)$$

the local time sequence of the walk stopped at an inverse local time as $y \in \mathbb{Z}$. It turns out that, in the edge repulsion case, for any fixed $(x,h) \in \mathbb{Z}\times\mathbb{Z}_+$, the process $y \mapsto \widetilde{\Lambda}_{x,h}(y) \in \mathbb{Z}_+$ is Markovian and it can be thoroughly analysed.

It is a fact that the similar reduction does not hold for the original MSAW with site repulsion. Here, the natural objects are defined in the same way:

$$T_{x,h} := \min\{n \ge 0 : \ell(n,x) \ge h\}$$
$$\Lambda_{x,h}(y) := \ell(T_{x,h}, y)$$

for any $x, y \in \mathbb{Z}$ and $h \in \mathbb{Z}_+$. The process $y \mapsto \Lambda_{x,h}(y) \in \mathbb{Z}_+$ (with fixed $(x,h) \in \mathbb{Z}\times\mathbb{Z}_+$) is not Markovian and thus the Ray – Knight type of approach fails. Hence no limit theorem is known for the original MSAW model with site repulsion in discrete time.

Later, in [TW98], Tóth and Werner constructed the *true self-repelling motion* which is believed to be the scaling limit of the one-dimensional MSAW. The construction of the process is intricate. It is based on an uncountable collection of coalescing Brownian motions starting from each point of the two-dimensional space-time. This system of trajectories was later called the Brownian web, see [FINR04]. The local time profile of the true self-repelling motion is constructed using the Brownian web, and the process itself can be recovered from that.

The true self-repelling motion possesses all the analytic and stochastic properties of an assumed scaling limit of $A^{-2/3}\widetilde{X}(\lfloor At\rfloor)$ as $A\to\infty$. The invariance principle for the MSAW model with edge repulsion defined in (1.5)–(1.6) has been clarified in [NR06].

The proofs of limit theorems in [T95] (and also in subsequent papers) have some built-in combinatorial elements which make it difficult (if possible at all) to extend these proofs robustly to a full class of 1d models of random motions pushed by the negative gradient of their occupation time measure. However, more recently, a robust proof was given for the super-diffusive behaviour of the 1d models: in [TTV11], inter alia, it is proved that for the $1d$ SRBP models $\underline{\lim}_{t\to\infty} t^{-5/4}\mathbf{E}\left(X(t)^2\right) > 0$ and $\overline{\lim}_{t\to\infty} t^{-3/2}\mathbf{E}\left(X(t)^2\right) < \infty$. These are robust super-diffusive bounds (not depending on microscopic details) but still far from the expected $t^{2/3}$ scaling.

### 1.1.4 Recent developments for different versions of MSAW and SRBP

We start the enumeration of results with two remarks on the definition of the MSAW. In many cases, it is more convenient to speak about MSAW in continuous time. One of the models of this thesis is also treated in the continuous time setting. The definition can be modified in a straightforward way, and it is given precisely later in Chapter 4.

It is also worth noting here that, as indicated in the informal introduction, the transition probabilities of the MSAW given by the last expression of (1.2) are indeed proportional to the exponential function of the negative discrete gradient of local times. We generalize the definition slightly by replacing the exponential function with an arbitrary non-decreasing function $w:\mathbb{R}\to\mathbb{R}_+$ with some mild technical assumptions imposed later. Formally, instead of (1.2), we use the definition

$$\mathbf{P}\left(X(n+1)=x+e \mid \mathcal{F}_n, X(n)=x\right) = \frac{w(\ell(n,x)-\ell(n,x+e))}{\sum_{|e'|=1} w(\ell(n,x)-\ell(n,x+e'))} \tag{1.7}$$

which gives back (1.2) with the choice $w(u) = e^{\beta u}$. For this generalized model, our methods remain applicable as it will be seen in Chapter 3 and Chapter 4.

Next, we present an overview about the most important results on the MSAW and SRBP which have appeared in the literature so far.

**One dimension:** $X(n)\sim n^{2/3}$ with non-Gaussian scaling limit conjectured;

- Conjectures, renormalization group arguments in [APP83], [PP87];
- MSAW with edge repulsion: Tóth in [T95] and explained above;
- Construction of the *true self-repelling motion* as a scaling limit of one-dimensional MSAW: Tóth and Werner in [TW98] and above;
- MSAW with *oriented* edge repulsion: Tóth and Vető in [TV08] and Chapter 3 of the present thesis;
- MSAW in continuous time with site repulsion: Tóth and Vető in [TV11] and Chapter 4 of the present thesis;
- On-line demonstration of the one-dimensional MSAW models: Vető in [V09];
- Super-diffusive bounds on the one-dimensional SRBP: Tarrès, Tóth and Valkó in [TTV11].

**Two dimensions:** $X(n) \sim n^{1/2} \log^{1/4} n$ with Gaussian scaling limit conjectured;

- Conjectures, renormalization group arguments in [APP83], [OP83];
- Super-diffusive bounds on the SRBP: Tóth and Valkó in [TV10].

**Three or higher dimensions:** $X(n) \sim n^{1/2}$ with Gaussian scaling limit conjectured;

- Conjectures, renormalization group arguments in [APP83], [OP83];
- Diffusive bounds and central limit theorem for the SRBP: Horváth, Tóth and Vető in [HTV11] and Chapter 5 of the present thesis;
- Diffusive bounds and central limit theorem for the MSAW: Horváth, Tóth and Vető in [HTV11]. This result is *not part of the present thesis.*

## 1.2 Overview of thesis

In this section, we summarize the results which are presented in details in later chapters of this thesis. The thesis is based on the papers [TV07], [TV08], [TV11] and [HTV11]. Each of the next four chapters describe the results of one of these papers.

At the beginning of the chapters, we give the appropriate definitions again in order to avoid confusions. We note here that the same letter may denote different objects in different chapters, but the notation within the chapters is consistent.

Compared to the journal papers which the chapters are based on, sketch proofs have been integrated to the text, see Section 2.2, 3.3, 4.2 and 5.3. After stating the main results in each chapter, the reader can find a separate section which contains an outline of the forthcoming rigorous proofs in one or two pages with references to the formulae of the proof. It enables the reader to skim through the thesis without all the details and also to keep track of the steps of the proofs.

The chapters are not based on each other. We tried to keep them self-contained in order to make them understandable on their own.

Finally, we give a short summary of recent developments about MSAW and SRBP and we finish with conclusion.

### 1.2.1 Reflected Brownian trajectories

In Chapter 2, we consider Brownian trajectories reflected on each other in the Skorohod sense. One motivation for studying these problems was that reflected Brownian trajectories come up as building blocks in the construction of the true self-repelling motion defined in [TW98] which is the limit object of the one-dimensional myopic self-avoiding walk.

As in [TV07], we consider independent Brownian motions $B(t)$, $X(t)$ and $Y(t)$ in one dimension. Let $X^+(t)$ and $Y^-(t)$ be the trajectories of $X(t)$ and $Y(t)$ pushed upwards and respectively downwards by $B(t)$ according to Skorohod reflection. We show that the distance of the reflected trajectories $X^+(t) - Y^-(t)$ is a three-dimensional Bessel process. By the time of publishing [TV07], it turned out that a more general theorem was proved parallelly in [W07], but we give a simpler elementary proof in Chapter 2 by using the discrete approximation of Brownian paths and Donsker's invariance principle.

### 1.2.2 One-dimensional results

In Chapter 3 and 4, we consider two different versions of the one-dimensional MSAW, and we describe the scaling limits in both models which are different from the usual diffusion processes. Limit theorems for the local time processes are also proved.

The definition of the model treated in Chapter 3 and in [TV08] is slightly different from the original MSAW, i.e. the local time is defined on *oriented edges* instead of local times on the vertices of $\mathbb{Z}$. (The unoriented edge version was described above.) This little change in the definition results in a surprisingly new phenomenon: we prove that the scaling behaviour is different from the other one-dimensional MSAW models. Instead of the 2/3th power, the proper scaling of the walk turns out to be square root of time. We use the general idea of Ray – Knight approach which appeared first in [T95] in this context for the proof of the edge repulsion case of the MSAW. We show that, after appropriate scaling, the local time process of the walk converges to a deterministic triangular shape.

We also give a local limit theorem for the position of the random walker after a large random number of steps. The rescaled limit distribution is uniform. It suggests that there cannot exist any continuous scaling limit of the walk. At the end of the chapter, we present computer simulations thich show an apparent agreement with our limit theorems.

In the continuous time version of the original MSAW, we prove in Chapter 4 and in [TV11] that the right scaling is 2/3th power of time in accordance with the physicists' conjecture, and we identify the scaling limit which is the true self-repelling motion defined in [TW98]. With the Ray – Knight method, we can describe the MSAW stopped at inverse local times. We give the limit of the rescaled local time profile of the stopped MSAW in terms of reflected and absorbed Brownian motions and a local limit theorem for the displacement. These are the first mathematically rigorous results for a self-repelling random walk model with site repulsion, which is the original formulation of the problem.

With these results, we contribute to the understanding of the one-dimensional MSAW, but the picture is not complete yet. We use the general Ray – Knight method of [T95] in our proofs for both models among other new ideas, but it does not seem to be applicable for the original model in discrete time for some combinatorial reasons. Nevertheless, it is commonly agreed that the original MSAW behaves also like the discrete time unoriented edge repulsion model such as the continuous time site repulsion model.

### 1.2.3 Transient dimensions

In the paper [HTV11], the MSAW and the SRBP model are investigated in three or more dimensions. Chapter 5 contains the results about the SRBP model with full proof, but those about the MSAW are *not part of the present thesis*, however the theorems are formulated in Chapter 6 in order to have a complete picture in high dimensions. We remark here that the central limit theorem for both models are based on similar ideas, but with essential differences. For the diffusive upper bound for the MSAW, the Brascamp – Lieb inequality was needed which we do not use in the thesis.

In Chapter 5, we work in three or higher dimensions where the simple symmetric random walk is transient. Heuristically, in this regime, in any bounded domain, only finite amount of time is spent, hence the effect of self-repellence causes only a local perturbation which disappears in the limit.

We give a rigorous proof of the central limit theorem for the SRBP in terms of the finite dimensional distributions, which means that the scaling is indeed diffusive with Gaussian limit as conjectured in the 1980's. In addition, diffusive upper and lower bounds are given on the variance of the displacement, which yield that the normal distribution in the limit is non-degenerate.

Considering the SRBP, the main difficulty is caused by its long memory, since the whole history of the process influences the evolution. With a natural observation due to Varadhan, if one considers the motion from the point of view of the moving particle, then the problem transforms to finding Gaussian behaviour for a certain additive functional of a Markov process, which has an extended theory in the literature.

The first result on central limit theorem for additive functionals of Markov chains dates back to 1986 due to Kipnis and Varadhan [KV86]. The main tool in their proof and later generalizations is approximation via a martingale with stationary increments. The theorem of Kipnis and Varadhan is only applicable if the Markov chain under discussion is reversible, which is not the case here. Extensions of this theory for the non-reversible case appeared in the literature. The most general sufficient condition, the the so-called *graded sector condition*, is given in [SVY00] for the central limit theorem to hold.

Checking the graded sector condition for the SRBP in three or more dimensions requires advanced functional analytic tools. The theory of Gaussian Hilbert spaces provides the framework of the computations. The proof relies on deep understanding of the resolvent calculus of the operator which appears here as infinitesimal generator. The computations are performed in the space of Fourier transforms where the operators can be handled.

In the proof of the diffusive bounds, the natural symmetries (most notably the Yaglom reversibility) of the model are used several times.

# Chapter 2

# Reflected Brownian trajectories

The study of 1d Brownian trajectories pushed up or down by Skorohod reflection on some other Borwnian trajectories (running backwards in time) was initiated in [STW00] and motivated in [TW98] by the construction of the object what is today called the Brownian web, see [FINR04]. It turns out that these Brownian paths, reflected on one another, have very interesting, sometimes surprising properties. For further studies of Skorohod reflection of Brownian paths on one another, see also [SW02], [BN02], [W07] etc. In particular, in [W07], Warren considers two interlaced families of Brownian trajectories with paths belonging to the second family reflected off by paths belonging to the first (in Skorohod's sense) and derives a determinantal formula for the distribution of coalescing Brownian motions.

A particular case of Warren's formula is the following: fix a Brownian path and let two other Brownian paths be pushed upwards and respectively downwards by Skorohod reflection on the trajectory of the first one. The difference of the last two will be a three-dimensional Bessel process. In the present chapter, we give an alternative, elementary proof of this fact.

## 2.1 Results on reflected Brownian paths

### 2.1.1 Skorohod reflection

Let $T \in (0,\infty)$ and $b, x : [0,T) \to \mathbb{R}$ be continuous functions. Assume $x(0) \geq b(0)$. The construction of the following proposition is due to Skorohod. Its proof can be found either in [RY99] (see Lemma 2.1 in Chapter VI) or in [STW00] (see Lemma 2 in Section 2.1)

**Proposition 2.1.1.** *(1) There exists a unique continuous function $x_{b\uparrow} : [0,T) \to \mathbb{R}$ with the following properties:*

- *The function $x_{b\uparrow} - b$ is non-negative.*
- *The function $x_{b\uparrow} - x$ is non-decreasing.*

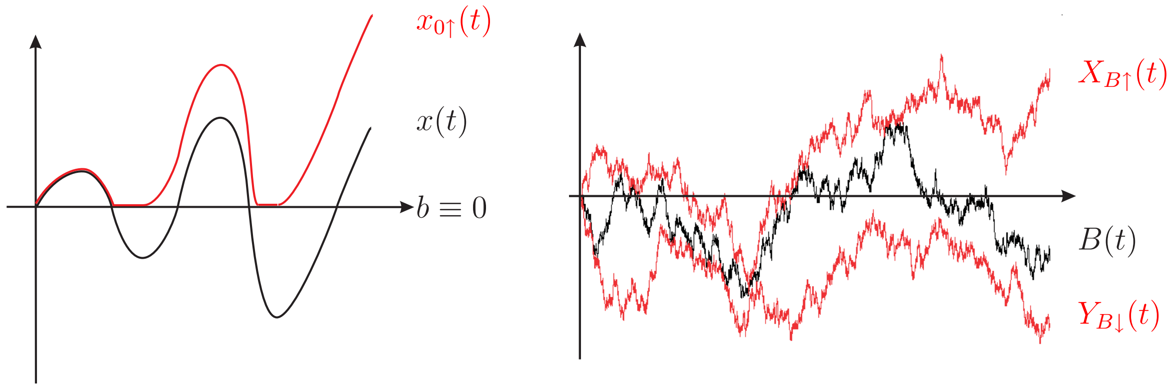


Figure 2.1: Example for Skorohod reflection and the reflected Brownian paths of Theorem 2.1.2

– *The function $x_{b\uparrow} - x$ increases only when $x_{b\uparrow} = b$. That is*

$$\int_0^T \mathbb{1}(x_{b\uparrow}(t) \neq b(t))\, \mathrm{d}(x_{b\uparrow}(t) - x(t)) = 0.$$

*(2) The function $t \mapsto x_{b\uparrow}(t)$ is given by the construction*

$$x_{b\uparrow}(t) = x(t) + \sup_{0 \le s \le t} \big(x(s) - b(s)\big)_{-}.$$

*(3) The map $C([0,T)) \times C([0,T)) \ni \big(b(\cdot), x(\cdot)\big) \mapsto \big(b(\cdot), x_{b\uparrow}(\cdot)\big) \in C([0,T)) \times C([0,T))$ is contunuous in supremum distance.*

We call the function $t \mapsto x_{b\uparrow}(t)$ the *upwards Skorohod reflection* of $x(\cdot)$ on $b(\cdot)$. As it is remarked in [STW00], the term *Skorohod pushup* of $x(\cdot)$ by $b(\cdot)$ would be more adequate. Skorohod reflection on paths $b(t) = \text{const.}$ plays a fundamental role in the proper formulation and proof of Tanaka's formula, see Chapter VI of [RY99]. See also Figure 2.1.

The downwards Skorohod reflection or Skorohod pushdown is defined for continuous functions $b, y : [0,T) \mapsto \mathbb{R}$ with $y(0) \le b(0)$ by

$$y_{b\downarrow} := -\left((-y)_{(-b)\uparrow}\right), \qquad y_{b\downarrow}(t) = y(t) - \sup_{0 \le s \le t} \big(y(s) - b(s)\big)_{+}.$$

Given three continuous trajectories $b, x, y : [0,T) \to \mathbb{R}$ with $y(0) \le b(0) \le x(0)$, the map $C([0,T)) \times C([0,T)) \times C([0,T)) \ni (b(\cdot), x(\cdot), y(\cdot)) \mapsto (b(\cdot), x_{b\uparrow}(\cdot), y_{b\downarrow}(\cdot)) \in C([0,T)) \times C([0,T)) \times C([0,T))$ is clearly continuous in supremum distance.

### 2.1.2 The result

Let $B(t)$, $X(t)$ and $Y(t)$ be independent standard 1d Brownian motions starting from 0 and define

$$X^+(t) := X_{B\uparrow}(t), \qquad \widehat{X}(t) := \quad X^+(t) - B(t), \tag{2.1}$$
$$Y^-(t) := Y_{B\downarrow}(t), \qquad \widehat{Y}(t) := -Y^-(t) + B(t) \tag{2.2}$$

as seen on Figure 2.1. We are interested in the difference process

$$Z(t) := X^+(t) - Y^-(t) = \widehat{X}(t) + \widehat{Y}(t). \tag{2.3}$$

It is straightforward that $2^{-1/2}\widehat{X}(t)$ and $2^{-1/2}\widehat{Y}(t)$ are both standard reflected Brownian motions. They are, of course, strongly dependent.

The following fact is a particular consequence of the main results in [W07]:

**Theorem 2.1.2.** *The process $2^{-1/2}Z(t)$ is* BES$^3$*, that is, standard 3d Bessel process:*

$$\mathrm{d}Z(t) = 2\frac{1}{Z(t)}\mathrm{d}t + \sqrt{2}\mathrm{d}W(t), \qquad Z(0) = 0. \tag{2.4}$$

In Section 2.3, we present an elementary proof of this fact.

## 2.2 Sketch of proof

Before the formal proof of Theorem 2.1.2, we describe the main steps and ideas in this short section.

The basic tools are the discrete approximation of Brownian trajectories and Donsker's invariance principle. The Brownian motions $B(t)$, $X(t)$ and $Y(t)$ are given as diffusive limits of the independent simple random walks $M(n)$, $U(n)$ and $L(n)$, see (2.6). For the random walk trajectories, the *discrete Skorohod reflection* is defined in Proposition 2.3.1. By Donsker's invariance principle, it follows that the difference process $Z(t)$ defined in (2.3) can be represented as the scaling limit of the distance of two random walk trajectories $U_{M\uparrow}(n)$ and $L_{M\downarrow}(n)$ reflected upwards and downwards on the trajectory of $M(n)$ in the *discrete* Skorohod sense, see (2.7). The discrete distance of $U_{M\uparrow}(n)$ and $L_{M\downarrow}(n)$ is called $2D_n$.

Lemma 2.3.2 tells us that $D_n$ is a Markov chain on its own, and the transition matrix is given by (2.8). It is enough to see that the scaling limit of $D_n$ is indeed BES$^3$, because the SDE (2.4) can be deduced by computing the expectation and variance of one step of $D_n$.

The actual computations for checking the transitions of $D_n$ are left to Lemma 2.3.3. Slightly more is proved there: besides identifying the transition matrix of $D_n$, it is also shown that the position of $M(n)$ between the reflected versions of $U(n)$ and $L(n)$ is uniformly distributed. The two statements are verified by a common induction using elementary observations.

## 2.3 Proof of the theorem

### 2.3.1 Discrete Skorohod reflection

Define the following square lattices embedded in $\mathbb{R} \times \mathbb{R}$:

$$\mathcal{L} := \{(t, x) \in \mathbb{Z} \times \mathbb{Z} : t + x \text{ is even}\}, \qquad \mathcal{L}^* := \{(t, x) \in \mathbb{Z} \times \mathbb{Z} : t + x \text{ is odd}\}. \tag{2.5}$$

In both of the lattices, the points $(t_1, x_1)$ and $(t_2, x_2)$ are connected with an edge if and only if $|t_1 - t_2| = |x_1 - x_2| = 1$. Note that $\mathcal{L}$ and $\mathcal{L}^*$ are Whitney duals of each other.

We define the discrete analogue of the Skorohod reflection in $\mathcal{L}$ and $\mathcal{L}^*$. Later on, we say that the function $y : [0, T] \cap \mathbb{Z} \to \mathbb{Z}$ is a *walk* in the lattice $\mathcal{L}$ or $\mathcal{L}^*$ if the consecutive elements of the sequence $(0, y(0)), (1, y(1)), \ldots, (T, y(T))$ are edges in $\mathcal{L}$ or $\mathcal{L}^*$.

Let $b : [0, T] \cap \mathbb{Z} \to \mathbb{Z}$ and $x : [0, T] \cap \mathbb{Z} \to \mathbb{Z}$ be two walks in the lattices $\mathcal{L}$ and $\mathcal{L}^*$, respectively. Assume that $x(0) \geq b(0)$. An analogue of Proposition 2.1.1 holds in this case, but the proof is even easier.

**Proposition 2.3.1.** *(1) There is a unique walk $x_{b\uparrow} : [0, T] \cap \mathbb{Z} \to \mathbb{Z}$ in $\mathcal{L}^*$ with the following properties:*

- *The function $x_{b\uparrow} - b$ is non-negative.*
- *The function $x_{b\uparrow} - x$ is non-decreasing.*
- *The function $x_{b\uparrow} - x$ increases only when $x_{b\uparrow} = b + 1$, i.e.*

$$\sum_{t=1}^{T} \mathbb{1}(x_{b\uparrow}(t) - b(t) > 1) \left[ \left( x_{b\uparrow}(t) - x(t) \right) - \left( x_{b\uparrow}(t-1) - x(t-1) \right) \right] = 0.$$

*(2) The function $t \mapsto x_{b\uparrow}(t)$ can be expressed as*

$$x_{b\uparrow}(t) = x(t) + \sup_{s \in [0,t] \cap \mathbb{Z}} (x(s) - b(s))_- + 1.$$

We call the function $t \mapsto x_{b\uparrow}(t)$ the *discrete upwards Skorohod reflection* of $x(\cdot)$ on $b(\cdot)$. The discrete downwards Skorohod reflection is defined similarly. If $y : [0, T] \cap \mathbb{Z} \to \mathbb{Z}$ is a walk in $\mathcal{L}^*$ and $b : [0, T] \cap \mathbb{Z} \to \mathbb{Z}$ is a walk in $\mathcal{L}$ with $y(0) \leq b(0)$, then

$$y_{b\downarrow} := -\left( (-y)_{(-b)\uparrow} \right), \qquad y_{b\downarrow}(t) = y(t) - \sup_{s \in [0,t] \cap \mathbb{Z}} (y(s) - b(s))_+ - 1.$$

See also Figure 2.2.

In this chapter, we use the same notation for the discrete Skorohod reflection and the continuous one (defined as Skorohod reflection), but it will be always clear from the context which is the adequate one.

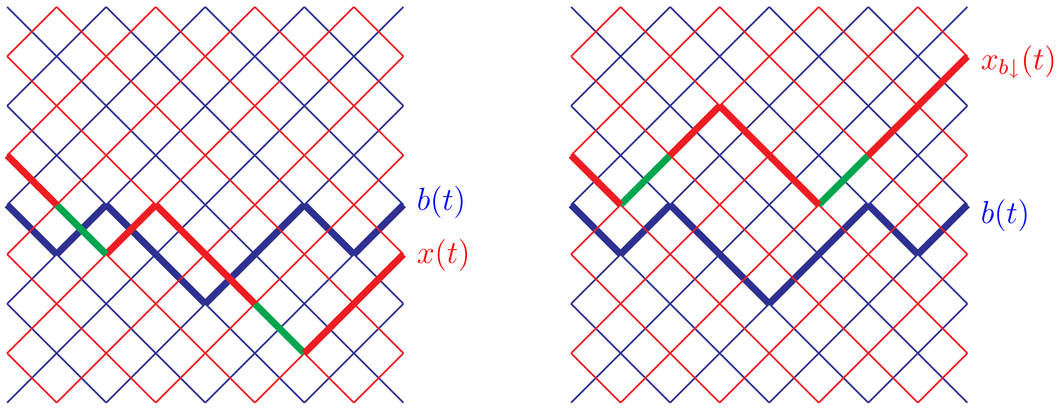


Figure 2.2: The discrete upwards Skorohod reflection of the path $x(t) \in \mathcal{L}$ on $b(t) \in \mathcal{L}^*$. The green edges have been modified

### 2.3.2 Approximation of reflected Brownian motions

Let $M(t)$ be a random walk on the lattice $\mathcal{L}$ with jumps from $(t, x)$ to $(t+1, x+1)$ or $(t+1, x-1)$ with probability $1/2 - 1/2$ and $M(0) = 0$. We define the random walks $U(t)$ and $L(t)$ on $\mathcal{L}^*$ with the same transition probabilities, which are independent of each other and of $M(t)$. The initial values are $U(0) = 1$ and $L(0) = -1$. We extend our walks for non-integral values of $t$ linearly, so the trajectories are continuous.

Since all these three random walks have steps with mean 0 and variance 1, it follows that

$$\left(\frac{M(nt)}{\sqrt{n}}, \frac{U(nt)}{\sqrt{n}}, \frac{L(nt)}{\sqrt{n}}\right)_{0\le t\le T} \Longrightarrow (B(t), X(t), Y(t))_{0\le t\le T} \tag{2.6}$$

weakly on $C[0,T]$ for any $T > 0$ as $n \to \infty$. We established earlier that the map $(b(\cdot), x(\cdot), y(\cdot)) \mapsto (b(\cdot), x_{b\uparrow}(\cdot), y_{b\downarrow}(\cdot))$ is continuous in supremum distance. From Donsker's invariance principle (see e.g. Section 7.6 of [D95] for the notion of weak convergence and Donsker's invariance principle), we conclude that

$$\left(\frac{M(nt)}{\sqrt{n}}, \frac{U_{M(n\cdot)\uparrow}(nt)}{\sqrt{n}}, \frac{L_{M(n\cdot)\downarrow}(nt)}{\sqrt{n}}\right)_{0\le t\le T} \Longrightarrow (B(t), X^+(t), Y^-(t))_{0\le t\le T} \tag{2.7}$$

weakly as $n \to \infty$. Note that we can use the discrete Skorohod reflection to transform $U$ and $L$, because the difference is only the addition of 1, which vanishes in the limit. At this point, it suffices to show that

$$2^{-1/2}\frac{U_{M(n\cdot)\uparrow}(nt) - L_{M(n\cdot)\downarrow}(nt)}{\sqrt{n}}$$

converges to a $\mathrm{BES}^3$ process.

For $x, y \in \mathbb{Z}^+$, we define the stochastic matrix

$$\mathbf{P}_{xy} = \frac{y}{x} \cdot \begin{cases} \frac{1}{2} & \text{if } y = x \\ \frac{1}{4} & \text{if } |y-x| = 1 \\ 0 & \text{otherwise} \end{cases}. \tag{2.8}$$

It is well known that if $X_n$ is a homogeneous Markov chain with transition probabilities $(\mathbf{P}_{xy})_{x,y\in\mathbb{Z}^+}$, then its diffusive limit is BES$^3$, i.e. for every $T > 0$, the process $\sqrt{2}(n^{-1/2}X_{nt})_{0\le t\le T}$ converges to a 3d Bessel process in the Skorohod topology as $n \to \infty$. So the proof of our theorem relies on the following

**Lemma 2.3.2.** $\frac{1}{2}(U_{M\uparrow}(t) - L_{M\downarrow}(t))$ *is a Markov chain and its transition matrix is given by* $(\mathbf{P}_{xy})_{x,y\in\mathbb{Z}^+}$ *where* $U_{M\uparrow}$ *and* $L_{M\downarrow}$ *are discrete Skorohod reflections.*

### 2.3.3 Markov property of the distance of the two reflected walks

We introduce a different notation for the triple $(M, U_{M\uparrow}, L_{M\downarrow})$, which is just a linear transformation. Let $K_n := L_{M\downarrow}(n)$ be the position of the lower reflected walk. With the definition $D_n := \frac{1}{2}(U_{M\uparrow}(n) - L_{M\downarrow}(n))$, the distance of the two reflected walks is $2D_n$. $P_n := \frac{1}{2}(M(n) - L_{M\downarrow}(n) - 1)$, which means that the position of $M$ related to the lower walk is $2P_n + 1$. The vector $(K_n, D_n, P_n)$ is clearly a Markov chain.

We are only interested in the coordinate $D_n$, which turns out to be also Markovian and to have transition matrix $(\mathbf{P}_{xy})_{xy\in\mathbb{Z}^+}$. To show this, we have to determine the conditional distribution of $P_n$, because in certain cases it modifies the transition rules of $D_n$.

**Lemma 2.3.3.** *The following identities hold*

$$\mathbb{P}\left(P_n = x \mid D_0^n\right) = \tfrac{1}{D_n}\mathbb{1}(x \in \{0, 1, \ldots, D_n - 1\}), \tag{2.9}$$

$$\mathbb{P}\left(D_{n+1} = y \mid D_0^n\right) = \mathbf{P}_{D_n y} \tag{2.10}$$

*where* $D_0^n$ *means the sequence of variables* $D_0, \ldots, D_n$.

*Proof.* The two identities (2.9), respectively, (2.10) of the lemma are proved by a common induction on $n$. Since $D_0 = 1$ and $P_0 = 0$, the case $n = 0$ is trivial.

For the induction step, we have to enumerate the possible transitions of the Markov chain $(K_n, D_n, P_n)$. For the sake of simplicity, we only prove for $D_n = D_{n-1} - 1$, the other cases are similar. It is easy to check that the transition $(k, d, p) \to (k + 1, d - 1, p)$ has probability $\frac{1}{8}\mathbb{1}(p \in \{0, 1, \ldots, d - 2\})$, this will be called type $A$ events. Type $B$ events are the transitions $(k, d, p) \to (k + 1, d - 1, p - 1)$, which happen with probability $\frac{1}{8}\mathbb{1}(p \in \{1, 2, \ldots, d - 1\})$. No other cases give $d \to d - 1$.

*Proof of* (2.9): Let $x, y \in \mathbb{Z}^+$. We suppose that $y = D_{n-1} - 1$.

$$\begin{aligned}
&\mathbb{P}\left(P_n = x \mid D_n = y, D_0^n\right) \\
&\quad= \sum_{z\in\mathbb{Z}} \mathbb{P}\left(P_n = x \mid P_{n-1} = z, D_n = y, D_0^{n-1}\right) \mathbb{P}\left(P_{n-1} = z \mid D_n = y, D_0^{n-1}\right) \\
&\quad= \sum_{z\in\mathbb{Z}} \frac{\mathbb{P}\left(P_n = x, D_n = y \mid P_{n-1} = z, D_0^{n-1}\right)}{\mathbb{P}\left(D_n = y \mid P_{n-1} = z, D_0^{n-1}\right)} \mathbb{P}\left(P_{n-1} = z \mid D_n = y, D_0^{n-1}\right) \\
&\quad= \sum_{z=x}^{x+1} \mathbb{P}\left(P_n = x, D_n = y \mid P_{n-1} = z, D_0^{n-1}\right) \frac{\mathbb{P}\left(P_{n-1} = z \mid D_0^{n-1}\right)}{\mathbb{P}\left(D_n = y \mid D_0^{n-1}\right)} \\
&\quad= \mathbb{P}\left(P_n = x, D_n = y \mid P_{n-1} = x, D_0^{n-1}\right) \frac{\mathbb{P}\left(P_{n-1} = x \mid D_0^{n-1}\right)}{\mathbb{P}\left(D_n = y \mid D_0^{n-1}\right)} \\
&\qquad+ \mathbb{P}\left(P_n = x, D_n = y \mid P_{n-1} = x+1, D_0^{n-1}\right) \frac{\mathbb{P}\left(P_{n-1} = x+1 \mid D_0^{n-1}\right)}{\mathbb{P}\left(D_n = y \mid D_0^{n-1}\right)} \\
&\quad= \frac{1}{8}\mathbb{1}(x \in \{0, 1, \ldots, D_{n-1} - 2\}) \frac{\frac{1}{D_{n-1}}\mathbb{1}(x \in \{0, 1, \ldots, D_{n-1} - 1\})}{\frac{1}{4}\frac{D_{n-1}-1}{D_{n-1}}} \\
&\qquad+ \frac{1}{8}\mathbb{1}(x \in \{0, 1, \ldots, D_{n-1} - 2\}) \frac{\frac{1}{D_{n-1}}\mathbb{1}(x \in \{-1, 0, \ldots, D_{n-1} - 2\})}{\frac{1}{4}\frac{D_{n-1}-1}{D_{n-1}}} \\
&\quad= \frac{1}{D_{n-1} - 1}\mathbb{1}(x \in \{0, \ldots, D_{n-1} - 2\}) = \frac{1}{y}\mathbb{1}(x \in \{0, 1, \ldots, y - 1\}).
\end{aligned} \tag{2.11}$$

First, we used the law of total probability and the definition of conditional probability and the identity $\mathbb{P}(E|F)/\mathbb{P}(F|E) = \mathbb{P}(E)/\mathbb{P}(F)$ on a conditional probability space. As remarked at the beginning of this proof, there are only two cases to reduce the value of $D$, so the sum has only two terms. Then, we used both inductional hypotheses to evaluate the conditional probabilities. The remaining steps are obvious.

*Proof of* (2.10): We spell out the proof for $D_{n+1} = D_n - 1$, the cases $D_{n+1} = D_n$ and $D_{n+1} = D_n + 1$ are similar.

$$\begin{aligned}
&\mathbb{P}\left(D_{n+1} = D_n - 1 \mid D_0^n\right) \\
&\quad= \sum_{x=0}^{D_n-1} \mathbb{P}\left(D_{n+1} = D_n - 1 \mid P_n = x, D_0^n\right) \mathbb{P}\left(P_n = x \mid D_0^n\right) \\
&\quad= \sum_{x=0}^{D_n-1} \left(\frac{1}{8}\mathbb{1}(x \in \{0, 1, \ldots, D_n - 2\}) + \frac{1}{8}\mathbb{1}(x \in \{1, 2, \ldots, D_n - 1\})\right) \frac{1}{D_n} \\
&\quad= \frac{1}{4}\frac{D_n - 1}{D_n} = \mathbf{P}_{D_n(D_n-1)}.
\end{aligned} \tag{2.12}$$

In the second step, only type $A$ and $B$ events can cause the transition $D_{n+1} = D_n - 1$. We applied the first part of this lemma to evaluate the second conditional probability factor.

□

As a consequence, we see that the distribution of $D_{n+1}$ conditioned on $D_0^n$ depends only on $D_n$, which means that $D_n$ is a Markov chain with transition matrix $(\mathbf{P}_{xy})_{xy}$. From this, the assertion of the theorem follows.

# Chapter 3

# One-dimensional myopic self-avoiding walk with oriented edge repulsion

In this chapter, we consider a variant of self-repelling random walk on the integer lattice $\mathbb{Z}$ where the self-repellence is defined in terms of the local time on *oriented* edges. This model is similar to the one examined in [T95], but the walker here is pushed by the local differences of occupation time measures on *oriented* rather than unoriented edges.

The phenomenological behaviour is surprisingly different from the unoriented case. We prove limit theorems for the local time process and for the position of the random walker under square-root-of-time (rather than time-to-the-2/3) space-scaling, but the limit laws are not the usual diffusive ones. The main ingredient is a Ray−Knight type of approach.

This chapter is organized as follows. We introduce the model formally in Section 3.1. In Section 3.2, we formulate the main results. In Section 3.3, we give a sketch of proof. In Section 3.4, we prove Theorem 3.2.2 about the convergence in sup-norm and in probability of the local time process stopped at inverse local times. As a consequence, we also prove convergence in probability of the inverse local times to *deterministic values*. In Section 3.5, we convert the limit theorems for the inverse local times to local limit theorems for the position of the random walker at independent random stopping times of geometric distribution with large expectation. Finally, in Section 3.6, we present some numerical simulations of the position and local time processes with particular choices of the weight function. The figures show the strange scaling behaviour of the walk considered.

## 3.1 Definition of the model

Let $w$ be a weight function which is non-decreasing and non-constant:

$$w : \mathbb{Z} \to \mathbb{R}_+, \qquad w(z+1) \geq w(z), \qquad \lim_{z\to\infty} \big(w(z) - w(-z)\big) > 0. \tag{3.1}$$

We will consider a nearest neighbour random walk $X(n)$, $n \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$, on the integer lattice $\mathbb{Z}$, starting from $X(0) = 0$, which is governed by its local time process through the function $w$ in the following way. Denote by $\ell^{\pm}(n, k)$, $(n, k) \in \mathbb{Z}_+ \times \mathbb{Z}$, the local time (that is: its occupation time measure) on oriented edges:

$$\ell^{\pm}(n, k) := \#\{0 \le j \le n-1 : X(j) = k, X(j+1) = k \pm 1\}$$

where $\#\{\dots\}$ denotes cardinality of the set. Note that

$$\ell^{+}(n, k) - \ell^{-}(n, k+1) = \begin{cases} +1 \text{ if } & 0 \le k < X(n), \\ -1 \text{ if } & X(n) \le k < 0, \\ \ \ 0 \text{ otherwise.} & \end{cases} \tag{3.2}$$

We will also use the notation

$$\ell(n, k) := \ell^{+}(n, k) + \ell^{-}(n, k+1) \tag{3.3}$$

for the local time spent on the unoriented edge $\langle k, k+1 \rangle$.

Our random walk is governed by the evolution rules

$$\begin{aligned} &\mathbf{P}\left(X(n+1) = X(n) \pm 1 \,\middle|\, \mathcal{F}_n\right) \\ &\quad = \frac{w(\mp(\ell^{+}(n, X(n)) - \ell^{-}(n, X(n))))}{w(\ell^{+}(n, X(n)) - \ell^{-}(n, X(n))) + w(\ell^{-}(n, X(n)) - \ell^{+}(n, X(n)))}, \\ &\ell^{\pm}(n+1, k) = \ell^{\pm}(n, x) + 1\!\!1(X(n) = k, X(n+1) = k \pm 1). \end{aligned} \tag{3.4}$$

That is: at each step, the walk prefers to choose that oriented edge pointing away from the actually occupied site which had been less visited in the past. In this way, it balances or smoothes out the roughness of the occupation time measure. We prove limit theorems for the local time process and for the position of the random walker at large times under *diffusive scaling*, that is: essentially for $n^{-1/2}\ell(n, \lfloor n^{1/2}x \rfloor)$ and $n^{-1/2}X(n)$, but with limit laws strikingly different from usual diffusions. See Theorem 3.2.2 and 3.2.4 for precise statements.

## 3.2 The main results

As in [T95], the key to the proof is a Ray–Knight approach. Let

$$T^{\pm}_{j,r} := \min\{n \ge 0 : \ell^{\pm}(n, j) \ge r\}, \qquad j \in \mathbb{Z}, \quad r \in \mathbb{Z}_+$$

be the so-called inverse local times and

$$\Lambda^{\pm}_{j,r}(k) := \ell(T^{\pm}_{j,r}, k) = \ell^{+}(T^{\pm}_{j,r}, k) + \ell^{-}(T^{\pm}_{j,r}, k+1), \qquad j, k \in \mathbb{Z}, \quad r \in \mathbb{Z}_+ \tag{3.5}$$

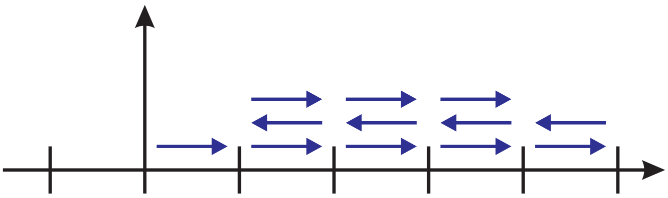


Figure 3.1: The local time sequence of the walk stopped at the inverse local time $T^+_{3,2} = 12$

the local time sequence (on unoriented edges) of the walk stopped at the inverse local times. See Figure 3.1. We denote by $\lambda^\pm_{j,r}$ and $\rho^\pm_{j,r}$ the leftmost, respectively, the rightmost edges visited by the walk before the stopping time $T^\pm_{j,r}$:

$$\begin{aligned}\lambda^\pm_{j,r} &:= \inf\{k \in \mathbb{Z} \,:\, \Lambda^\pm_{j,r}(k) > 0\},\\ \rho^\pm_{j,r} &:= \sup\{k \in \mathbb{Z} \,:\, \Lambda^\pm_{j,r}(k) > 0\}.\end{aligned}$$

The next proposition states that the random walk is recurrent in the sense that it visits infinitely often every site and edge of $\mathbb{Z}$.

**Proposition 3.2.1.** *Let $j \in \mathbb{Z}$ and $r \in \mathbb{Z}_+$ be fixed. We have*

$$\max\left\{T^\pm_{j,r}, \rho^\pm_{j,r} - \lambda^\pm_{j,r}, \sup_k \Lambda^\pm_{j,r}(k)\right\} < \infty$$

*almost surely.*

Actually, we will see from the proofs of our theorems that the quantities in Proposition 3.2.1 are finite, and much stronger results are true for them, so we do not give a separate proof of this statement.

### 3.2.1 Limit theorem for the local time process

The main result concerning the local time process stopped at inverse local times is the following:

**Theorem 3.2.2.** *Let $x \in \mathbb{R}$ and $h \in \mathbb{R}_+$ be fixed. Then*

$$A^{-1}\lambda^\pm_{\lfloor Ax\rfloor,\lfloor Ah\rfloor} \xrightarrow{\mathbf{P}} -|x| - 2h, \tag{3.6}$$

$$A^{-1}\rho^\pm_{\lfloor Ax\rfloor,\lfloor Ah\rfloor} \xrightarrow{\mathbf{P}} |x| + 2h, \tag{3.7}$$

*and*

$$\sup_{y\in\mathbb{R}} \left| A^{-1}\Lambda^\pm_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(\lfloor Ay\rfloor) - (|x| - |y| + 2h)_+ \right| \xrightarrow{\mathbf{P}} 0 \tag{3.8}$$

*as $A \to \infty$.*

Note that

$$T^{\pm}_{j,r} = \sum_{k=\lambda^{\pm}_{j,r}}^{\rho^{\pm}_{j,r}} \Lambda^{\pm}_{j,r}(k).$$

Hence, it follows immediately from Theorem 3.2.2 that

**Corollary 3.2.3.** *With the notations of Theorem 3.2.2,*

$$A^{-2}T^{\pm}_{\lfloor Ax\rfloor,\lfloor Ah\rfloor} \xrightarrow{\mathbf{P}} (|x|+2h)^2 \tag{3.9}$$

*as* $A\to\infty$.

Theorem 3.2.2 and Corollary 3.2.3 will be proved in Section 3.4.

**Remark.** Note that the local time process and the inverse local times converge in probability to deterministic objects rather than converging weakly in distribution to genuinely random variables. This makes the present case somewhat similar to the weakly reinforced random walks studied in [T97].

### 3.2.2 Limit theorem for the position of the walker

According to the arguments in [T95], [T99] and [T01], from the limit theorems

$$A^{-1/\nu}T^{\pm}_{\lfloor Ax\rfloor,\lfloor A^{(1-\nu)/\nu}h\rfloor} \Rightarrow \mathcal{T}_{x,h} \tag{3.10}$$

valid for any $(x,h)\in\mathbb{R}\times\mathbb{R}_+$, one can essentially derive the limit theorem for the one-dimensional marginals of the position process:

$$A^{-\nu}X(\lfloor At\rfloor) \Rightarrow \mathcal{X}(t).$$

Indeed, the summation arguments, given in detail in the papers quoted above, indicate that

$$\varphi(t,x) := 2\frac{\partial}{\partial t}\int_0^\infty \mathbf{P}\left(\mathcal{T}_{x,h} < t\right)\mathrm{d}h \tag{3.11}$$

is the good candidate for the the density of the distribution of $\mathcal{X}(t)$, with respect to Lebesgue measure. The scaling relation

$$A^{1/\nu}\varphi(At, A^{1/\nu}x) = \varphi(t,x) \tag{3.12}$$

clearly holds. In some cases (see e.g. [T95]), it is not trivial to check that $x\mapsto\varphi(t,x)$ is a bona fide probability density of total mass 1. (However, a Fatou argument easily shows that its total mass is not more than 1.) But in our present case, this fact drops out from explicit formulas. Indeed, the weak limits (3.9) hold, which, by straightforward computation, imply

$$\varphi(t,x) = \frac{1}{2\sqrt{t}}\mathbb{1}(|x|\le\sqrt{t}).$$

Actually, in order to prove limit theorem for the position of the random walker, some smoothening in time is needed, which is realized through the Laplace transform. Let

$$\widehat{\varphi}(s,x) := s\int_0^\infty e^{-st}\varphi(t,x)\mathrm{d}t = \sqrt{s\pi}(1-\Phi(\sqrt{2s}|x|))$$

where

$$\Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\mathrm{d}y$$

is the standard normal distribution function.

We prove the following *local limit theorem* for the position of the random walker stopped at an independent geometrically distributed stopping time of large expectation:

**Theorem 3.2.4.** *Let $s \in \mathbb{R}_+$ be fixed and $\theta_{s/A}$ a random variable with geometric distribution*

$$\mathbf{P}\left(\theta_{s/A} = n\right) = (1-e^{-s/A})e^{-sn/A} \tag{3.13}$$

*which is independent of the random walk $X(n)$. Then, for almost all $x \in \mathbb{R}$,*

$$A^{1/2}\mathbf{P}\left(X(\theta_{s/A}) = \lfloor A^{1/2}x\rfloor\right) \to \widehat{\varphi}(s,x)$$

*as $A \to \infty$.*

From the above local limit theorem, the integral limit theorem follows immediately:

$$\lim_{A\to\infty}\mathbf{P}\left(A^{-1/2}X(\theta_{s/A}) < x\right) = \int_{-\infty}^{x}\widehat{\varphi}(s,y)\mathrm{d}y.$$

From (3.6) and (3.7), the tightness of the distributions $(A^{-1/2}X(\lfloor At\rfloor))_{A\geq 1}$ follows easily. Theorem 3.2.4 yields that if the random walk $X(\cdot)$ has any scaling limit, then $\varphi(t,\cdot)$ given in (3.11) is indeed the density of the scaling limit, that is,

$$A^{-1/2}X(\lfloor At\rfloor) \Longrightarrow \mathrm{UNI}(-\sqrt{t},\sqrt{t}) \tag{3.14}$$

as $A \to \infty$ holds where $\mathrm{UNI}(-\sqrt{t},\sqrt{t})$ stands for the uniform distribution on the interval $(-\sqrt{t},\sqrt{t})$.

The proof of Theorem 3.2.4 is presented in Section 3.5.

## 3.3 Sketch proof of the limit theorems

We present the main ideas of the proofs of Theorem 3.2.2 and 3.2.4. Our aim is to let the reader understand the structures of the proofs without the technical details which are left to Section 3.4 and 3.5.

### 3.3.1 Theorem 3.2.2

The proof of Theorem 3.2.2 is organized as follows. We introduce independent auxiliary Markov chains $\eta_{j,\pm}$ in (3.25)–(3.28) associated to the vertices $j \in \mathbb{Z}$ with the same distribution. These Markov chains are essentially the differences of local times on adjacent edges. The law of these Markov chains is already analysed in Subsection 3.4.1. Lemma 3.4.1 determines the stationary distribution $\rho$ of these Markov chains and it gives an exponential bound on the rate of convergence to the stationary measure. It turns out that the stationary mean (3.24) is $-1/2$.

Proposition 3.4.2 tells us that these auxiliary Markov chains are indeed independent and have the same distribution which enables us to represent the local time sequence $L_{j,r}$ as a random walk. More precisely, we follow a Ray – Knight approach which means that we consider the local times of the random walk $X$ stopped at an inverse local time $T^{\pm}_{j,r}$, i.e. the sequence $(L_{j,r}(k))_{k\in\mathbb{Z}}$ with some $j \in \mathbb{Z}_-$ and $r \in \mathbb{Z}_+$, see (3.15).

The starting point of the random walk representation (3.29) follows by definition. On the other hand, equations (3.30)–(3.32) can also be seen easily from (3.2) and the definitions (3.15) and (3.25)–(3.28). These formulas yield that starting from position $j \in \mathbb{Z}_-$, one might generate the sequence of local times $L_{j,r}(k)$ step by step by always adding a new independent random variable, that is, as a random walk. Note that the expected jump size is $1/2$ between $j$ and 0, $-1/2$ above 0 and $-1/2$ below $j$ backwards. It is in accordance with (3.17). At this point, we have to show that the two-sided random walk $L_{j,r}(k)$ does not differ too much from the expected behaviour.

For taking the scaling limit, we set $j = \lfloor Ax \rfloor$ and $r = \lfloor Ah \rfloor$. By Lemma 3.4.1, as long as $L_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(k) > A^{1/2+\varepsilon}$, the Markov chains $\eta_{k,\pm}$ which appear in the increments $L_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(k) - L_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(k-1)$ are exponentially close to the stationary distribution. This enables us to couple them efficiently to the i.i.d. copies $\widetilde{\eta}_k(m)$ of the *stationary* Markov chain, that is, for which $\mathbf{P}(\widetilde{\eta}_k(m) = x) = \rho(x)$ for all $k \in \mathbb{Z}$ and $m \in \mathbb{Z}$.

We build up the walk $\widetilde{L}_{j,r}(k)$ in (3.36) similarly to (3.29)–(3.32) where the increments of $\widetilde{L}_{j,r}$ follow exactly the stationary distribution $\rho$, see (3.37). It is a standard large deviation estimate that the fluctuation of the random walk $\widetilde{L}_{j,r}$ around its mean value is uniformly smaller than $A^{1/2+\varepsilon}$ with high probability, see (3.39). By the construction of the coupling, it can be seen that $L_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(k) = \widetilde{L}_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(k)$ with high probability as long as both are above the threshold $A^{1/2+\varepsilon}$ as given in (3.45). On the other hand, (3.46) yields that, close to $k = \pm(A(|x|+2h))$, the random walk $L_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(k)$ approaches 0 with high probability.

The main part of Subsection 3.4.4 is Lemma 3.4.3 which gives an upper bound on the expected hitting time of 0 for $L$ starting from any level $r \in \mathbb{Z}_+$. By applying Markov's inequality, we get that if $L$ is below the threshold given in (3.46), then it hits 0 within time $A^{1/2+\varepsilon}$ with high probability, see (3.47) and (3.48). This completes the proof of Theorem

3.2.2.

### 3.3.2 Theorem 3.2.4

With the definitions (3.58)–(3.60), the assertion of Theorem 3.2.4 can be expressed as (3.61) with a pointwise convergence of Laplace transforms. The crucial identity here is (3.63) which connects the distribution of the displacement $X(n)$ with that of the inverse local times $T^{\pm}_{k,m}$. Therefore, we can use the limit theorem stated in Corollary 3.2.3 which is equivalent to (3.67) by the definitions (3.62) and (3.65). Hence the pointwise limit of the rescaled Laplace transform $\widehat{\varphi}_A$ can be given by using Fatou's lemma, and the result of Theorem 3.2.4 follows.

## 3.4 Proof of the limit theorem for the local times

In this section, we first introduce independent auxiliary Markov chains associated to the vertices of $\mathbb{Z}$ in such a way that the value of the local time at the edges can be expressed with a sum of such Markov chains. It turns out that the auxiliary Markov chains converge exponentially fast to their common unique stationary distribution. It allows us to couple the local time process of the self-repelling random walk with the sum of i.i.d. random variables. The coupling yields that the law of large numbers for i.i.d. variables can be applied for the behaviour of the local time, with high probability. The coupling argument breaks down when the local time approaches 0. We show in Subsection 3.4.4, how to handle this case.

Let

$$L_{j,r}(k) := \ell^{+}(T^{+}_{j,r}, k). \tag{3.15}$$

Mind that due to (3.2), (3.3) and (3.5)

$$\left|\Lambda^{+}_{j,r}(k) - 2L_{j,r}(k)\right| \leq 1. \tag{3.16}$$

We give the proof of (3.6), (3.7) and

$$\sup_{y\in\mathbb{R}} \left| A^{-1} L_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(\lfloor Ay\rfloor) - \left(\frac{|x|-|y|}{2} + h\right)_{+} \right| \xrightarrow{\mathbf{P}} 0 \tag{3.17}$$

as $A \to \infty$, which, due to (3.16), implies (3.8) for $\Lambda^{+}$. The case of $\Lambda^{-}$ can be done similarly. Without loss of generality, we can suppose that $x \leq 0$.

### 3.4.1 Auxiliary Markov chains

First, we define the $\mathbb{Z}$-valued Markov chain $l \mapsto \xi(l)$ with the following transition probabilities:

$$\mathbf{P}\left(\xi(l+1) = x+1 \,\middle|\, \xi(l) = x\right) = \frac{w(-x)}{w(x)+w(-x)} =: p(x), \tag{3.18}$$

$$\mathbf{P}\left(\xi(l+1) = x-1 \,\middle|\, \xi(l) = x\right) = \frac{w(x)}{w(x)+w(-x)} =: q(x). \tag{3.19}$$

Let $\tau_\pm(m)$, $m = 0, 1, 2, \ldots$ be the stopping times of consecutive upwards/downwards steps of $\xi$:

$$\tau_\pm(0) := 0, \qquad \tau_\pm(m+1) := \min\left\{l > \tau_\pm(m) : \xi(l) = \xi(l-1) \pm 1\right\}.$$

Then, clearly, the processes

$$\eta_+(m) := -\xi(\tau_+(m)), \qquad \eta_-(m) := +\xi(\tau_-(m))$$

are themselves Markov chains on $\mathbb{Z}$. Due to the $\pm$ symmetry of the process $\xi$, the Markov chains $\eta_+$ and $\eta_-$ have the same law. In the present subsection, we simply denote them by $\eta$ neglecting the subscripts $\pm$. The transition probabilities of this process are

$$P(x,y) := \mathbf{P}\left(\eta(m+1) = y \,\middle|\, \eta(m) = x\right) = \begin{cases} \prod_{z=x}^{y} p(z) q(y+1) & \text{if} \quad y \geq x-1, \\ 0 & \text{if} \quad y < x-1. \end{cases} \tag{3.20}$$

In the following lemma, we collect the technical ingredients of the forthcoming proof of our limit theorems. We identify the stationary measure of the Markov chain $\eta$, state exponential tightness of the distributions of $\big(\eta(m) \,\big|\, \eta(0) = 0\big)$ uniformly in $m$ and state exponentially fast convergence to stationarity. The proof of the lemma is postponed to Subsection 3.4.5.

**Lemma 3.4.1.** *(i) The unique stationary measure of the Markov chain $\eta$ is*

$$\rho(x) = Z^{-1} \prod_{z=1}^{\lfloor |2x+1|/2 \rfloor} \frac{w(-z)}{w(z)} \quad \textit{with} \quad Z := 2 \sum_{x=0}^{\infty} \prod_{z=1}^{x} \frac{w(-z)}{w(z)}. \tag{3.21}$$

*(ii) There exist constants $C < \infty$, $\beta > 0$ such that for all $m \in \mathbb{N}$*

$$P^m(0,y) \leq C e^{-\beta |y|}. \tag{3.22}$$

*(iii) There exist constants $C < \infty$ and $\beta > 0$ such that for all $m \geq 0$*

$$\sum_{y \in \mathbb{Z}} |P^m(0,y) - \rho(y)| < C e^{-\beta m}. \tag{3.23}$$

**Remark on notation:** We shall use the generic notation for a function $f$

$$f(Y) \leq Ce^{-\beta Y}$$

for exponentially strong bounds. The constants $C < \infty$ and $\beta > 0$ will vary at different occurrences and they may (and will) depend on various fixed parameters but of course not on quantities appearing in the expression $Y$. There will be no cause for confusion.

Note that for any choice of the weight function $w$

$$\sum_{x=-\infty}^{+\infty} x\rho(x) = -\frac{1}{2}. \tag{3.24}$$

### 3.4.2 The basic construction

For $j \in \mathbb{Z}$, denote the inverse local times (times of jumps leaving site $j \in \mathbb{Z}$)

$$\gamma_j(l) := \min\left\{n : \ell^+(n,j) + \ell^-(n,j) \geq l\right\}, \tag{3.25}$$

and

$$\xi_j(l) := \ell^+(\gamma_j(l), j) - \ell^-(\gamma_j(l), j), \tag{3.26}$$

$$\tau_{j,\pm}(0) := 0, \qquad \tau_{j,\pm}(m+1) := \min\{l > \tau_{j,\pm}(m) : \xi_j(l) = \xi_j(l-1) \pm 1\}, \tag{3.27}$$

$$\eta_{j,+}(m) := -\xi_j(\tau_{j,+}(m)), \qquad \eta_{j,-}(m) := +\xi_j(\tau_{j,-}(m)). \tag{3.28}$$

The following proposition is the key to the Ray−Knight approach.

**Proposition 3.4.2.** *(i) The processes $l \mapsto \xi_j(l)$, $j \in \mathbb{Z}$, are independent copies of the Markov chain $l \mapsto \xi(l)$, defined in Subsection 3.4.1, starting with initial conditions $\xi_j(0) = 0$.*

*(ii) As a consequence: the processes $k \mapsto \eta_{j,\pm}(k)$, $j \in \mathbb{Z}$, are independent copies of the Markov chain $m \mapsto \eta_\pm(m)$, if we consider exactly one of $\eta_{j,+}$ and $\eta_{j,-}$ for each $j$. The initial conditions are $\eta_{j,\pm}(0) = 0$.*

The statement is intuitively clear. The mathematical content of the driving rules (3.4) of the random walk $X(n)$ is exactly this: whenever the walk visits a site $j \in \mathbb{Z}$, the probability of jumping to the left or to the right (i.e. to site $j-1$ or to site $j+1$), conditionally on the whole past, will depend only on the difference of the number of past jumps from $j$ to $j-1$, respectively, from $j$ to $j+1$, and independent of what had happened at other sites. The more lengthy formal proof goes through exactly the same steps as the corresponding statement in [T95]. We omit here the formal proof.

Fix now $j \in \mathbb{Z}_-$ and $r \in \mathbb{N}$. The definitions (3.15) and (3.25)–(3.28) imply that

$$L_{j,r}(j) = r, \tag{3.29}$$

$$L_{j,r}(k+1) = L_{j,r}(k) + 1 + \eta_{k+1,-}(L_{j,r}(k)+1), \qquad j \le k < 0, \tag{3.30}$$

$$L_{j,r}(k+1) = L_{j,r}(k) + \eta_{k+1,-}(L_{j,r}(k)), \qquad 0 \le k < \infty, \tag{3.31}$$

$$L_{j,r}(k-1) = L_{j,r}(k) + \eta_{k,+}(L_{j,r}(k)), \qquad -\infty < k \le j. \tag{3.32}$$

Similar formulas are found for $j \in \mathbb{Z}_+$ and $r \in \mathbb{N}$.

Note that if $L_{j,r}(k_0) = 0$ for some $k_0 \ge 0$ (respectively, for some $k_0 \le j$) then $L_{j,r}(k) = 0$ for all $k \ge k_0$ (respectively, for all $k \le k_0$).

The idea of the further steps of proof can be summarized in terms of the above setup. With fixed $x \in \mathbb{R}_-$ and $h \in \mathbb{R}_+$, we choose $j = \lfloor Ax \rfloor$ and $r = \lfloor Ah \rfloor$ with the scaling parameter $A \to \infty$ at the end. We know from Lemma 3.4.1 that the Markov chains $\eta_{j,\pm}$ converge exponentially fast to their stationary distribution $\rho$. This allows us to couple efficiently the increments $L_{\lfloor Ax \rfloor,\lfloor Ah \rfloor}(k+1) - L_{\lfloor Ax \rfloor,\lfloor Ah \rfloor}(k)$ with properly chosen i.i.d. random variables as long as the value of $L_{\lfloor Ax \rfloor,\lfloor Ah \rfloor}(k) > A^{1/2+\varepsilon}$ and to use the law of large numbers. This coupling does not apply when the value of $L_{\lfloor Ax \rfloor,\lfloor Ah \rfloor}(k) < A^{1/2+\varepsilon}$. We prove that once the value of $L_{\lfloor Ax \rfloor,\lfloor Ah \rfloor}(k)$ drops below this threshold, $L_{\lfloor Ax \rfloor,\lfloor Ah \rfloor}(k)$ hits zero (and sticks there) in $o(A)$ time, with high probability. These steps of the proof are presented in the next two subsections.

### 3.4.3 Coupling

We are in the context of the representation (3.29)–(3.32) with $j = \lfloor Ax \rfloor$, $r = \lfloor Ah \rfloor$. Due to Lemma 3.4.1, we can realize jointly the *pairs of coupled processes*

$$\big(m \mapsto (\eta_{k,-}(m), \widetilde{\eta}_k(m))\big)_{k>j}, \qquad \big(m \mapsto (\eta_{k,+}(m), \widetilde{\eta}_k(m))\big)_{k\le j} \tag{3.33}$$

with the following properties.

– The pairs of coupled processes with different $k$-indices are independent.

– The processes $\big(m \mapsto \eta_{k,-}(m)\big)_{k>j}$ and $\big(m \mapsto \eta_{k,+}(m)\big)_{k\le j}$ are those of the previous subsection, i.e. they are independent copies of the Markov chain $m \mapsto \eta(m)$ with initial conditions $\eta_{k,\pm}(0) = 0$.

– The processes $\big(m \mapsto \widetilde{\eta}_k(m)\big)_{k\in\mathbb{Z}}$ are independent copies of the *stationary* process $m \mapsto \eta(m)$, i.e. these processes are initialized independently with $\mathbf{P}\left(\widetilde{\eta}_k(0) = x\right) = \rho(x)$ and run independently of one another.

– The pairs of coupled processes $m \mapsto (\eta_{k,\pm}(m), \widetilde{\eta}_k(m))$ are coalescing. This means the following: we define the coalescence time

$$\mu_k := \inf\{m \ge 0 : \eta_{k,\pm}(m) = \widetilde{\eta}_k(m)\}. \tag{3.34}$$

Then, for $m \geq \mu_k$, the two processes stick together: $\eta_{k,\pm}(m) = \widetilde{\eta}(m)$. Mind that the random variables $\mu_k$, $k \in \mathbb{Z}$ are i.i.d.

– The tail of the distribution of the coalescence times decays exponentially fast:

$$\mathbf{P}\left(\mu_k > m\right) < Ce^{-\beta m}. \tag{3.35}$$

We define the processes $k \mapsto \widetilde{L}_{j,r}(k)$ similarly to the processes $k \mapsto L_{j,r}(k)$ in (3.29)–(3.32) with the $\eta$'s replaced by the $\widetilde{\eta}$'s:

$$\begin{aligned}
&\widetilde{L}_{j,r}(j) = r, \\
&\widetilde{L}_{j,r}(k+1) = \widetilde{L}_{j,r}(k) + 1 + \widetilde{\eta}_{k+1,-}(\widetilde{L}_{j,r}(k)+1), && j \leq k < 0, \\
&\widetilde{L}_{j,r}(k+1) = \widetilde{L}_{j,r}(k) + \widetilde{\eta}_{k+1,-}(\widetilde{L}_{j,r}(k)), && 0 \leq k < \infty, \\
&\widetilde{L}_{j,r}(k-1) = \widetilde{L}_{j,r}(k) + \widetilde{\eta}_{k,+}(\widetilde{L}_{j,r}(k)), && -\infty < k \leq j.
\end{aligned} \tag{3.36}$$

Note that the increments of this process are *independent* with distribution

$$\begin{aligned}
&\mathbf{P}\left(\widetilde{L}_{j,r}(k+1) - \widetilde{L}_{j,r}(k) = z\right) = \rho(z-1), && j \leq k < 0, \\
&\mathbf{P}\left(\widetilde{L}_{j,r}(k+1) - \widetilde{L}_{j,r}(k) = z\right) = \rho(z), && 0 \leq k < \infty, \\
&\mathbf{P}\left(\widetilde{L}_{j,r}(k-1) - \widetilde{L}_{j,r}(k) = z\right) = \rho(z), && -\infty < k \leq j.
\end{aligned} \tag{3.37}$$

Hence, from (3.24), it follows that for any $K < \infty$

$$\sup_{|y| \leq K} \left| A^{-1}\widetilde{L}_{\lfloor Ax \rfloor, \lfloor Ah \rfloor}(\lfloor Ay \rfloor) - \left((|x| - |y|)/2 + h\right) \right| \xrightarrow{\mathbf{P}} 0. \tag{3.38}$$

Actually, by Doob's inequality, the following large deviation estimate holds: for any $x \in \mathbb{R}$, $h \in \mathbb{R}_+$ and $K < \infty$ fixed

$$\mathbf{P}\left( \sup_{|k| \leq AK} \left| \widetilde{L}_{\lfloor Ax \rfloor, \lfloor Ah \rfloor}(k) - \left((A|x| - |k|)/2 + Ah\right) \right| > A^{1/2+\varepsilon} \right) < Ce^{-\beta A^{2\varepsilon}}. \tag{3.39}$$

(The constants $C < \infty$ and $\beta > 0$ do depend on the fixed parameters $x$, $h$ and $K$.) Denote now

$$\begin{aligned}
\kappa^+_{j,r} &:= \min\{k \geq j : L_{j,r}(k) \neq \widetilde{L}_{j,r}(k)\}, \\
\kappa^-_{j,r} &:= \max\{k \leq j : L_{j,r}(k) \neq \widetilde{L}_{j,r}(k)\},
\end{aligned}$$

i.e. the coupling of $L_{j,r}$ and $\widetilde{L}_{j,r}$ works on the interval $(\kappa^-_{j,r}, \kappa^+_{j,r})$. Then, for $k \geq j$:

$$\begin{aligned}
&\mathbf{P}\left(\kappa^+_{j,r} \leq k+1\right) - \mathbf{P}\left(\kappa^+_{j,r} \leq k\right) \\
&\quad = \mathbf{P}\left(\kappa^+_{j,r} = k+1, \widetilde{L}_{j,r}(k) \leq A^{1/2+\varepsilon}\right) + \mathbf{P}\left(\kappa^+_{j,r} = k+1, \widetilde{L}_{j,r}(k) \geq A^{1/2+\varepsilon}\right) \\
&\quad \leq \mathbf{P}\left(\widetilde{L}_{j,r}(k) \leq A^{1/2+\varepsilon}\right) + \mathbf{P}\left(\kappa^+_{j,r} = k+1 \;\middle|\; \kappa^+_{j,r} > k, L_{j,r}(k) = \widetilde{L}_{j,r}(k) \geq A^{1/2+\varepsilon}\right).
\end{aligned} \tag{3.40}$$

Similarly, for $k \leq j$:

$$\begin{aligned}
&\mathbf{P}\left(\kappa^-_{j,r} \geq k-1\right) - \mathbf{P}\left(\kappa^-_{j,r} \geq k\right) \\
&\quad = \mathbf{P}\left(\kappa^-_{j,r} = k-1, \widetilde{L}_{j,r}(k) \leq A^{1/2+\varepsilon}\right) + \mathbf{P}\left(\kappa^-_{j,r} = k-1, \widetilde{L}_{j,r}(k) \geq A^{1/2+\varepsilon}\right) \\
&\quad \leq \mathbf{P}\left(\widetilde{L}_{j,r}(k) \leq A^{1/2+\varepsilon}\right) + \mathbf{P}\left(\kappa^-_{j,r} = k-1 \;\middle|\; \kappa^-_{j,r} < k, L_{j,r}(k) = \widetilde{L}_{j,r}(k) \geq A^{1/2+\varepsilon}\right).
\end{aligned} \tag{3.41}$$

Now, from (3.39), it follows that for $|k| \leq A(|x|+2h) - 4A^{1/2+\varepsilon}$

$$\mathbf{P}\left(\widetilde{L}_{j,r}(k) \leq A^{1/2+\varepsilon}\right) \leq Ce^{-\beta A^{2\varepsilon}}. \tag{3.42}$$

On the other hand, from (3.35),

$$\mathbf{P}\left(\kappa^+_{j,r} = k+1 \;\middle|\; \kappa^+_{j,r} > k, L_{j,r}(k) = \widetilde{L}_{j,r}(k) \geq A^{1/2+\varepsilon}\right) \leq Ce^{-\beta A^{1/2+\varepsilon}}, \tag{3.43}$$

$$\mathbf{P}\left(\kappa^-_{j,r} = k-1 \;\middle|\; \kappa^-_{j,r} < k, L_{j,r}(k) = \widetilde{L}_{j,r}(k) \geq A^{1/2+\varepsilon}\right) \leq Ce^{-\beta A^{1/2+\varepsilon}} \tag{3.44}$$

with some constants $C < \infty$ and $\beta > 0$ which do depend on all fixed parameters and may vary from formula to formula.

Putting together (3.40), (3.42), (3.43), respectively, (3.41), (3.42), (3.44) and noting that $\mathbf{P}\left(\kappa^+_{j,r} = j\right) = 0$, we conclude that

$$\mathbf{P}\left(\min\left\{|k| : L_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(k) \neq \widetilde{L}_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(k)\right\} \leq A(|x|+2h) - 4A^{1/2+\varepsilon}\right) \leq CAe^{-\beta A^{2\varepsilon}}, \tag{3.45}$$

the coupling does not break down between $-A(|x|+2h)+4A^{1/2+\varepsilon}$ and $A(|x|+2h)-4A^{1/2+\varepsilon}$. On the other hand, at these two values of $k$, $L_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(k)$ is not too large:

$$\mathbf{P}\left(L_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(\pm\lfloor A(|x|+2h) - 4A^{1/2+\varepsilon}\rfloor) \geq 3A^{1/2+\varepsilon}\right) \leq Ce^{-\beta A^{2\varepsilon}}. \tag{3.46}$$

### 3.4.4 Hitting of 0

It follows from Lemma 3.4.1 that all moments of the distributions $P^n(0,\cdot)$ converge to the corresponding moments of $\rho$. In particular, for any $\delta > 0$, there exists $n_\delta < \infty$ such that

$$\sum_{x\in\mathbb{Z}} P^n(0,x)x \leq -\frac{1}{2+\delta}$$

holds if $n \geq n_\delta$.

Consider now the Markov chains defined by (3.31) or (3.32) (the two are identical in law):

$$L(k+1) = L(k) + \eta_{k+1}(L(k)), \qquad L(0) = r \in \mathbb{N},$$

where $m \mapsto \eta_k(m)$, $k = 1,2,3,\ldots$ are i.i.d. copies of the Markov chain $m \mapsto \eta(m)$ with initial conditions $\eta_k(0) = 0$. Define the stopping times

$$\tau_x := \min\{k : L(k) \leq x\}, \qquad x = 0,1,2,\ldots.$$

**Lemma 3.4.3.** *For any $\delta > 0$ there exists $K_\delta < \infty$ such that for any $r \in \mathbb{N}$:*

$$\mathbf{E}\left(\tau_0 \mid L(0) = r\right) \le (2+\delta)r + K_\delta.$$

*Proof.* Clearly,

$$\mathbf{E}\left(\tau_0 \mid L(0) = r\right) \le \mathbf{E}\left(\tau_{n_\delta} \mid L(0) = r\right) + \max_{0 \le s \le n_\delta} \mathbf{E}\left(\tau_0 \mid L(0) = s\right).$$

Now, by optional stopping,

$$\mathbf{E}\left(\tau_{n_\delta} \mid L(0) = r\right) \le (2+\delta)r,$$

and obviously

$$K_\delta := \max_{0 \le s \le n_\delta} \mathbf{E}\left(\tau_0 \mid L(0) = s\right) < \infty.$$

□

In particular, choosing $\delta = 1$ and applying Markov's inequality, it follows that

$$\begin{aligned}\mathbf{P}\Big(\rho^+_{\lfloor Ax\rfloor,\lfloor Ax\rfloor} > A(|x|+2h) + A^{1/2+2\varepsilon} \;\Big|\; L_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(\lfloor A(|x|+2h) - 4A^{1/2+\varepsilon}\rfloor) \le 3A^{1/2+\varepsilon}\Big) \\ \le \frac{9A^{1/2+\varepsilon} + K_1}{5A^{1/2+2\varepsilon}} < 10A^{-\varepsilon},\end{aligned} \tag{3.47}$$

and similarly

$$\begin{aligned}\mathbf{P}\Big(\lambda^+_{\lfloor Ax\rfloor,\lfloor Ax\rfloor} < -A(|x|+2h) - A^{1/2+2\varepsilon} \;\Big|\; L_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(-\lfloor A(|x|+2h) + 4A^{1/2+\varepsilon}\rfloor) \le 3A^{1/2+\varepsilon}\Big) \\ \le \frac{9A^{1/2+\varepsilon} + K_1}{5A^{1/2+2\varepsilon}} < 10A^{-\varepsilon}.\end{aligned} \tag{3.48}$$

Eventually, Theorem 3.2.2 follows from (3.38), (3.45), (3.46), (3.47) and (3.48).

### 3.4.5 Proof of Lemma 3.4.1

*Proof.* The following proof is reminiscent of the proof of Lemmas 1 and 2 from [T95]. It is somewhat streamlined and weaker conditions are assumed.

(i) The irreducibility of the Markov chain $\eta$ is straightforward. One can easily rewrite (3.20) using (3.21) as

$$P(x,y) = \begin{cases} \frac{1}{\rho(x)}\left(p(x)\prod_{z=x+1}^{y+1} q(z)\right)\rho(y) & \text{if} \quad y \ge x-1, \\ 0 & \text{if} \quad y < x-1. \end{cases}$$

It yields that $\rho$ is indeed stationary distribution for $\eta$, because

$$\sum_{x\in\mathbb{Z}} \rho(x)P(x,y) = \left(\sum_{x \le y+1} p(x) \prod_{z=x+1}^{y+1} q(z)\right)\rho(y) = \rho(y)$$

where the last equality holds, because $\lim_{z\to-\infty}\prod_{u=z}^{y+1} q(u) = 0$.

(ii) The stationarity of $\rho$ implies that

$$P^n(0,y) \le \frac{\rho(y)}{\rho(0)} = \prod_{z=1}^{\lfloor|2y+1|/2\rfloor} \frac{w(-z)}{w(z)} \le Ce^{-\beta|y|}. \tag{3.49}$$

The exponential bound follows from (3.1). As a consequence, we get finite expectations in the forthcoming steps of the proofs below.

(iii) Define the stopping times

$$\begin{aligned} \theta_+ &= \min\{n \ge 0 : \eta(n) \ge 0\}, \\ \theta_0 &= \min\{n \ge 0 : \eta(n) = 0\}. \end{aligned}$$

From Theorem 6.14 and Example 5.5(a) of [N84], we can conclude the exponential convergence (3.23), if for some $\gamma > 0$

$$\mathbf{E}\left(\exp(\gamma\theta_0) \;\middle|\; \eta(0) = 0\right) < \infty \tag{3.50}$$

holds.

The following decomposition is true, because the Markov chain $\eta$ can jump at most one step to the left.

$$\begin{aligned} \mathbf{E}\left(\exp(\gamma\theta_0) \;\middle|\; \eta(0) = 0\right) &= e^\gamma \sum_{y\ge 0} P(0,y)\mathbf{E}\left(\exp(\gamma\theta_0) \;\middle|\; \eta(0) = y\right) \\ &+ e^\gamma P(0,-1) \sum_{y\ge 0} \mathbf{E}\left(\exp(\gamma\theta_+)1\!\!1(\eta(\theta_+) = y) \;\middle|\; \eta(0) = -1\right) \mathbf{E}\left(\exp(\gamma\theta_0) \;\middle|\; \eta(0) = y\right). \end{aligned} \tag{3.51}$$

One can easily check that, given $\eta(0) = -1$, the random variables $\theta_+$ and $\eta(\theta_+)$ are independent by the form of (3.20), and the distribution of $\eta(\theta_+)$ is the same as that of $\eta(1)$ conditionally given $\eta(0) = 0$ and $\eta(1) \ge 0$. Hence for $y \ge 0$

$$\mathbf{E}\left(\exp(\gamma\theta_+)1\!\!1(\eta(\theta_+) = y) \;\middle|\; \eta(0) = -1\right) = \frac{P(0,y)}{1-P(0,-1)}\mathbf{E}\left(\exp(\gamma\theta_+) \;\middle|\; \eta(0) = -1\right). \tag{3.52}$$

Combining (3.51) and (3.52) gives us

$$\begin{aligned} &\mathbf{E}\left(\exp(\gamma\theta_0) \;\middle|\; \eta(0) = 0\right) \\ &= e^\gamma \sum_{y\ge 0} P(0,y)\mathbf{E}\left(\exp(\gamma\theta_0) \;\middle|\; \eta(0) = y\right) \left(1 + \frac{P(0,-1)}{1-P(0,-1)}\mathbf{E}\left(\exp(\gamma\theta_+) \;\middle|\; \eta(0) = -1\right)\right). \end{aligned} \tag{3.53}$$

So, in order to get the result, we need to prove that for properly chosen $\gamma > 0$

$$\mathbf{E}\left(\exp(\gamma\theta_+) \;\middle|\; \eta(0) = -1\right) < \infty \tag{3.54}$$

and

$$\mathbf{E}\left(\exp(\gamma\theta_0) \;\middle|\; \eta(0) = y\right) \le e^{\frac{\beta}{2}y} \qquad \text{for } y \in \mathbb{Z}_+ \tag{3.55}$$

where $\beta$ is the constant in (3.22).

In order to make the argument shorter, we make the assumption

$$w(-1) < w(+1),$$

or, equivalently,

$$p(1) = \frac{w(-1)}{w(+1) + w(-1)} < \frac{1}{2} < \frac{w(+1)}{w(+1) + w(-1)} = q(1).$$

The proof can be easily extended for the weaker assumption (3.1), but the argument is somewhat longer.

First, we prove (3.54). Let $x < 0$ and $x - 1 \le y < 0$. Then the following stochastic domination holds:

$$\sum_{z \ge y} P(x, z) = \prod_{z=x}^{y} p(z) \ge p(-1)^{y-x+1} = q(1)^{y-x+1}. \tag{3.56}$$

Let $\zeta(r)$, $r = 1, 2, \dots$ be i.i.d. random variables with geometric law:

$$\mathbf{P}(\zeta = z) = q(1)^{z+1} p(1), \quad z = -1, 0, 1, 2, \dots,$$

and

$$\widetilde{\theta} := \min\left\{t \ge 0 : \sum_{s=1}^{t} \zeta(s) \ge 1\right\}.$$

Note that $\mathbf{E}(\zeta) > 0$. From the stochastic domination (3.56), it follows that

$$\mathbf{P}\left(\theta_+ > t \;\middle|\; \eta(0) = -1\right) \le \mathbf{P}\left(\widetilde{\theta} > t\right),$$

and hence

$$\mathbf{E}\left(\exp(\gamma\theta_+) \;\middle|\; \eta(0) = -1\right) \le \mathbf{E}\left(\exp(\gamma\widetilde{\theta})\right) < \infty$$

for sufficiently small $\gamma > 0$.

Now, we turn to (3.55). Let now $0 \le x - 1 \le y$. In this case, the following stochastic domination is true:

$$\sum_{z \ge y} P(x, z) = \prod_{z=x}^{y} p(z) \le p(1)^{y-x+1}. \tag{3.57}$$

Let now $\zeta(r)$, $r = 1, 2, \dots$ be i.i.d. random variables with geometric law:

$$\mathbf{P}(\zeta = z) = p(1)^{z+1} q(1), \quad z = -1, 0, 1, 2, \dots,$$

and for $y \ge 0$

$$\widetilde{\theta}_y := \min\left\{t \ge 0 : \sum_{s=1}^{t} \zeta(s) \le -y\right\}.$$

Note that now $\mathbf{E}(\zeta) < 0$. From the stochastic domination (3.57), it follows now that with $y \geq 0$

$$\mathbf{P}\left(\theta_0 > t \mid \eta(0) = y\right) \leq \mathbf{P}\left(\widetilde{\theta}_y > t\right),$$

and hence

$$\mathbf{E}\left(\exp(\gamma\theta_0) \mid \eta(0) = y\right) \leq \mathbf{E}\left(\exp(\gamma\widetilde{\theta}_y)\right) \leq Ce^{\frac{\beta}{2}y},$$

for sufficiently small $\gamma > 0$. □

## 3.5 Proof of the theorem for the position of the walker

First, we introduce the following notations. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, let

$$P(n,k) := \mathbf{P}\left(X(n) = k\right) \tag{3.58}$$

be the distribution of the position of the random walker. For $s \in \mathbb{R}_+$,

$$R(s,k) := (1 - e^{-s}) \sum_{n=0}^{\infty} e^{-sn} P(n,k) \tag{3.59}$$

is the distribution of $X(\theta_s)$ where $\theta_s$ has geometric distribution (3.13) and it is independent of $X(n)$.

Also (3.12) tells us that the proper definition of the rescaled distribution is

$$\varphi_A(t,x) := A^{1/2} P(\lfloor At \rfloor, \lfloor A^{1/2} x \rfloor),$$

if $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Let

$$\widehat{\varphi}_A(s,x) := A^{1/2} R(A^{-1}s, \lfloor A^{1/2} x \rfloor) \tag{3.60}$$

which is asymptotically the Laplace transform of $\varphi_A$ as $A \to \infty$.

With these definitions, the statement of Theorem 3.2.4 is equivalent to

$$\widehat{\varphi}_A(s,x) \to \widehat{\varphi}(s,x), \tag{3.61}$$

which is proved below.

We will need the Laplace transform of the limit of the inverse local times $\mathcal{T}_{x,h}$ given in (3.10). In this case, $\mathcal{T}_{x,h}$ is deterministic by Corollary 3.2.3. Hence the Laplace transform is given by

$$\widehat{\rho}(s,x,h) = s\,\mathbf{E}\left(\exp\left(-s\,\mathcal{T}_{x,h}\right)\right) = se^{-s(|x|+2h)^2}, \tag{3.62}$$

for which

$$\widehat{\varphi}(s,x) = 2\int_0^{\infty} \widehat{\rho}(s,|x|,h)\mathrm{d}h$$

holds.

*Proof of Theorem 3.2.4.* Fix $x \geq 0$. We can proceed in the case $x \leq 0$ similarly. We start with the identity

$$P(n,k) = \mathbf{P}\left(X(n) = k\right) = \sum_{m=0}^{\infty} \left(\mathbf{P}\left(T^{+}_{k-1,m} = n\right) + \mathbf{P}\left(T^{-}_{k+1,m} = n\right)\right), \tag{3.63}$$

which holds, because the event $\{T^{+}_{k-1,m} = n\}$ means that the $n$th step of the walker $X$ is exactly the $m$th $k-1 \to k$ jump. From the definitions (3.59) and (3.60),

$$\begin{aligned} \widehat{\varphi}_A(s,x) &= \frac{1-e^{-s/A}}{s/A} s \sum_{m=0}^{\infty} \frac{1}{\sqrt{A}} e^{-ns/A} P(n, \lfloor A^{1/2} x \rfloor) \\ &= \frac{1-e^{-s/A}}{s/A} s \sum_{m=0}^{\infty} \frac{1}{\sqrt{A}} \left(\mathbf{E}\left(e^{-T^{+}_{\lfloor A^{1/2}x\rfloor -1,m} s/A}\right) + \mathbf{E}\left(e^{-T^{-}_{\lfloor A^{1/2}x\rfloor +1,m} s/A}\right)\right) \end{aligned} \tag{3.64}$$

where we used (3.63) in the second equality. Let

$$\widehat{\rho}^{\pm}_A(s,x,h) = s\,\mathbf{E}\left(\exp\left(-\frac{s}{A} T^{\pm}_{\lfloor A^{1/2}x\rfloor, \lfloor A^{1/2}h\rfloor}\right)\right). \tag{3.65}$$

Then (3.64) can be written as

$$\widehat{\varphi}_A(s,x) = \frac{1-e^{-s/A}}{s/A} \int_0^{\infty} \left(\widehat{\rho}^{+}_A(s, x - A^{-1/2}, h) + \widehat{\rho}^{-}_A(s, x + A^{-1/2}, h)\right) \mathrm{d}h. \tag{3.66}$$

It follows from (3.9) that, for all $s > 0$, $x \in \mathbb{R}$ and $h > 0$,

$$\widehat{\rho}^{\pm}_A(s,x,h) \to \widehat{\rho}(s,x,h) \tag{3.67}$$

as $A \to \infty$. Applying Fatou's lemma in (3.66) yields

$$\liminf_{A\to\infty} \widehat{\varphi}_A(s,x) \geq 2\int_0^{\infty} \widehat{\rho}(s,x,h)\mathrm{d}h = \widehat{\varphi}(s,x).$$

If we use Fatou's lemma again, we get

$$1 = \int_{-\infty}^{\infty} \widehat{\varphi}(s,x)\mathrm{d}x \leq \int_{-\infty}^{\infty} \liminf_{A\to\infty} \widehat{\varphi}_A(s,x)\mathrm{d}x \leq \liminf_{A\to\infty} \int_{-\infty}^{\infty} \widehat{\varphi}_A(s,x)\mathrm{d}x = 1,$$

which gives for all $s \in \mathbb{R}$ that

$$\widehat{\varphi}(s,x) = \liminf_{A\to\infty} \widehat{\varphi}_A(s,x) \tag{3.68}$$

holds for almost all $x \in \mathbb{R}$. Note that (3.68) is also true for any subsequence $A_k \to \infty$, which implies the assertion of Theorem 3.2.4. □

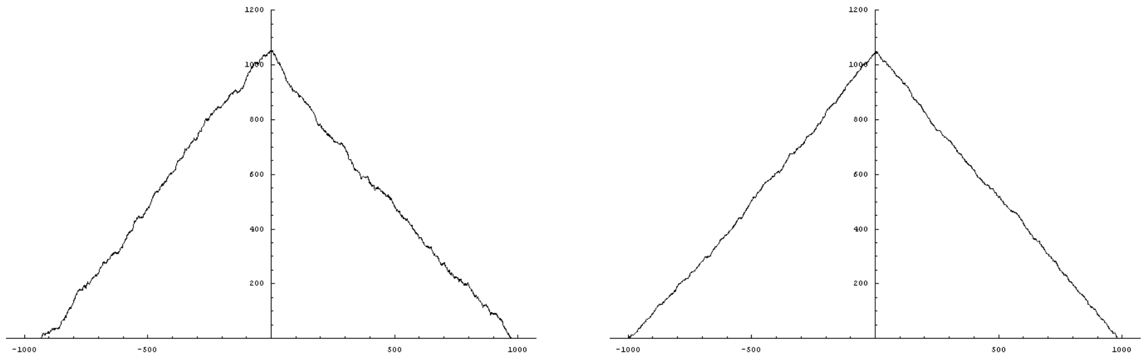

Figure 3.2: The local time process of the random walk with $w(k) = 2^k$ and $w(k) = 10^k$

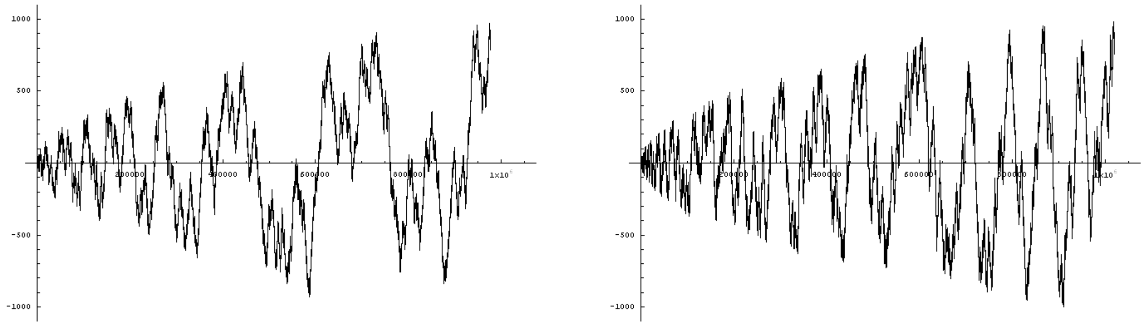

Figure 3.3: The trajectories of the random walk with $w(k) = 2^k$ and $w(k) = 10^k$

## 3.6 Computer simulations

We have prepared computer simulations with exponential weight functions $w(k) = 2^k$ and $w(k) = 10^k$.

Note that the limit objects in our theorems do not depend on the choice of the weight function $w$. Therefore, we expect that the behaviour of the local time and the trajectories is qualitatively similar, and we will find only quantitative differences.

Figure 3.2 shows the local time process of the random walk after approximately $10^6$ steps. More precisely, we have plotted the value of $\Lambda^+_{100,800}$ with $w(k) = 2^k$ and $w(k) = 10^k$ respectively. One can see that the limits are the same in the two cases – according to Theorem 3.2.2 – but the rate of convergence does depend on the choice of $w$. We can conclude the empirical rule that the faster the weight function grows at infinity, the faster the convergence of the local time process is.

The difference between the trajectories of random walks generated with various weights is more conspicuous. On Figure 3.3, the trajectories of the walks with $w(k) = 2^k$ and $w(k) = 10^k$ are illustrated, respectively. The number of steps is random, it is about $10^6$. The data comes from the same sample as the one shown on Figure 3.2.

The first thing that we can observe on Figure 3.3 is that the trajectories draw a sharp upper and lower hull according to $\sqrt{t}$ and $-\sqrt{t}$, which agrees with our expectations after (3.14). On the other hand, the trajectories oscillate very heavily between their extreme values, especially in the case $w(k) = 10^k$, there are almost but not quite straight crossings from $\sqrt{t}$ to $-\sqrt{t}$ and back. It shows that there is no continuous scaling limit of the self-repelling random walk with directed edges.

The shape of the trajectories are slightly different in the cases $w(k) = 2^k$ and $w(k) = 10^k$. The latter has heavier oscillations, because it corresponds to a higher rate of growth of the weight function. Note that despite this difference in the oscillation, the large scale behaviour is the same on the two pictures on Figure 3.3. The reason for this is that if the random walk explores a new region, e.g. it exceeds its earlier maximum, then the probability of the reversal does not depend on $w$, since both outgoing edges have local time 0. It can be a heuristic argument, why the upper and lower hulls $\sqrt{t}$ and $-\sqrt{t}$ are universal.

# Chapter 4

# One-dimensional myopic self-avoiding walk with site repulsion in continuous time

In the present chapter, we consider the continuous time version of the ‘true’ or ‘myopic’ self-avoiding random walk (MSAW) with site repulsion in $1d$. The Ray−Knight type method which was applied first in [T95] and also in Chapter 3 of this thesis to the discrete time and (unoriented and oriented) edge repulsion case is applicable to this model with some modifications. We present a limit theorem for the local time of the walk and a local limit theorem for the displacement.

The main ideas of this chapter are similar to those of [T95], but there are essential differences, too. Those parts of the proofs which are the same as in [T95] will not be spelled out explicitly. E.g. the full proof of Theorem 4.3.6 is omitted altogether. We put the emphasis on those arguments which differ genuinely from [T95]. In particular, we present some new coupling arguments.

This chapter is organized as follows. First, we describe the model which we will study and present our theorems. Then, we outline the proof of them in Section 4.2. In Section 4.3, we give the full proof of Theorem 4.1.1 in three steps. First, we introduce the main technical tools, i.e. some auxiliary Markov processes. Then we state technical lemmas which are all devoted to check the conditions of Theorem 4.3.6 cited from [T95]. Finally, we complete the proof using the lemmas. The proof of these lemmas are postponed until Section 4.5. The proof of Theorem 4.1.3 can be found in Section 4.4.

## 4.1 The random walk considered and the main results

Now, we define a version of myopic self-avoiding random walk in continuous time which is a counterpart of the one given in (1.7) and for which the Ray−Knight type method is applicable. Let $X(t)$, $t \in \mathbb{R}_+$ be a *continuous time* random walk on $\mathbb{Z}$ starting from $X(0) = 0$ and having right continuous paths. Denote by $\ell(t, x)$, $(t, x) \in \mathbb{R}_+ \times \mathbb{Z}$ its local time (occupation time measure) on sites:

$$\ell(t, x) := |\{s \in [0, t) \,:\, X(s) = x\}| \tag{4.1}$$

where $|\{\dots\}|$ denotes Lebesgue measure of the set indicated. Let $w : \mathbb{R} \to (0, \infty)$ be an almost arbitrary rate function. We assume that it is non-decreasing and not constant.

The law of the random walk is governed by the following jump rates and differential equations (for the local time increase):

$$\mathbf{P}\left(X(t + \mathrm{d}t) = x \pm 1 \;\middle|\; \mathcal{F}_t,\ X(t) = x\right) = w(\ell(t, x) - \ell(t, x \pm 1))\,\mathrm{d}t + o(\mathrm{d}t), \tag{4.2}$$

$$\dot{\ell}(t, x) = 1\!\!1(X(t) = x) \tag{4.3}$$

with initial conditions

$$X(0) = 0, \qquad \ell(0, x) = 0.$$

The dot in (4.3) denotes time derivative. Note that, for the choice of exponential weight function $w(u) = \exp\{\beta u\}$, this means exactly that conditionally on a jump occurring at the instant $t$, the random walker jumps to right or left from its actual position with probabilities $e^{-\beta\ell(t,x\pm1)}/(e^{-\beta\ell(t,x+1)} + e^{-\beta\ell(t,x-1)})$, just like in (1.2). It will turn out that in the long run the holding times remain of order one.

Fix $j \in \mathbb{Z}$ and $r \in \mathbb{R}_+$. We consider the random walk $X(t)$ running from $t = 0$ up to the stopping time

$$T_{j,r} = \inf\{t \geq 0 : \ell(t, j) \geq r\} \tag{4.4}$$

which is the inverse local time for our model. Define

$$\Lambda_{j,r}(k) := \ell(T_{j,r}, k) \qquad k \in \mathbb{Z} \tag{4.5}$$

the local time process of $X$ stopped at the inverse local time.

Let

$$\lambda_{j,r} := \inf\{k \in \mathbb{Z} : \Lambda_{j,r}(k) > 0\},$$
$$\rho_{j,r} := \sup\{k \in \mathbb{Z} : \Lambda_{j,r}(k) > 0\}.$$

Fix $x \in \mathbb{R}$ and $h \in \mathbb{R}_+$. Consider the two-sided reflected Brownian motion $W_{x,h}(y)$, $y \in \mathbb{R}$ with starting point $W_{x,h}(x) = h$. See Figure 4.1. Define the times of the first

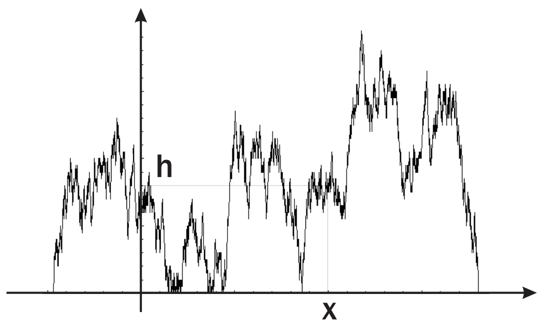


Figure 4.1: The two-sided reflected and absorbed Brownian motion $W_{x,h}$

hitting of 0 outside the interval $[0, x]$ or $[x, 0]$ with

$$\mathfrak{l}_{x,h} := \sup\{y < 0 \wedge x : W_{x,h}(y) = 0\},$$
$$\mathfrak{r}_{x,h} := \inf\{y > 0 \vee x : W_{x,h}(y) = 0\}$$

where $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$, and let

$$\mathcal{T}_{x,h} := \int_{\mathfrak{l}_{x,h}}^{\mathfrak{r}_{x,h}} W_{x,h}(y)\,\mathrm{d}y. \tag{4.6}$$

The main result of this chapter is

**Theorem 4.1.1.** *Let $x \in \mathbb{R}$ and $h \in \mathbb{R}_+$ be fixed. Then*

$$A^{-1}\lambda_{\lfloor Ax\rfloor,\lfloor\sqrt{A}\sigma h\rfloor} \Longrightarrow \mathfrak{l}_{0\wedge x,h}, \tag{4.7}$$
$$A^{-1}\rho_{\lfloor Ax\rfloor,\lfloor\sqrt{A}\sigma h\rfloor} \Longrightarrow \mathfrak{r}_{0\vee x,h}, \tag{4.8}$$

*and*

$$\left(\frac{\Lambda_{\lfloor Ax\rfloor,\lfloor\sqrt{A}\sigma h\rfloor}(\lfloor Ay\rfloor)}{\sigma\sqrt{A}}, \frac{\lambda_{\lfloor Ax\rfloor,\lfloor\sqrt{A}\sigma h\rfloor}}{A} \le y \le \frac{\rho_{\lfloor Ax\rfloor,\lfloor\sqrt{A}\sigma h\rfloor}}{A}\right) \Longrightarrow (W_{x,h}(y), \mathfrak{l}_{0\wedge x,h} \le y \le \mathfrak{r}_{0\vee x,h}) \tag{4.9}$$

*as $A \to \infty$ where $\sigma^2 = \int_{-\infty}^{\infty} u^2 \rho(\mathrm{d}u) \in (0, \infty)$ with $\rho$ defined by* (4.26) *and* (4.22) *later.*

**Corollary 4.1.2.** *For any $x \in \mathbb{R}$ and $h \ge 0$,*

$$\frac{T_{\lfloor Ax\rfloor,\lfloor\sqrt{A}\sigma h\rfloor}}{\sigma A^{3/2}} \Longrightarrow \mathcal{T}_{x,h}. \tag{4.10}$$

For stating Theorem 4.1.3, we need some more definitions. It follows from (4.6) that $\mathcal{T}_{x,h}$ has an absolutely continuous distribution. Let

$$\omega(t, x, h) := \frac{\partial}{\partial t}\mathbf{P}\left(\mathcal{T}_{x,h} < t\right) \tag{4.11}$$

be the density of the distribution of $\mathcal{T}_{x,h}$. Define

$$\varphi(t,x) := \int_0^\infty \omega(t,x,h)\,\mathrm{d}h.$$

Theorem 2 of [T95] gives that, for fixed $t > 0$, $\varphi(t,\cdot)$ is a density function, i.e.

$$\int_{-\infty}^\infty \varphi(t,x)\,\mathrm{d}x = 1. \tag{4.12}$$

One could expect that $\varphi(t,\cdot)$ is the density of the limit distribution of $X(At)/A^{2/3}$ as $A \to \infty$, but we prove a similar statement for their Laplace transform. We denote by $\widehat{\varphi}$ the Laplace transforms of $\varphi$:

$$\widehat{\varphi}(s,x) := s \int_0^\infty e^{-st}\varphi(t,x)\,\mathrm{d}t. \tag{4.13}$$

**Theorem 4.1.3.** *Let $s \in \mathbb{R}_+$ be fixed and $\theta_{s/A}$ a random variable of exponential distribution with mean $A/s$ which is independent of the random walk $X(t)$. Then, for almost all $x \in \mathbb{R}$,*

$$A^{2/3}\mathbf{P}\left(X(\theta_{s/A}) = \lfloor A^{2/3}x \rfloor\right) \to \widehat{\varphi}(s,x) \tag{4.14}$$

*as $A \to \infty$.*

From this local limit theorem, the integral limit theorem follows immediately:

$$\lim_{A\to\infty} \mathbf{P}\left(A^{-2/3}X(\theta_{s/A}) < x\right) = \int_{-\infty}^x \widehat{\varphi}(s,y)\,\mathrm{d}y.$$

## 4.2 Sketch proof of limit theorems

The proofs of limit theorems of the present chapter follow the same structure as in Chapter 3, see Section 3.3 for a sketch. We point out here the differences from those proofs. We also give some hints about the proof of one of the technical lemmas which contains a new coupling argument.

Similarly to [T95] and Chapter 3, we apply a Ray – Knight approach, and we introduce auxiliary Markov processes. Adapted to the present setup, these processes evolve in continuous time, and they are associated to the *edges* of $\mathbb{Z}$. By definitions (4.15)–(4.18) and (4.24)–(4.25), $\eta_{k,\pm}$ is essentially the difference of local times at *vertices* $k$ and $k+1$ which turn out to be Markovian. Proposition 4.3.1 gives that these Markov processes are independent, which is a crucial point for the random walk representation, because it provides the independence of the steps. The initial distributions are also given.

The key step of the proof of Theorem 4.1.1 is the random walk representation of the sequence of local times $(\Lambda_{j,r}(k))_{k\in\mathbb{Z}}$. The equations (4.40) are the continuous time counterparts of (3.29)–(3.32), but formulae are simpler this time. Theorem 4.3.6 is used without proof, because an analogous result has already appeared in [T95]. It says that, under some technical assumptions, the random walk representations (4.41) and (4.42) both forward and backward converge to a reflected and absorbed Brownian motion $W_{x,h}$ under the proper scaling, which immediately yields Theorem 4.1.1. The technical assumptions are satisfied by the Lemmas 4.3.2–4.3.5 which are proved in Section 4.5.

The proof of Theorem 4.1.3 is rather similar to that of Theorem 3.2.4. The main difference is caused by considering continuous time. Therefore, the key equation (4.45) is given by integration as opposed to (3.63). In the remaining steps, convergence of Laplace transforms is checked.

From the proofs of technical lemmas, the most interesting might be that of Lemma 4.3.4 which is given in Subsection 4.5.2. We prove that a copy of the auxiliary Markov process $\eta_1$ starting from 0 converges exponencially fast to the stationary distribution $\rho$. We couple it with a *stationary* process $\eta_2$ which evolves according to the same transition rules. The two processes merge at the random time $T$, and the right tail of $T$ provides an upper bound on the variational distance in (4.61).

It is easily seen that the two processes can be coupled in such a way that if both are in the interval $(-b, b)$ with some $b > 0$ to be chosen at the end, then the rate of marge can be bounded from below by some $\beta(b) > 0$, see (4.62). The rest of the proof is devoted to show that, for a positive proportion of time, both processes are in the interval $(-b, b)$ with high probability if $b$ is large enough. It completes the proof.

## 4.3 Ray – Knight construction

The aim of this section is to give a random walk representation of the local time sequence $\Lambda_{j,r}$. Therefore, we introduce auxiliary Markov processes corresponding to each edge of $\mathbb{Z}$. The process corresponding to the edge $e$ is defined in such a way that its value is the difference of local times of $X(T_{j,r})$ on the two vertices adjacent to $e$ where $X(T_{j,r})$ is the process $X(t)$ stopped at an inverse local time. It turns out that the auxiliary Markov processes are independent. Hence, by induction, the sequence of local times can be given as partial sums of independent auxiliary Markov processes. The proof of Theorem 4.1.1 relies exactly on this observation.

### 4.3.1 The basic construction

Let

$$\tau(t, k) := \ell(t, k) + \ell(t, k + 1) \tag{4.15}$$

be the local time spent on (the endpoints of) the edge $\langle k, k + 1\rangle$, $k \in \mathbb{Z}$, and

$$\theta(s, k) := \inf\{t \geq 0 \,:\, \tau(t, k) > s\} \tag{4.16}$$

its inverse. Further on, define

$$\xi_k(s) := \ell(\theta(s,k), k+1) - \ell(\theta(s,k), k), \tag{4.17}$$
$$\alpha_k(s) := \mathbb{1}(X(\theta(s,k)) = k+1) - \mathbb{1}(X(\theta(s,k)) = k). \tag{4.18}$$

A crucial observation is that, for each $k \in \mathbb{Z}$, $s \mapsto (\alpha_k(s), \xi_k(s))$ is a Markov process on the state space $\{-1,+1\} \times \mathbb{R}$. The transition rules are

$$\mathbf{P}\left(\alpha_k(t+\mathrm{d}t) = -\alpha_k(t) \mid \mathcal{F}_t^k\right) = w(\alpha_k(t)\xi_k(t))\,\mathrm{d}t + o(\mathrm{d}t), \tag{4.19}$$
$$\dot{\xi}_k(t) = \alpha_k(t), \tag{4.20}$$

with some initial state $(\alpha_k(0), \xi_k(0))$. The $\sigma$-algebra $\mathcal{F}_t^k$ contains the history of $(\alpha_k(s), \xi_k(s))$ up to time $t$. Remember that due to the time change defined in (4.15)–(4.18), $t$ here is the time that the walker had spent on one of the endpoints of the edge $(k, k+1)$.

Furthermore, these processes are independent. In plain words:

1. $\xi_k(t)$ is the difference of time spent by $\alpha_k$ in the states $+1$ and $-1$, alternatively, the difference of time spent by the walker on the sites $k+1$ and $k$;

2. $\alpha_k(t)$ changes sign with rate $w(\alpha_k(t)\xi_k(t))$ since the walker jumps between $k$ and $k+1$ with these rates.

The common infinitesimal generator of these processes is

$$(Gf)(\pm 1, u) = \pm f'(\pm 1, u) + w(\pm u)\big(f(\mp 1, u) - f(\pm 1, u)\big)$$

where $f'(\pm 1, u)$ is the derivative with respect to the second variable. It is an easy computation to check that these Markov processes are ergodic and their common unique stationary measure is

$$\mu(\pm 1, \mathrm{d}u) = \frac{1}{2Z} e^{-W(u)}\,\mathrm{d}u \tag{4.21}$$

where

$$W(u) := \int_0^u (w(v) - w(-v))\,\mathrm{d}v \quad \text{and} \quad Z := \int_{-\infty}^{\infty} e^{-W(v)}\,\mathrm{d}v. \tag{4.22}$$

Mind that, due to the condition imposed on $w$ (non-decreasing and non-constant),

$$\lim_{|u|\to\infty} \frac{W(u)}{|u|} = \lim_{v\to\infty} (w(v) - w(-v)) > 0, \tag{4.23}$$

and thus $Z < \infty$ and $\mu(\pm 1, \mathrm{d}u)$ is indeed a probability measure on $\{-1,+1\} \times \mathbb{R}$.

Let

$$\beta_{\pm}(t,k) := \inf\left\{ s \geq 0 : \int_0^s \mathbb{1}(\alpha_k(u) = \pm 1)\,\mathrm{d}u \geq t \right\} \tag{4.24}$$

be the inverse local times of $(\alpha_k(t), \xi_k(t))$. With the use of them, we can define the processes

$$\eta_{k,-}(t) := \xi_k(\beta_-(t,k)), \qquad \eta_{k,+}(t) := -\xi_k(\beta_+(t,k)) \tag{4.25}$$

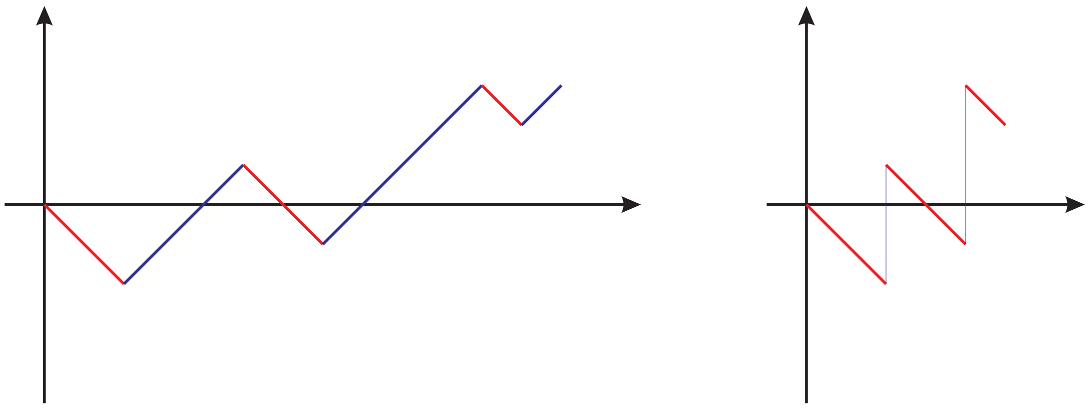

Figure 4.2: A possible trajectory of $\xi_k(t)$ and $\eta_{k,\pm}(t)$

which are also Markovian. By symmetry, the processes with different sign have the same law. The infinitesimal generator of $\eta_{k,\pm}$ is

$$(Hf)(u) = -f'(u) + w(u) \int_u^\infty e^{-\int_u^v w(s)\,\mathrm{d}s} w(v)(f(v) - f(u))\,\mathrm{d}v.$$

For a typical trajectory of $\xi_k(t)$ and $\eta_{k,\pm}(t)$, see Figure 4.2.

It is easy to see that the Markov processes $\eta_{k,\pm}$ are ergodic and their common unique stationary distribution is

$$\rho(\mathrm{d}u) := \frac{1}{Z} e^{-W(u)}\,\mathrm{d}u \tag{4.26}$$

with the notations (4.22). The stationarity of $\rho$ is not surprising after (4.21), but a straightforward calculation yields it also.

Denote by $P^t = e^{tH}$ the transition kernel of $\eta_{k,\pm}$. For any $x \in \mathbb{R}$, define the probability measure

$$Q(x, \mathrm{d}y) := \begin{cases} \exp(-\int_x^y w(u)\,\mathrm{d}u) w(y)\,\mathrm{d}y & \text{if } y \geq x, \\ 0 & \text{if } y < x \end{cases}$$

which is the conditional distribution of the endpoint of a jump of $\eta_{k,\pm}$ provided that $\eta_{k,\pm}$ jumps from $x$. Hence $Q(x, \cdot)$ is a transition kernel.

The main point is the following

**Proposition 4.3.1.** *1. The processes $s \mapsto (\alpha_k(s), \xi_k(s))$, $k \in \mathbb{Z}$ are independent Markov process with the same law given in (4.19)–(4.20). They start from the initial states $\xi_k(0) = 0$ and*

$$\alpha_k(0) = \begin{cases} +1 & \text{if} \quad k < 0, \\ -1 & \text{if} \quad k \geq 0. \end{cases}$$

*2. The processes $s \mapsto \eta_{k,\pm}(s)$, $k \in \mathbb{Z}$ are independent Markov processes if we consider* *exactly one of* $\eta_{k,+}$ *and* $\eta_{k,-}$ *for each* $k$*. The initial distributions are*

$$\mathbf{P}\left(\eta_{k,+}(0) \in A\right) = \begin{cases} Q(0,A) & \text{if} \quad k \geq 0, \\ \mathbb{1}(0 \in A) & \text{if} \quad k < 0, \end{cases} \tag{4.27}$$

$$\mathbf{P}\left(\eta_{k,-}(0) \in A\right) = \begin{cases} \mathbb{1}(0 \in A) & \text{if} \quad k \geq 0, \\ Q(0,A) & \text{if} \quad k < 0. \end{cases} \tag{4.28}$$

### 4.3.2 Technical lemmas

The lemmas of this subsection descibe the behaviour of the auxiliary Markov processes $\eta_{k,\pm}$. Since they all have the same law, we denote them by $\eta$ to keep the notation simple, and it means that the statement is true for all $\eta_{k,\pm}$.

Fix $b \in \mathbb{R}$. Define the stopping times

$$\theta_+ := \inf\{t > 0 : \eta(t) \geq b\}, \tag{4.29}$$

$$\theta_- := \inf\{t > 0 : \eta(t) \leq b\}. \tag{4.30}$$

In our lemmas, $\gamma$ will always be a positive constant which is considered as being a small exponent, and $C$ will be a finite constant considered as being large. To simplify the notation, we will use the same letter for constants at different points of our proof. The notation does not emphasizes, but their values depend on $b$.

First, we estimate the exponential moments of $\theta_-$ and $\theta_+$.

**Lemma 4.3.2.** *There are* $\gamma > 0$ *and* $C < \infty$ *such that, for all* $y \geq b$*,*

$$\mathbf{E}\left(\exp(\gamma\theta_-) \;\middle|\; \eta(0) = y\right) \leq \exp(C(y-b)). \tag{4.31}$$

**Lemma 4.3.3.** *There exists* $\gamma > 0$ *such that*

$$\mathbf{E}\left(\exp(\gamma\theta_+) \;\middle|\; \eta(0) = b\right) < \infty. \tag{4.32}$$

We show that the Markov process $\eta$ converges exponentially fast to its stationary distribution $\rho$ defined by (4.26) if the initial distribution is 0 with probability 1 or $Q(0,\cdot)$.

**Lemma 4.3.4.** *There are* $C < \infty$ *and* $\gamma > 0$ *such that*

$$\left\| P^t(0,\cdot) - \rho \right\| < C\exp(-\gamma t) \tag{4.33}$$

*and*

$$\left\| Q(0,\cdot)P^t - \rho \right\| < C\exp(-\gamma t). \tag{4.34}$$

We give a bound on the decay of the tails of $P^t(0,\cdot)$ and $Q(0,\cdot)P^t$ uniformly in $t$.

**Lemma 4.3.5.** *There are constants $C < \infty$ and $\gamma > 0$ such that*

$$P^t(0,(x,\infty)) \leq Ce^{-\gamma x} \tag{4.35}$$

*and*

$$Q(0,\cdot)P^t(0,(x,\infty)) \leq Ce^{-\gamma x} \tag{4.36}$$

*for all $x \geq 0$ and for any $t > 0$ uniformly, i.e. the values of $C$ and $\gamma$ do not depend on $x$ and $t$.*

We introduce some notation from [T95] and cite a theorem, which will be the main ingredient of our proof. Let $A > 0$ be the scaling parameter, and let

$$S_A(l) = S_A(0) + \sum_{j=1}^{l} \xi_A(j) \qquad l \in \mathbb{N}$$

be a discrete time random walk on $\mathbb{R}_+$ with the law

$$\mathbf{P}\left(\xi_A(l) \in \mathrm{d}x \;\middle|\; S_A(l-1) = y\right) = \pi_A(\mathrm{d}x, y, l)$$

for each $l \in \mathbb{N}$ with

$$\int_{-y}^{\infty} \pi_A(\mathrm{d}x, y, l) = 1.$$

Define the following stopping time of the random walk $S_A(\cdot)$:

$$\omega_{[Ar]} = \inf\{l \geq [Ar] : S_A(l) = 0\}.$$

We give the following theorem without proof, because this is the continuous analogue of Theorem 4 in [T95] and its proof is essentially identical to that of the corresponding statement in [T95].

**Theorem 4.3.6.** *Suppose that the following conditions hold:*

1. *The step distributions $\pi_A(\cdot, y, l)$ converge exponentially fast as $y \to \infty$ to a common asymptotic distribution $\pi$. That is, for each $l \in \mathbb{Z}$,*

$$\int_{\mathbb{R}} |\pi_A(\mathrm{d}x, y, l) - \pi(\mathrm{d}x)| < Ce^{-\gamma y}.$$

2. *The asymptotic distribution is symmetric: $\pi(-\mathrm{d}x) = \pi(\mathrm{d}x)$, and its moments are finite, in particular, denote*

$$\sigma^2 := \int_{\mathbb{R}} x^2 \pi(\mathrm{d}x). \tag{4.37}$$

3. *Uniform decay of the step distributions: for each $l \in \mathbb{Z}$,*

$$\pi_A((x,\infty), y, l) \leq Ce^{-\gamma x}.$$

4. *Uniform non-trapping condition: The random walk is not trapped in a bounded domain or in a domain away from the origin. That is, there is* $\delta > 0$ *such that*

$$\int_{\delta}^{\infty} \pi_A(\mathrm{d}x, y, l) > \delta \quad \text{or} \quad \int_{x=\delta}^{\infty}\int_{z=-\infty}^{\infty} \pi_A(\mathrm{d}x - z, y+z, l+1)\pi_A(\mathrm{d}z, y, l) > \delta \tag{4.38}$$

*and*

$$\int_{-\infty}^{-(\delta\wedge y)} \pi_A(\mathrm{d}x, y, l) > \delta.$$

*Under these conditions, if*

$$\frac{S_A(0)}{\sigma\sqrt{A}} \to h,$$

*then*

$$\left(\frac{\omega_{[Ar]}}{A}, \frac{S_A([Ay])}{\sigma\sqrt{A}} : 0 \le y \le \frac{\omega_{[Ar]}}{A}\right) \Longrightarrow \left(\omega_r^W, |W_y| : 0 \le y \le \omega_r^W \;\middle|\; |W_0| = h\right) \tag{4.39}$$

*in* $\mathbb{R}_+ \times D[0,\infty)$ *as* $A \to \infty$ *where*

$$\omega_r^W = \inf\{s > r : W_s = 0\}$$

*with a standard Brownian motion* $W$ *and* $\sigma$ *is given by* (4.37).

### 4.3.3 Proof of the limit theorem for local times

Using the auxiliary Markov processes introduced in Subsection 4.3.1, we can build up the local time sequence as a random walk. This Ray−Knight type construction is the main idea of the following proof.

*Proof of Theorem 4.1.1.* Fix $j \in \mathbb{Z}$ and $r \in \mathbb{R}_+$. Using the definition (4.5) and the construction of $\eta_{k,\pm}$ (4.15)–(4.25), we can formulate the following recursion for $\Lambda_{j,r}$:

$$\begin{aligned} &\Lambda_{j,r}(j) = r, \\ &\Lambda_{j,r}(k+1) = \Lambda_{j,r}(k) + \eta_{k,-}(\Lambda_{j,r}(k)) && \text{if} \quad k \ge j, \\ &\Lambda_{j,r}(k-1) = \Lambda_{j,r}(k) + \eta_{k-1,+}(\Lambda_{j,r}(k)) && \text{if} \quad k \le j. \end{aligned} \tag{4.40}$$

It means that the processes $(\Lambda_{j,r}(j-k))_{k=0}^{\infty}$ and $(\Lambda_{j,r}(j+k))_{k=0}^{\infty}$ are random walks on $\mathbb{R}_+$, they start from $\Lambda_{j,r}(j) = r$, and the distribution of the following step always depends on the actual position of the walker. In order to apply Theorem 4.3.6, we rewrite (4.40):

$$\Lambda_{j,r}(j+k) = r + \sum_{i=0}^{k-1} \eta_{j+i,-}(\Lambda_{j,r}(j+i)) \qquad k = 0, 1, 2, \ldots, \tag{4.41}$$

$$\Lambda_{j,r}(j-k) = r + \sum_{i=0}^{k-1} \eta_{j-i-1,+}(\Lambda_{j,r}(j-i)) \qquad k = 0, 1, 2, \ldots. \tag{4.42}$$

The step distributions of these random walks are

$$\pi_A(\mathrm{d}x, y, l) = \begin{cases} P^y(0, \mathrm{d}x) \\ Q(0,\cdot)P^y(\mathrm{d}x) \end{cases}$$

according to (4.27)–(4.28).

The exponential closeness of the step distribution to the stationary distribution is shown by Lemma 4.3.4. One can see from (4.26) and (4.22) that the distribution $\rho$ is symmetric and it has a non-zero finite variance. Lemma 4.3.5 gives a uniform exponential bound on the tail of the distributions $P^t(0,\cdot)$ and $Q(0,\cdot)P^t$.

Since we only consider $[\lambda_{j,r}, \rho_{j,r}]$, that is, the time interval until $\Lambda_{j,r}$ hits 0, we can force the walk to jump to 1 in the next step after hitting 0, which does not influence our investigations. It means that $\pi_A(\{1\}, 0, l) = 1$ for $l \in \mathbb{Z}$, and with this, the non-trapping condition (4.38) fulfils. Therefore, Theorem 4.3.6 is applicable for the forward and the backward walks, and Theorem 4.1.1 is proved. □

## 4.4 The position of the random walker

We turn to the proof of Theorem 4.1.3. First, we introduce the rescaled distribution

$$\varphi_A(t,x) := A^{2/3}\mathbf{P}\left(X(\lfloor At\rfloor) = \lfloor A^{2/3}x\rfloor\right)$$

where $t, x \in \mathbb{R}_+$. We define the Laplace transform of $\varphi_A$ with

$$\widehat{\varphi}_A(s,x) = s\int_0^\infty e^{-st}\varphi_A(t,x)\,\mathrm{d}t \tag{4.43}$$

which is the position of the random walker at an independent random time of exponential distribution with mean $A/s$.

We denote by $\widehat{\omega}$ the Laplace transforms of $\omega$ defined in (4.11) and we rewrite (4.13):

$$\widehat{\omega}(s,x,h) := s\int_0^\infty e^{-st}\omega(t,x,h)\,\mathrm{d}t = s\,\mathbf{E}\left(e^{-s\mathcal{T}_{x,h}}\right),$$
$$\widehat{\varphi}(s,x) = s\int_0^\infty e^{-st}\varphi(t,x)\,\mathrm{d}t = \int_0^\infty \widehat{\omega}(s,x,h)\,\mathrm{d}h.$$

Note that the scaling relations

$$\alpha\omega(\alpha t, \alpha^{2/3}x, \alpha^{1/3}h) = \omega(t,x,h),$$
$$\alpha^{2/3}\widehat{\varphi}(\alpha^{-1}s, \alpha^{2/3}x) = \widehat{\varphi}(s,x) \tag{4.44}$$

hold because of the scaling property of the Brownian motion.

*Proof of Theorem 4.1.3.* The first observation for the proof is the identity

$$\mathbf{P}\left(X(t) = k\right) = \int_{h=0}^\infty \mathbf{P}\left(T_{k,h} \in (t, t+\mathrm{d}h)\right), \tag{4.45}$$

which follows from (4.4). If we insert it to the definition of $\widehat{\varphi}_A$ (4.43), then we get

$$\begin{aligned}\widehat{\varphi}_A(s,x) &= sA^{-1/3}\int_0^\infty e^{-st/A}\mathbf{P}\left(X(t)=\lfloor A^{2/3}x\rfloor\right)\mathrm{d}t \\ &= sA^{-1/3}\int_0^\infty e^{-st/A}\int_{h=0}^\infty \mathbf{P}\left(T_{\lfloor A^{2/3}x\rfloor,h}\in(t,t+\mathrm{d}h)\right)\mathrm{d}t \\ &= sA^{-1/3}\int_0^\infty \mathbf{E}\left(e^{-sT_{\lfloor A^{2/3}x\rfloor,h}/A}\right)\mathrm{d}h\end{aligned} \tag{4.46}$$

using (4.45). Defining

$$\widehat{\omega}_A(s,x,h) = s\mathbf{E}\left(\exp(-sT_{\lfloor A^{2/3}x\rfloor,\lfloor A^{1/3}\sigma h\rfloor}/(\sigma A))\right)$$

gives us

$$\widehat{\varphi}_A(s,x) = \int_0^\infty \widehat{\omega}_A(\sigma s,x,h)\,\mathrm{d}h \tag{4.47}$$

from (4.46). From Corollary 4.1.2, it follows that, for any $s>0$, $x\geq 0$ and $h>0$,

$$\widehat{\omega}_A(s,x,h)\to\widehat{\omega}(s,x,h).$$

Applying Fatou's lemma in (4.47), one gets

$$\liminf_{A\to\infty}\widehat{\varphi}_A(s,x)\geq\int_0^\infty\widehat{\omega}(\sigma s,x,h)\,\mathrm{d}h=\sigma^{2/3}\widehat{\varphi}(s,\sigma^{2/3}x), \tag{4.48}$$

where we used (4.44) in the last equation. A consequence of (4.12), (4.48) integrated and a second application of Fatou's lemma yield

$$1=\int_{-\infty}^\infty\widehat{\varphi}(s,x)\,\mathrm{d}x\leq\int_{-\infty}^\infty\liminf_{A\to\infty}\widehat{\varphi}_A(s,x)\,\mathrm{d}x\leq\liminf_{A\to\infty}\int_{-\infty}^\infty\widehat{\varphi}_A(s,x)\,\mathrm{d}x=1,$$

which gives that, for fixed $s\in\mathbb{R}_+$, $\widehat{\varphi}_A(s,x)\to\widehat{\varphi}(s,x)$ holds for almost all $x\in\mathbb{R}$, indeed. □

## 4.5 Proof of lemmas

### 4.5.1 Exponential moments of the return times

*Proof of Lemma 4.3.2.* Consider the Markov process $\zeta_b(t)$ which decreases with constant speed 1, it has upwards jumps with homogeneous rate $w(-b)$, and the distribution of the size of a jump is the same as that of $\eta$, provided that the jump starts from $b$. In other words, the infinitesimal generator of $\zeta_b$ is

$$(Z_bf)(u)=-f'(u)+w(-b)\int_0^\infty e^{-\int_0^v w(b+s)\,\mathrm{d}s}w(b+v)(f(u+v)-f(u))\,\mathrm{d}v.$$

Note that, by the monotonicity of $w$, $\eta$ and $\zeta_b$ can be coupled in such a way that they start from the same position and, as long as $\eta \geq b$ holds, $\zeta_b \geq \eta$ is true almost surely. It means that it suffices to prove (4.31) with

$$\theta'_- := \inf\{t > 0 : \zeta_b(t) \leq b\} \tag{4.49}$$

instead of $\theta_-$. But the transitions of $\zeta_b$ are homogeneous in space, which yields that (4.31) follows from the finiteness of

$$\mathbf{E}\left(\exp(\gamma\theta'_-) \,\middle|\, \zeta_b(0) = b+1\right). \tag{4.50}$$

In addition to this, $\zeta_b$ is a supermartingale with stationary increments, which gives us

$$\mathbf{E}\left(\zeta_b(t)\right) = b + 1 - ct$$

with some $c > 0$, if the initial condition is $\zeta_b(0) = b + 1$. For $\alpha \in \left(-\infty, \lim_{u\to\infty} \frac{W(u)}{u}\right)$ (c.f. (4.23)), the expectation

$$\log \mathbf{E}\left(e^{\alpha(\zeta_b(t)-\zeta_b(0))}\right)$$

is finite, and negative for some $\alpha > 0$. Hence, the martingale

$$M(t) = \exp\left(\alpha(\zeta_b(t) - \zeta_b(0)) - t\log\mathbf{E}\left(e^{\alpha(\zeta_b(1)-\zeta_b(0))}\right)\right) \tag{4.51}$$

stopped at $\theta'_-$ gives that the expectation in (4.50) is finite with $\gamma = -\log\mathbf{E}\left(e^{\alpha(\zeta_b(1)-\zeta_b(0))}\right)$. □

*Proof of Lemma 4.3.3.* First, we prove for negative $b$, more precisely, for which $w(-b) > w(b)$. In this case, define the homogeneous process $\kappa_b$ with $\kappa_b(0) = b$ and generator

$$K_b f(u) = -f'(u) + w(-b)\int_0^\infty e^{-w(b)s} w(b)(f(u+s) - f(u))\,\mathrm{d}s.$$

It is easy to see that there is a coupling of $\eta$ and $\kappa_b$, for which $\eta \geq \kappa_b$ as long as $\eta \leq b$. Therefore, it is enough to show (4.32) with

$$\theta'_+ := \inf\{t > 0 : \kappa_b(t) \geq b\}$$

instead of $\theta_+$.

But $\kappa_b$ is a submartingale with stationary increments, for which

$$\log\mathbf{E}\left(e^{\alpha(\kappa_b(t)-\kappa_b(0))}\right)$$

is finite if $\alpha \in (-\infty, w(b))$, and negative for some $\alpha < 0$. The statement follows from the same idea as in the proof of Lemma 4.3.2.

Now, we prove the lemma for the remaining case. Fix $b$, for which we already know (4.32), and chose $b_1 > b$ arbitrarily. We start $\eta$ from $\eta(0) = b_1$, and we decompose its trajectory into independent excursions above and below $b$, alternatingly. Let

$$Y_0 := \inf\{t \geq 0 : \eta(t) \leq b\}, \tag{4.52}$$

and by induction, define

$$X_k := \inf\left\{t > 0 : \eta\left(\sum_{j=1}^{k-1} X_j + \sum_{j=0}^{k-1} Y_j + t\right) \geq b\right\}, \tag{4.53}$$

$$Y_k := \inf\left\{t \geq 0 : \eta\left(\sum_{j=1}^{k} X_j + \sum_{j=0}^{k-1} Y_j + t\right) \leq b\right\} \tag{4.54}$$

if $k = 1, 2, \ldots$. Note that $(X_k, Y_k)_{k=1,2,\ldots}$ is an i.i.d. sequence of pairs of random variables. Finally, let

$$Z_k := X_k + Y_k \qquad k = 1, 2, \ldots. \tag{4.55}$$

With this definition, the $Z_k$'s are the lengths of the epochs in a renewal process. Lemma 4.3.2 tells us that $Y_0$ has finite exponential moment. The same holds for $X_1, X_2, \ldots$ because of the first part of this proof for the case of small $b$. Note that the distribution of the upper endpoint of a jump of $\eta$ conditionally given that $\eta$ jumps above $b$ is exactly $Q(b, \cdot)$. Since $Q(b, \cdot)$ decays exponentially fast, we can use Lemma 4.3.2 again to conclude that $\mathbf{E}\left(\exp(\gamma Y_k)\right) < \infty$ for $\gamma > 0$ small enough. Define

$$\nu_t := \max\left\{n \geq 0 : \sum_{k=1}^{n} Z_k \leq t\right\} \tag{4.56}$$

in the usual way. The following decomposition is true:

$$\begin{aligned}\mathbf{P}&\left(\frac{\sum_{k=1}^{\nu_t+1} Y_k}{t} < \varepsilon\right)\\ &\leq \mathbf{P}\left(\frac{\nu_t+1}{t} < \frac{1}{2}\frac{1}{\mathbf{E}\left(Z_1\right)}\right) + \mathbf{P}\left(\frac{\sum_{k=1}^{\nu_t+1} Y_k}{t} < \varepsilon, \frac{\nu_t+1}{t} \geq \frac{1}{2}\frac{1}{\mathbf{E}\left(Z_1\right)}\right).\end{aligned} \tag{4.57}$$

Lemma 4.1 of [BT91] gives a large deviation principle for the renewal process $\nu_t$, hence

$$\mathbf{P}\left(\frac{\nu_t+1}{t} < \frac{1}{2}\frac{1}{\mathbf{E}\left(Z_1\right)}\right) \leq \mathbf{P}\left(\frac{\nu_t}{t} < \frac{1}{2}\frac{1}{\mathbf{E}\left(Z_1\right)}\right) < e^{-\gamma t} \tag{4.58}$$

with some $\gamma > 0$. For the second term on the right-hand side in (4.57),

$$\begin{aligned}\mathbf{P}&\left(\frac{\sum_{k=1}^{\nu_t+1} Y_k}{t} < \varepsilon, \frac{\nu_t+1}{t} \geq \frac{1}{2}\frac{1}{\mathbf{E}\left(Z_1\right)}\right)\\ &= \mathbf{P}\left(\frac{\sum_{k=1}^{\nu_t+1} Y_k}{\nu_t+1} < \varepsilon\frac{t}{\nu_t+1}, \frac{\nu_t+1}{t} \geq \frac{1}{2}\frac{1}{\mathbf{E}\left(Z_1\right)}\right)\\ &\leq \mathbf{P}\left(\frac{\sum_{k=1}^{\nu_t+1} Y_k}{\nu_t+1} < 2\varepsilon\mathbf{E}\left(Z_1\right), \frac{\nu_t+1}{t} \geq \frac{1}{2}\frac{1}{\mathbf{E}\left(Z_1\right)}\right)\\ &\leq \max_{n \geq \frac{1}{2}\frac{1}{\mathbf{E}(Z_1)}t} \mathbf{P}\left(\frac{\sum_{k=1}^{n} Y_k}{n} < 2\varepsilon\mathbf{E}\left(Z_1\right)\right),\end{aligned} \tag{4.59}$$

which is exponentially small for some $\varepsilon > 0$ by standard large deviation theory, and the same holds for the probability estimated in (4.57), which means that $\eta$ spends at least $\varepsilon t$ time above $b$ with overwhelming probability.

The inequality

$$\mathbf{P}\left(\theta_+ > t \mid \eta(0) = b_1\right) \le \mathbf{P}\left(\sum_{k=1}^{\nu_t+1} Y_k < \varepsilon t\right) + \mathbf{P}\left(\theta_+ > t \,\middle|\, \eta(0) = b_1, \sum_{k=1}^{\nu_t+1} Y_k > \varepsilon t\right)$$

is obvious. The first term on the right-hand side is exponentially small by (4.57)–(4.59). In order to bound the second term, denote by $J(t)$ the number of jumps when $\eta(s) \ge b$. The condition $\sum_{k=1}^{\nu_t+1} Y_k > \varepsilon t$ means that this is the case in an at least $\varepsilon$ portion of $[0,t]$. The rate of these jumps are at least $w(-b)$ by the monotonicity of $w$. Note that $J(t)$ dominates stochastically a Poisson random variable $L(t)$ with mean $w(-b)t$. Hence,

$$\mathbf{P}\left(J(t) < \frac{1}{2}w(-b)t\right) \le \mathbf{P}\left(L(t) < \frac{1}{2}w(-b)t\right) < e^{-\gamma t} \tag{4.60}$$

for $t$ large enough with some $\gamma > 0$ by a standard large deviation estimate.

Note that $Q$ is also monotone in the sense that

$$\int_{b_1}^{\infty} Q(x_1, \mathrm{d}y) < \int_{b_1}^{\infty} Q(x_2, \mathrm{d}y)$$

if $x_1 < x_2$. Therefore, a jump of $\eta$, which starts above $b$, exits $(-\infty, b_1]$ with probability at least

$$r = \int_{b_1}^{\infty} Q(b, \mathrm{d}y) > 0.$$

Finally,

$$\begin{aligned}
&\mathbf{P}\left(\theta_+ > t \,\middle|\, \eta(0) = b_1, \sum_{k=1}^{\nu_t+1} Y_k > \varepsilon t\right) \\
&\quad \le \mathbf{P}\left(J(t) < \frac{1}{2}w(-b)\varepsilon t\right) + \mathbf{P}\left(\theta_+ > t \,\middle|\, J(t) \ge \frac{1}{2}w(-b)\varepsilon t, \eta(0) = b_1, \sum_{k=1}^{\nu_t+1} Y_k > \varepsilon t\right) \\
&\quad \le e^{-\gamma t} + (1-r)^{\frac{1}{2}w(-b)\varepsilon t}
\end{aligned}$$

by (4.60), which is an exponential decay, as required. □

### 4.5.2 Exponential convergence to the stationarity

*Proof of Lemma 4.3.4.* First, we prove (4.33). We couple two copies of $\eta$, say $\eta_1$ and $\eta_2$. Suppose that

$$\eta_1(0) = 0 \qquad \text{and} \qquad \mathbf{P}\left(\eta_2(0) \in A\right) = \rho(A).$$

Their distribution after time $t$ are obviously $P^t(0, \cdot)$ and $\rho$, respectively. We use the standard coupling lemma to estimate their variation distance:

$$\left\| P^t(0, \cdot) - \rho \right\| \le \mathbf{P}\left(T > t\right) \tag{4.61}$$

where $T$ is the random time when the two processes merge.

Assume that $\eta_1 = x_1$ and $\eta_2 = x_2$ with fixed numbers $x_1, x_2 \in \mathbb{R}$. Consider the corresponding processes $(\xi_i(t), \alpha_i(t))$ where $i = 1, 2$ with distribution given by (4.19)–(4.20), and suppose that $\eta_1$ and $\eta_2$ are derived from them similarly to (4.24)–(4.25). Then $\xi_1 = x_1$, $\xi_2 = x_2$ and $\alpha_i = -1$ holds $i = 1, 2$. Our aim is to couple them in such a way that $\eta_1$ and $\eta_2$ merge with a certain positive rate. $\alpha_i$ turn to $+1$ with rate $w(-x_i)$ with $i = 1, 2$, hence both turn to $+1$ with rate $w(-x_1 \vee x_2)$ in a proper coupling. After that it happens, the probability that one of the $\xi_i$'s grows from level $x_1 \wedge x_2$ up to level $x_1 \vee x_2$ without interruption (that is, $\alpha_i$ turning back to $-1$) is $\exp\left(-\int_{x_1 \wedge x_2}^{x_1 \vee x_2} w(z)\,\mathrm{d}z\right)$. Note that, if it happens, then $\eta_1$ and $\eta_2$ merge. Therefore, there is a coupling where the rate of merge is

$$c(x_1, x_2) := w(-x_1 \vee x_2) \exp\left(-\int_{x_1 \wedge x_2}^{x_1 \vee x_2} w(z)\,\mathrm{d}z\right).$$

Consider the interval $I_b = (-b, b)$ where $b$ will be chosen later appropriately. If $\eta_1 = x_1$ and $\eta_2 = x_2$ where $x_1, x_2 \in I_b$, then for the rate of merge

$$c(x_1, x_2) \geq w(-b) \exp\left(-\int_{-b}^{b} w(z)\,\mathrm{d}z\right) =: \beta(b) > 0 \tag{4.62}$$

holds if $w(x) > 0$ for all $x \in \mathbb{R}$.

Let $\vartheta$ be the time spent in $I_b$, more precisely,

$$\begin{aligned} \vartheta_i(t) &:= |\{0 \leq s \leq t : \eta_i(s) \in I_b\}| \qquad i = 1, 2, \\ \vartheta_{12}(t) &:= |\{0 \leq s \leq t : \eta_1(s) \in I_b, \eta_2(s) \in I_b\}|. \end{aligned}$$

The estimate

$$\mathbf{P}(T > t) \leq \mathbf{P}\left(\vartheta_{12}(t) < \frac{t}{2}\right) + \mathbf{P}\left(T > t \;\middle|\; \vartheta_{12}(t) \geq \frac{t}{2}\right)$$

is clearly true. Note that

$$\mathbf{P}\left(T > t \;\middle|\; \vartheta_{12}(t) \geq \frac{t}{2}\right) \leq \exp\left(-\frac{1}{2}\beta(b) t\right)$$

follows from (4.62).

By the inclusion relation

$$\left\{\vartheta_{12}(t) < \frac{t}{2}\right\} \subset \left\{\vartheta_1(t) < \frac{3}{4}t\right\} \cup \left\{\vartheta_2 < \frac{3}{4}t\right\}, \tag{4.63}$$

it suffices to prove that the tails of $\mathbf{P}\left(\vartheta_i(t) < \frac{3}{4}t\right)$ decay exponentially $i = 1, 2$, if $b$ is large enough.

We will show that

$$\mathbf{P}\left(\frac{|\{0 \leq s \leq t : \eta(s) < b\}|}{t} < \frac{7}{8}\right) \leq e^{-\gamma t}. \tag{4.64}$$

A similar statement can be proved for the time spent above $-b$, therefore another inclusion relation like (4.63) gives the lemma.

First, we verify that the first hitting of level $b$

$$\inf\{s > 0 : \eta_i(s) = b\}$$

has finite exponential moment, hence, it is negligible with overwhelming probability and we can suppose that $\eta_i(0) = b$. Indeed, for any fixed $\varepsilon > 0$, the measures $\rho$ and $Q(b, \cdot)$ assign exponentially small weight to the complement of the interval $[-\varepsilon t, \varepsilon t]$ as $t \to \infty$. From now on, we suppress the subscript of $\eta_i$, we forget about the initial values, and assume only that $\eta(0) \in [-\varepsilon t, \varepsilon t]$.

If $\eta(0) \in [b, \varepsilon t]$, then recall the proof Lemma 4.3.2. There, we could majorate $\eta$ with a homogeneous process $\zeta_b$. If we define

$$a := \mathbf{E}\left(\theta'_- \;\middle|\; \zeta_b(0) = b + 1\right)$$

with the notation (4.49), which is finite by Lemma 4.3.2, then from a large deviation principle,

$$\mathbf{P}\left(\theta_-(t) > 2a\varepsilon t \;\middle|\; \eta(0) \in [b, \varepsilon t]\right) \le \mathbf{P}\left(\theta'_-(t) > 2a\varepsilon t \;\middle|\; \eta(0) \in [b, \varepsilon t]\right) < e^{-\gamma t} \tag{4.65}$$

with some $\gamma > 0$.

If $\eta(0) \in [-\varepsilon t, b]$, then we can neglect that piece of the trajectory of $\eta$ which falls into the interval $[0, \theta_+]$, because without this, $\vartheta(t)$ decreases and the bound on (4.64) becomes stronger. Since $\eta$ jumps at $\theta_+$ a.s. and the distribution of $\eta(\theta_+)$ is $Q(b, \cdot)$, we can use the previous observations concerning the case $\eta(0) \in [b, \varepsilon t]$.

Using (4.65), it is enough to prove that

$$\mathbf{P}\left(\frac{|\{0 \le s \le t : \eta(s) < b\}|}{t} < \frac{7}{8} + 2a\varepsilon\right) \le e^{-\gamma t}$$

with the initial condition $\eta(0) = b$ where the value of $b$ is not specified yet. We introduce $X_k, Y_k, Z_k$ and $\nu_t$ as in (4.53)–(4.56) with $Y_0 \equiv 0$. The only difference is that here we want to ensure a given portion of time spent below $b$ with high probability with the appropriate choice of $b$. With the same idea as in the proof of Lemma 4.3.3 in (4.57)–(4.59), we can show that

$$\mathbf{P}\left(\frac{\sum_{k=1}^{\nu_t+1} X_k}{t} \le \frac{7}{8} + 2a\varepsilon\right)$$

is exponentially small by large deviation theory if we choose $b$ large enough to set $\mathbf{E}(X_1)/\mathbf{E}(Z_1)$ (the expected portion of time spent below $b$) sufficiently close to 1. With this, the proof of (4.33) is complete, that of (4.34) is similar. □

### 4.5.3 Decay of the transition kernel

*Proof of Lemma 4.3.5.* We return to the idea that the partial sums of $Z_k$'s form a renewal process. Remember the definitions (4.53)–(4.56). This proof relies on the estimate

$$|\eta(t)| \le Z_{\nu_t+1},$$

which is true, because the process $\eta$ can decrease with speed at most 1. Therefore, it suffices to prove the exponential decay of the tail of $Z_{\nu_t+1}$.

Define the *renewal measure* with

$$U(A) := \sum_{n=0}^{\infty} \mathbf{P}\left(\sum_{k=1}^{n} Z_k \in A\right)$$

for any $A \subset \mathbb{R}$. We consider the *age* and the *residual waiting time*

$$A_t := t - \sum_{k=1}^{\nu_t} Z_k,$$
$$R_t := \sum_{k=1}^{\nu_t+1} Z_k - t$$

separately. For the distribution of the former $H(t,x) := \mathbf{P}(A_t > x)$, the renewal equation

$$H(t,x) = (1 - F(t))\mathbb{1}(t > x) + \int_0^t H(t-s,x)\,\mathrm{d}F(s) \tag{4.66}$$

holds where $F(x) = \mathbf{P}(Z_1 < x)$. (4.66) can be deduced by conditioning on the time of the first renewal, $Z_1$. From Theorem (4.8) in [D95], it follows that

$$H(t,x) = \int_0^t (1 - F(t-s))\mathbb{1}(t-s > x)U(\mathrm{d}s). \tag{4.67}$$

As explained after (4.55), Lemma 4.3.2 and Lemma 4.3.3 with $b = 0$ together imply that $1 - F(x) \le Ce^{-\gamma x}$ with some $C < \infty$ and $\gamma > 0$. On the other hand,

$$U([k,k+1]) \le U([0,1])$$

is true, because, in the worst case, there is a renewal at time $k$. Otherwise, the distribution of renewals in $[k,k+1]$ can be obtained by shifting the renewals in $[0,1]$ with $R_k$. We can see from (4.67) by splitting the integral into segments with unit length that

$$H(t,x) \le U([0,1]) \sum_{k=\lfloor x \rfloor}^{\infty} Ce^{-\gamma k},$$

which is uniform in $t > 0$.

With the equation

$$\{R_t > x\} = \{A_{t+x} \ge x\} = \{\text{no renewal in } (t, t+x]\},$$

a similar uniform exponential bound can be deduced for the tail $\mathbf{P}(R_t > x)$. Since $Z_{\nu_t+1} = A_t + R_t$, the proof is complete. □

# Chapter 5

# Central limit theorem for the self-repellent Brownian polymer in three or higher dimensions

As a continuous space-time counterpart of the myopic self-avoiding walk, we investigate the asymptotic behaviour of the self-repellent Brownian polymer (SRBP) in the non-recurrent dimensions. First, extending $1d$ results from [TTV11], we identify a natural time-stationary and ergodic distribution of the environment (essentially smeared-out occupation time measure of the process) as seen from the moving particle. As a main result, we prove that, in three and more dimensions, in this stationary (and ergodic) regime, the displacement of the moving particle scales diffusively and its finite dimensional distributions converge to those of a Wiener process.

The results of the present chapter settle parts of the conjectures in [APP83], which are also described in Section 1.1. These conjectures were based on scaling and renormalization group arguments, but we present a rigorous proof which is a first mathematical result in the high dimensional regime.

The main tool is the non-reversible version of the Kipnis – Varadhan type central limit theorem for additive functionals of ergodic Markov processes and the *graded sector condition* of Sethuraman, Varadhan and Yau, see [SVY00].

## 5.1 Definition of model and background

The self-repellent Brownian polymer (SRBP) model is defined as follows. We give a more precise definition here than the one in (1.3)–(1.4). Let $V : \mathbb{R}^d \to \mathbb{R}_+$ be an approximate identity, that is, a smooth ($C^\infty$), spherically symmetric function with sufficiently fast decay at infinity, and

$$F : \mathbb{R}^d \to \mathbb{R}^d, \qquad F(x) := -\text{grad } V(x).$$

For reasons which will be clarified later, we also impose the condition of *positive definiteness* of $V$:

$$\widehat{V}(p) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ip\cdot x} V(x)\,\mathrm{d}x \geq 0.$$

A particular choice could be $V(x) := \exp\{-|x|^2/2\}$.

Let $t \mapsto B(t) \in \mathbb{R}^d$ be standard $d$-dimensional Brownian motion and define the stochastic process $t \mapsto X(t) \in \mathbb{R}^d$ as the solution of the SDE

$$X(t) = B(t) + \int_0^t \int_0^s F(X(s) - X(u))\,\mathrm{d}u\,\mathrm{d}s, \tag{5.1}$$

or

$$\mathrm{d}X(t) = \mathrm{d}B(t) + \left(\int_0^t F(X(t) - X(u))\,\mathrm{d}u\right)\mathrm{d}t. \tag{5.2}$$

**Remark.** Other types of self-interaction functions $F$ give rise to various different asymptotics. For the few rigorous results (mostly in 1d), see [NRW87], [DR92], [CL95], [CM96] and in particular [MT08] which also contains a survey of the results. Recent 1d results appear in [TTV11].

Now, introducing the occupation time measure

$$\ell(t, A) := \ell(0, A) + |\{0 < s \leq t : X(s) \in A\}| \tag{5.3}$$

where $A \subset \mathbb{R}^d$ is any measurable domain, and $\ell(0, A)$ is some signed initialization, we can rewrite the SDE (5.2) as follows:

$$\mathrm{d}X(t) = \mathrm{d}B(t) - \operatorname{grad}\big(V * \ell(t, \cdot)\big)(X(t))\,\mathrm{d}t \tag{5.4}$$

where $*$ stands for convolution in $\mathbb{R}^d$. We assume that $\ell(0, A)$ is a signed Borel measure on $\mathbb{R}^d$ with slow increase: for any $\varepsilon > 0$

$$\lim_{N\to\infty} N^{-(d+\varepsilon)} |\ell| (0, [-N, N]^d) = 0.$$

The form (5.4), compared with (1.7), shows explicitly the phenomenological similarity with the MSAW.

The non-rigorous scaling and renormalization group arguments originally formulated for the MSAW are equally well applicable to the SRBP. They suggest that the the dimension-dependent asymptotic scaling behaviour given in Section 1.1 is also valid here.

In the present chapter, we address the three or higher dimensional case of the SRBP model. We identify a stationary and ergodic distribution of the environment as seen from the position of the moving point where the environment is the smeared-out local time profile on $\mathbb{R}^d$. In this particular stationary regime, we prove *diffusive limit* (that is non-degenerate central limit theorem with normal scaling) for the displacement.

Our general approach is that of martingale approximation for additive functionals of ergodic Markov processes initiated for reversible processes in the classic Kipnis – Varadhan paper [KV86] and extended to non-reversible cases in [T86], [V96] and [SVY00]. We shall refer to this approach as the *Kipnis – Varadhan theory*. In particular, validity of the efficient martingale approximation will rely on checking the *graded sector condition* of [SVY00].

Next, we describe our results in plain words. For precise formulations, see subsection 5.2.4.

As a first step, we note that the environment profile appearing on the right-hand side of (5.2) and (5.4) as seen in a moving coordinate frame tied to the current position of the process $\eta(t) = (\eta(t,x))_{x\in\mathbb{Z}}$ with

$$\eta(t,x) := \eta(0, X(t)+x) + \int_0^t V(X(t)+x-X(u))\,\mathrm{d}u = \big(V * \ell(t,\cdot)\big)(X(t)+x) \qquad (5.5)$$

is a Markov process in a properly chosen function space $\Omega$, to be specified later, see (5.16), (5.18) and (5.22). As a first step, we identify a natural *time-stationary and ergodic distribution* of this process (5.5). Rather surprisingly, this is the Gaussian (scalar) field $x \mapsto \omega(x) \in \mathbb{R}$ with expectation and covariances

$$\mathbf{E}\left(\omega(x)\right) = 0, \qquad C(x-y) := \mathbf{E}\left(\omega(x)\omega(y)\right) = g * V(x-y) \qquad (5.6)$$

where

$$g : \mathbb{R}^d \to \mathbb{R}, \qquad g(x) := |x|^{2-d}$$

is the Green function of the Laplacian in $\mathbb{R}^d$. Note that throughout this chapter, $d \geq 3$. This is the *massless free Gaussian field* whose ultraviolet singularity is smeared out by convolution with the smooth and rapidly decaying approximate identity $V$. The Fourier transform of the covariance is

$$\widehat{C}(p) = |p|^{-2}\,\widehat{V}(p). \qquad (5.7)$$

See Theorem 5.2.1. All further results will be meant for the process being in this stationary regime. From this result, by ergodicity, the law of large numbers for the process $X(t)$ drops out, see Corollary 5.2.2.

The main result of this chapter refers to the *diffusive limit* of the process $t \mapsto X(t)$. From (5.1) and (5.5), it arises that the displacement is written as

$$X(t) = B(t) + \int_0^t \varphi(\eta(s))\,\mathrm{d}s \qquad (5.8)$$

where $\varphi : \Omega \to \mathbb{R}^d$ is a function of the state of the stationary and ergodic Markov process $t \mapsto \eta(t)$:

$$\varphi(\omega) = -\mathrm{grad}\,\omega(0). \qquad (5.9)$$

So, the natural approach to the diffusive limit of $X(t)$ is the Kipnis – Varadhan theory. We will prove validity of an efficient martingale approximation by checking Sethuraman – Varadhan – Yau's *graded sector condition*, cf. [SVY00]. It is easy to see that due to the spherical symmetry of the problem, we get

$$\mathbf{E}\left(X_k(t)X_l(t)\right) = \delta_{k,l} d^{-1} \mathbf{E}\left(|X(t)|^2\right).$$

We prove that, for $d \geq 3$, the limiting variance

$$\sigma^2 := d^{-1} \lim_{t\to\infty} t^{-1} \mathbf{E}\left(|X(t)|^2\right) \in (0,\infty) \tag{5.10}$$

exists and the finite dimensional marginals of the diffusively rescaled process

$$X_N(t) := \frac{X(Nt)}{\sigma\sqrt{N}} \tag{5.11}$$

converge to those of a standard $d$-dimensional Brownian motion. See Theorem 5.2.3. The main result shows similarity in spirit and techniques with those of [KO03], but the differences are also clear.

The results are meant *in probability with respect to the initial profile* $\eta(0,x)$ sampled from the stationary (and ergodic) initial distribution hinted at above. Recent results by Cuny and Peligrad [CP10] raise the hope that the Kipnis – Varadhan theory could be enhanced to central limit theorem for *almost all* initial conditions sampled according to the stationary distribution.

The rest of the chapter is structured as follows: in Section 5.2, we give the formal definitions, introduce notations, identify the stationary measure and formulate our main results precisely. Section 5.3 contains a sketch proof of the central limit theorem. In Section 5.4, we give the functional analytic background: the Hilbert spaces and the (bounded and unbounded) linear operators involved and the infinitesimal generator is presented. Ergodicity and law of large numbers for the displacement drop out for free. The short Section 5.5 is devoted to recalling the martingale approximation in Kipnis – Varadhan theory and the *graded sector condition* of [SVY00]. Finally, in Section 5.6, we check the abstract functional analytic conditions for our particular problem, and we conclude with the full proof of the central limit theorem for the displacement.

## 5.2 Formal setup and results

### 5.2.1 The stationary measure

We start with the *Ansatz* that the stationary distribution of the process $t \mapsto \eta(t,\cdot)$ is translation invariant zero mean Gaussian scalar with some covariance

$$\mathbf{E}\left(\eta(t,x)\eta(t,y)\right) = C(y-x) \tag{5.12}$$

to be identified at the end of the following computations.

In order to prove that this is indeed time-stationary, we have to show that, for any test function $x \to u(x) \in \mathbb{R}$, the moment generating functional

$$\varphi(t,u) := \mathbf{E}\left(\exp\{\langle u, \eta(t)\rangle\}\right)$$

is actually constant in time. In the present subsection, we use the notation

$$\langle u, v\rangle := \int_{\mathbb{R}^d} v(x)u(x)\,\mathrm{d}x.$$

In the forthcoming computations of the present section all *repeated subscripts* are summed from 1 to $d$. Using (5.5), note that, by standard Itô calculus,

$$\mathrm{d}\langle u, \eta(t)\rangle = -\langle \partial_l u, \eta(t)\rangle\,\mathrm{d}B_l(t) + \frac{1}{2}\langle \partial^2_{ll}u, \eta(t)\rangle\,\mathrm{d}t - \langle \partial_l u, \eta(t)\rangle \partial_l \eta(t,0)\,\mathrm{d}t + \langle u, V\rangle\,\mathrm{d}t.$$

Hence

$$\begin{aligned}&\mathbf{E}\left(\mathrm{d}\exp\{\langle u, \eta(t)\rangle\} \;\middle|\; \mathcal{F}_t\right)\\ &\quad= \exp\{\langle u, \eta(t)\rangle\}\left(\frac{1}{2}\langle \partial^2_{ll}u, \eta(t)\rangle + \frac{1}{2}\langle \partial_l u, \eta(t)\rangle^2 - \langle \partial_l u, \eta(t)\rangle\partial_l\eta(t,0) + \langle u, V\rangle\right)\mathrm{d}t.\end{aligned} \tag{5.13}$$

Now, using the Ansatz that $x \mapsto \eta(t,x)$ (with $t$ fixed) is a Gaussian field with covariance (5.12), by standard computations of Gaussian expectations, from (5.13), we obtain

$$\begin{aligned}\frac{\mathrm{d}\mathbf{E}\left(\exp\{\langle u, \eta(t)\rangle\}\right)}{\mathrm{d}t} &= \exp\{\langle u, C * u\rangle/2\}\Big(\frac{1}{2}\langle \partial^2_{ll}u, C*u\rangle + \frac{1}{2}\langle \partial_l u, C*\partial_l u\rangle +\\ &+ \frac{1}{2}\langle \partial_l u, C*u\rangle^2 - \langle \partial_l u, \partial_l C\rangle - \langle \partial_l u, C*u\rangle\langle u, \partial_l C\rangle + \langle u, V\rangle\Big)\,\mathrm{d}t.\end{aligned} \tag{5.14}$$

On the right-hand side the first two terms cancel out by an integration by parts. The third and fifth terms cancel one by one due to the simple fact that for any test function $u$,

$$\langle \partial_l u, C*u\rangle = 0.$$

Thus, the right-hand side of (5.14) is canceled out completely if and only if

$$V = -\partial^2_{ll}C. \tag{5.15}$$

This is equivalent to (5.6) or (5.7). Note that $d \geq 3$ was assumed.

### 5.2.2 State space and Gaussian measure

The proper state space of our basic processes will be the space of smooth scalar fields of slow increase at infinity:

$$\Omega := \left\{\omega \in C^\infty(\mathbb{R}^d \to \mathbb{R}) : \|\,\omega\,\|_{m,r} < \infty\right\} \tag{5.16}$$

where $\| \omega \|_{m,r}$ are the seminorms

$$\| \omega \|_{m,r} := \sup_{x \in \mathbb{R}^d} \big(1 + |x|\big)^{-1/r} \left|\partial^{|m|}_{m_1,\dots,m_d} \omega(x)\right|$$

defined for the multiindices $m = (m_1, \dots, m_d)$, $m_j \geq 0$; and $r \geq 1$. The space $\Omega$ endowed with these seminorms is a Fréchet space.

From Minlos's theorem (see [S74]), it follows that there exists a unique Gaussian probability measure $\pi(\mathrm{d}\omega)$ on the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d \to \mathbb{R})$ with characteristic functional

$$\mathbf{E}\left(\exp\{i\langle u, \omega\rangle\}\right) = \exp\left\{-\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(x)C(x-y)u(y)\,\mathrm{d}x\,\mathrm{d}y\right\},$$

and from smoothness of the covariance function $C(x)$, it follows that the probability measure $\pi(\mathrm{d}\omega)$ is actually concentrated on the space $\Omega \subset \mathcal{S}'(\mathbb{R}^d \to \mathbb{R})$ and (5.6) holds.

The Gaussian field $\omega(x)$ is realized e.g. as a moving average of white noise:

$$\omega(x) = \int_{\mathbb{R}^d} U(x-y)w(y)\,\mathrm{d}y,$$

where $U$ is the unique positive definite function for which $U * U = V$ and $w$ is $d$-dimensional white noise.

The group of spatial translations

$$\mathbb{R}^d \ni z \mapsto \tau_z : \Omega \to \Omega, \qquad (\tau_z\omega)(x) := \omega(x+z)$$

acts naturally on $\Omega$ and preserves the probability measure $\pi(\mathrm{d}\omega)$. Actually, the dynamical system $(\Omega, \pi(\mathrm{d}\omega), \tau_z : z \in \mathbb{R}^d)$ is *ergodic*.

### 5.2.3 Processes

First, we consider the process $t \mapsto (X(t), \zeta(t,\cdot)) \in \mathbb{R}^d \times \Omega$ defined as follows:

$$X(t) = X(0) + B(t) - \int_0^t \operatorname{grad}\zeta(s, X(s))\,\mathrm{d}s, \tag{5.17}$$

$$\zeta(t,x) = \zeta(0,x) + \int_0^t V(x - X(s))\,\mathrm{d}s \tag{5.18}$$

where $t \mapsto B(t)$ is a standard $d$-dimensional Brownian motion, and $X(0) \in \mathbb{R}^d$, $\zeta(0,\cdot) \in \Omega$ are the initial data for the process $t \mapsto (X(t), \zeta(t,\cdot))$.

Written as a single SDE for the process $t \mapsto X(t)$, we get from (5.17) and (5.18) that

$$X(t) = X(0) + B(t) + \int_0^t \left\{\zeta(0, X(s)) + \int_0^s F(X(s) - X(u))\,\mathrm{d}u\right\}\mathrm{d}s. \tag{5.19}$$

The SDE (5.19) differs from the original SDE (5.1) only by the presence of the initial profile $\zeta(0,x)$, which is a natural modification.

From (5.17) and (5.18), it follows that

$$X(t_0+t) = X(t_0) + \big(B(t+t_0) - B(t_0)\big) - \int_{t_0}^{t_0+t} \operatorname{grad}\zeta(s, X(s))\,\mathrm{d}s, \tag{5.20}$$

$$\zeta(t_0+t, x) = \zeta(t_0, x) + \int_{t_0}^{t_0+t} V(x - X(s))\,\mathrm{d}s. \tag{5.21}$$

From this form, it is apparent that the process $t \mapsto (X(t), \zeta(t,\cdot)) \in \mathbb{R}^d \times \Omega$ is Markovian.

The environment profile as seen from the moving point $X(t)$ is

$$x \mapsto \eta(t,x) := \zeta(t, X(t)+x). \tag{5.22}$$

From (5.20) and (5.21), we readily obtain that $t \mapsto \eta(t) := \eta(t,\cdot)$ is itself a Markov process on the state space $\Omega$.

We define the function $\varphi : \Omega \to \mathbb{R}^d$ by (5.9), and from (5.17), (5.18) and (5.22), we readily get (5.8).

### 5.2.4 Results

**Theorem 5.2.1.** *The Gaussian probability measure $\pi(\mathrm{d}\omega)$ on $\Omega$ with mean $0$ and covariances (5.6) is time-invariant and ergodic for the $\Omega$-valued Markov process $t \mapsto \eta(t)$.*

**Corollary 5.2.2.** *For $\pi$-almost all initial profiles $\zeta(0,\cdot)$,*

$$\lim_{t\to\infty} \frac{X(t)}{t} = 0 \quad \text{a.s.} \tag{5.23}$$

**Remark.** It is clear that, in dimensions $d \geq 3$, other stationary distributions of the process $t \mapsto \eta(t)$ exist. In particular, due to transience of the process $t \mapsto X(t)$, the stationary measure (presumably) reached from starting with "empty" initial conditions $\eta(0,x) \equiv 0$ certainly differs from our $\mathrm{d}\pi$. Our methods and results are valid for the particular stationary distribution $\mathrm{d}\pi$.

The main result of the present chapter is the following theorem:

**Theorem 5.2.3.** *In dimensions $d \geq 3$, the following hold:*

1. *The limiting variance*

$$\sigma^2 := d^{-1} \lim_{t\to\infty} t^{-1}\mathbf{E}\left(|X(t)|^2\right)$$

*exists and*

$$1 \leq \sigma^2 \leq 1 + \rho^2 \tag{5.24}$$

*where*

$$\rho^2 := d^{-1} \int_{\mathbb{R}^d} |p|^{-2}\, \widehat{V}(p)\,\mathrm{d}p < \infty. \tag{5.25}$$

2. *The finite dimensional marginal distributions of the diffusively rescaled process*

$$X_N(t) := \frac{X(Nt)}{\sigma\sqrt{N}} \tag{5.26}$$

*converge to those of a standard d-dimensional Brownian motion. The convergence is meant in probability with respect to the starting state $\eta(0)$ sampled according to* $d\pi$.

Theorem 5.2.3 will be proved by use of the martingale approximation of the Kipnis – Varadhan theory and the so-called *graded sector condition* of [SVY00].

## 5.3 Sketch proof of the central limit theorem

In Section 5.4, we introduce the Gaussian Hilbert space $\mathcal{H} = \mathcal{L}^2(\Omega, \pi)$ first which the infinitesimal generator of the Markov process $\eta(t)$ acts on. This is a graded Hilbert space, that is, a direct sum of infinitely many orthogonal subspaces as given in (5.28). There are two other isomorphic representations of $\mathcal{H}$ which are used in the computations later. The basic linear operators (differentiations, creations and annihilations) are defined in Subsection 5.4.2 and their action is given in all the three representations in Subsection 5.4.3.

The infinitesimal generator $G$ of the environment process $\eta(t)$ is computed in (5.42). It is rewritten in (5.43) using the basic operators. The self-adjoint and skew self-adjoint parts of $G$ are (5.45) and (5.46) which are compatible with the grading of the Hilbert space in the sense of (5.47).

A natural symmetry holds in our model, $\eta(t)$ has the *Yaglom reversibility* property, i.e., starting from the stationary distribution, the flipped and time-reversed process $\widetilde{\eta}(t)$ in (5.50) is equal with $\eta(t)$ in law. In terms of the genarator, it is equivalent to (5.49). This property will be useful later.

For the proof of Theorem 5.2.1, stationarity of $d\pi$ was verified in Subsection 5.2.1. The Dirichlet form of the process $\eta(t)$ is given in (5.51). Ergodicity drops out from that easily.

The starting point for proving central limit theorem for the displacement $X(t)$ of SRBP is the equation (5.8). In this representation, $B(t)$ is a Brownian motion, the integral term is an additive functional of the Markov process $\eta(t)$. Hence our aim is to use a version of the Kipnis – Varadhan theorem.

Section 5.5 is devoted to recall the general Kipnis – Varadhan theory and the *graded sector condition*, which will be used for the SRBP later. The first version of the Kipnis – Varadhan theorem in [KV86] is valid for reversible Markov processes, i.e. if the skew self-adjoint part of the generator $A = 0$ vanishes. Theorem 5.5.1 from [T86] gives sufficient conditions (5.55)–(5.56) for the central limit theorem for additive functionals of *non-reversible* Markov processes to hold. In the presence of grading (5.58), Theorem 5.5.2 can be applied. One assumes that the operators are compatible with the grading, i.e. (5.47) holds. Then the conditions (5.59) and (5.60) yield the central limit theorem for the functional (5.52) through Theorem 5.5.1. (5.59) refers to the operator, and it is the *graded sector condition*, (5.60) is the diffusive upper bound or $H_{-1}$-bound on the function $f$.

Section 5.6 contains the rigorous proof of Theorem 5.2.3. The central limit theorem is proved using Theorem 5.5.2, hence diffusive upper bound and graded sector condition are checked. For showing (5.24), diffusive lower bound is also needed. With the latter, we rule out the case when the diffusive limit of the integral term is (5.8) cancels out with the martingale (in this case Brownian) term. In principle, it could happen just like in the famous case of the one-dimensional nearest neighbour symmetric simple exclusion process, see [A83].

The diffusive lower bound follows from the Yaglom reversibility of the SRBP model, namely the process $M(s,t)$ defined in (5.61) is a forward and backward martingale as stated in Lemma 5.6.1. It is enough to see that $M(s,t)$ and the integral

$$\int_s^t \varphi(\eta(u))\,\mathrm{d}u$$

are uncorrelated. This provides a diffusive lower bound on the variance of the displacement $X(t)$, see (5.64).

For the diffusive upper bound, Lemma 5.6.2 from [SVY00] is used. It says that the limiting variance of the integral term in (5.8) is bounded by that of the integral of the same function along the trajectory of a different Markov process $\xi(t)$ on the same state space. The process $\xi(t)$ is a reversible Markov process with infinitesimal generator $-S$, i.e. the symmetric part of the generator $G$ of $\eta(t)$. In the case of the SRBP, the process $\xi(t)$ is a diffusion in random scenery, for which the limiting variance of the integral is computed in (5.66), and it is finite in three or higher dimensions.

For checking the graded sector condition, one needs to prove a bound (5.67) with $\gamma < 1$. The exact form of the operator to be bounded is given in (5.68). It is already established in (5.38) that the operator $\nabla_l|\Delta|^{-1/2}$ is bounded, hence it is enough to calculate the norm of $|\Delta|^{-1/2}a_l^*$. It is done in (5.69)–(5.70). The result is (5.71), that is, the graded sector condition is satisfied with $\gamma = 1/2$. We use in the computations again that the dimension is at least 3. With this, the proof of Theorem 5.2.3 is complete.

## 5.4 Spaces and operators

The natural formalism for the proofs of our theorems is that of Fock spaces and Gaussian Hilbert spaces, and linear operators over them. For basics of Gaussian Hilbert spaces and

Wick products, see [J97] and [S74]. Our main Hilbert space is $\mathcal{H} := \mathcal{L}^2(\Omega, \pi)$. This is a Gaussian Hilbert space, and has very natural unitary equivalent representations as Fock spaces. We follow the usual notation of Euclidean quantum field theory, see e.g. [S74]. In subsection 5.4.1, we give formal definition of the three unitary equivalent representations of the Hilbert space $\mathcal{L}^2(\Omega, \pi)$. In subsection 5.4.2, we define the linear operators which are relevant for our purposes and we present their action on the three unitary equivalent formulations in Subsection 5.4.3. In Subsection 5.4.4, the infinitesimal generator of the semigroup of the stationary Markov process $t \mapsto \eta(t,\cdot) \in \Omega$, acting on $\mathcal{L}^2(\Omega, \pi)$, and its adjoint are computed and the first consequences (ergodicity, law of large numbers) are settled.

### 5.4.1 Spaces

Throughout this chapter, we use the convention of unitary Fourier transform

$$\widehat{u}(p) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ip\cdot x} u(x)\,\mathrm{d}x \tag{5.27}$$

and the shorthand notation

$$\begin{aligned} &\mathbf{x} = (x_1, \dots, x_n) \in \left(\mathbb{R}^d\right)^n, && x_m = (x_{m1}, \dots, x_{md}) \in \mathbb{R}^d, && \partial_{ml} := \frac{\partial}{\partial x_{ml}}, \\ &\mathbf{p} = (p_1, \dots, p_n) \in \left(\mathbb{R}^d\right)^n, && p_m = (p_{m1}, \dots, p_{md}) \in \mathbb{R}^d, \end{aligned}$$

$m = 1, \dots, n$, $l = 1, \dots, d$.

We denote by $\mathcal{S}_n$, respectively, $\widehat{\mathcal{S}}_n$, the *symmetric* Schwartz spaces

$$\begin{aligned} \mathcal{S}_n :=& \{u : \mathbb{R}^{dn} \to \mathbb{C} : u(\varpi \mathbf{x}) = u(\mathbf{x}),\ \varpi \in \mathrm{Perm}(n)\}, \\ \widehat{\mathcal{S}}_n :=& \{\widehat{u} : \mathbb{R}^{dn} \to \mathbb{C} : \widehat{u}(\varpi \mathbf{p}) = \widehat{u}(\mathbf{p}),\ \varpi \in \mathrm{Perm}(n)\}. \end{aligned}$$

In the preceding formulas $\mathrm{Perm}(n)$ denotes the group of permutations on the $n$ indices.

The spaces $\mathcal{S}_n$, respectively, $\widehat{\mathcal{S}}_n$ are endowed with the following scalar products

$$\begin{aligned} \langle u, v\rangle &:= \int_{\mathbb{R}^{dn}} \int_{\mathbb{R}^{dn}} \overline{u(\mathbf{x})} C(\mathbf{x} - \mathbf{y}) v(\mathbf{y})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}, \\ \langle \widehat{u}, \widehat{v}\rangle &:= \int_{\mathbb{R}^{dn}} \overline{\widehat{u}(\mathbf{p})} \widehat{C}(\mathbf{p}) \widehat{v}(\mathbf{p})\,\mathrm{d}\mathbf{p} \end{aligned}$$

where

$$C(\mathbf{x} - \mathbf{y}) := \prod_{m=1}^{n} C(x_m - y_m), \qquad \widehat{C}(\mathbf{p}) := \prod_{m=1}^{n} \widehat{C}(p_m).$$

Let $\mathcal{K}_n$ and $\widehat{\mathcal{K}}_n$ be the closures of $\mathcal{S}_n$, respectively, $\widehat{\mathcal{S}}_n$ with respect to the Euclidean norms defined by these inner products. The Fourier transform (5.27) realizes an isometric isomorphism between the Hilbert spaces $\mathcal{K}_n$ and $\widehat{\mathcal{K}}_n$.

These Hilbert spaces are actually the symmetrized $n$-fold tensor products

$$\mathcal{K}_n := \mathrm{symm}\big(\mathcal{K}_1^{\otimes n}\big), \qquad \mathcal{K}_n := \mathrm{symm}\big(\widehat{\mathcal{K}}_1^{\otimes n}\big).$$

Finally, the full Fock spaces are

$$\mathcal{K} := \overline{\oplus_{n=0}^{\infty}\mathcal{K}_n}, \qquad \widehat{\mathcal{K}} := \overline{\oplus_{n=0}^{\infty}\widehat{\mathcal{K}}_n}.$$

The Hilbert space of our true interest is $\mathcal{H} = \mathcal{L}^2(\Omega, \pi)$. This is itself a graded Gaussian Hilbert space

$$\mathcal{H} = \overline{\oplus_{n=0}^{\infty}\mathcal{H}_n} \tag{5.28}$$

where the subspaces $\mathcal{H}_n$ are isometrically isomorphic with the subspaces $\mathcal{K}_n$ of $\mathcal{K}$ through the identification

$$\varphi_n : \mathcal{K}_n \to \mathcal{H}_n, \quad \varphi_n(u) := \frac{1}{\sqrt{n!}} \int_{\mathbb{R}^{dn}} u(\mathbf{x}) :\!\omega(x_1) \ldots \omega(x_n)\!: \, \mathrm{d}\mathbf{x}. \tag{5.29}$$

Here and in the rest of this chapter, we denote by $:\!X_1 \ldots X_n\!:$ the Wick product of the jointly Gaussian random variables $(X_1, \ldots, X_n)$. In order to ease notation, the mapping $\varphi_1 : \mathcal{K}_1 \to \mathcal{H}_1$ will be simply denoted by $\varphi$.

As the graded Hilbert spaces

$$\mathcal{H} := \overline{\oplus_{n=0}^{\infty}\mathcal{H}_n}, \quad \mathcal{K} := \overline{\oplus_{n=0}^{\infty}\mathcal{K}_n}, \quad \widehat{\mathcal{K}} := \overline{\oplus_{n=0}^{\infty}\widehat{\mathcal{K}}_n}$$

are isometrically isomorphic in a natural way, we shall move freely between the various representations.

### 5.4.2 General notation of operators

We use the standard notation of Fock spaces. First, we give a general framework of notation and identities formulated over the Gaussian Hilbert space $\mathcal{H}$. Then, we turn to our relevant linear operators and we give their representations in all three Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ and $\widehat{\mathcal{K}}$.

The action of linear operators over $\mathcal{H} = \overline{\oplus_{n=0}^{\infty}\mathcal{H}_n}$ will be typically given in terms of Wick monomials. It is understood that their action is extended by linearity and graph closure.

Given a (bounded or unbounded) densely defined and closed linear operator $A$ over the basic Hilbert space $\mathcal{K}_1$, its second quantized version acting over the graded Gaussian Hilbert space $\mathcal{H}$ will be denoted by $\mathrm{d}\Gamma(A)$. This latter one acts over Wick monomials as follows, $\mathrm{d}\Gamma(A) : \mathcal{H}_n \to \mathcal{H}_n$,

$$\mathrm{d}\Gamma(A) :\!\varphi(v_1) \cdots \varphi(v_n)\!: = \sum_{m=1}^{n} :\!\varphi(v_1) \cdots \varphi(Av_m) \cdots \varphi(v_n)\!: \; .$$

Given a vector $u$ from the basic Hilbert space $\mathcal{K}_1$, the creation and annihilation (raising and lowering) operators associated to it, acting over the Gaussian Hilbert space $\mathcal{H}$, will be denoted by $a^*(u) : \mathcal{H}_n \to \mathcal{H}_{n+1}$, respectively, $a(u) : \mathcal{H}_n \to \mathcal{H}_{n-1}$ acting on Wick monomials as

$$\begin{aligned} a^*(u) :\varphi(v_1)\dots\varphi(v_n): &= :\varphi(u)\varphi(v_1)\dots\varphi(v_n):, \\ a(u) :\varphi(v_1)\dots\varphi(v_n): &= \sum_{m=1}^{n} \langle u, v_m\rangle :\varphi(v_1)\dots\varphi(v_{m-1})\varphi(v_{m+1})\dots\varphi(v_n): . \end{aligned}$$

For basics about creation, annihilation and second quantized operators, see e.g. [S74] or [J97].

We also define the unitary involution $J$ on $\mathcal{H}$:

$$Jf(\omega) := f(-\omega), \qquad J\restriction_{\mathcal{H}_n} = (-1)^n I \restriction_{\mathcal{H}_n} .$$

The well-known canonical commutation relations between the operators introduced are:

$$[a(u), a(v)] = 0, \qquad [a^*(u), a^*(v)] = 0, \qquad [a(u), a^*(v)] = \langle u, v\rangle I, \tag{5.30}$$

$$[\mathrm{d}\Gamma(A), a^*(u)] = a^*(Au), \qquad [\mathrm{d}\Gamma(A), a(u)] = -a(A^*u), \tag{5.31}$$

$$[J, \mathrm{d}\Gamma(A)] = 0, \qquad \{J, a^*(u)\} = 0, \qquad \{J, a(u)\} = 0. \tag{5.32}$$

Two more operators will be needed: given an element $u \in \mathcal{K}_1$, *multiplication by* $\varphi(u)$ will be denoted $M(u)$, that is, formally, for $f \in \mathcal{L}^2(\Omega, \pi)$,

$$\big(M(u)f\big)(\omega) := \varphi(u)(\omega)f(\omega).$$

Finally, for a fixed element $\vartheta \in \Omega$, we introduce *differentiation in the direction* $\vartheta$: formally

$$D_\vartheta f(\omega) := \lim_{\varepsilon\to 0} \varepsilon^{-1}\big(f(\omega + \varepsilon\vartheta) - f(\omega)\big).$$

Both operators are well-defined on Wick monomials, and are extended by linearity and graph closure.

Given $u \in \mathcal{K}_1$ the identities (5.33) and (5.34) below hold:

(1) The multiplication operator $M(u)$ is actually

$$M(u) = a^*(u) + a(u). \tag{5.33}$$

(2) If $C * u \in \Omega$ then

$$D_{C*u} = a(u). \tag{5.34}$$

Both identities are checked by direct computation on Wick monomials. The identity (5.34) is a particular case of the *directional derivative* of Malliavin calculus, see [J97].

### 5.4.3 Specific linear operators

The most relevant operators for our present purposes are

$$\nabla_l := \mathrm{d}\Gamma(\partial_l), \qquad \Delta := \sum_{l=1}^{d} \nabla_l^2, \qquad a_l := a(\partial_l \delta_0), \qquad a_l^* := a^*(\partial_l \delta_0)$$

where $\partial_l = \frac{\partial}{\partial x_l}$ and $\delta_0$ is Dirac's delta concentrated on $0 \in \mathbb{R}^d$. Note that $\delta_0$ and all its partial derivatives are in the Hilbert space $\mathcal{K}_1$.

We give now their action on the spaces $\mathcal{H}_n$, $\mathcal{K}_n$ and $\widehat{\mathcal{K}}_n$. The point is that we are interested primarily in their action on the space $\mathcal{L}^2(\Omega, \pi) = \overline{\oplus_{n=0}^{\infty} \mathcal{H}_n}$, but explicit computations in later sections are handy in the unitary equivalent representations over the space $\widehat{\mathcal{K}} = \overline{\oplus_{n=0}^{\infty} \widehat{\mathcal{K}}_n}$. The action of various operators over $\mathcal{H}_n$ will be given in terms of the Wick monomials $:\omega(x_1)\ldots\omega(x_n):$ and it is understood that the operators are extended by linearity and graph closure.

- The operators $\nabla_l$, $l = 1, \ldots, d$:

$$\nabla_l : \mathcal{H}_n \to \mathcal{H}_n, \qquad \nabla_l :\omega(x_1)\ldots\omega(x_n): = -\sum_{m=1}^{n} :\omega(x_1)\ldots\partial_l\omega(x_m)\ldots\omega(x_n): ,$$

$$\nabla_l : \mathcal{K}_n \to \mathcal{K}_n, \qquad \nabla_l u(\mathbf{x}) = \sum_{m=1}^{n} \frac{\partial u}{\partial x_{ml}}(\mathbf{x}),$$

$$\nabla_l : \widehat{\mathcal{K}}_n \to \widehat{\mathcal{K}}_n, \qquad \nabla_l \widehat{u}(\mathbf{p}) = i\Big(\sum_{m=1}^{n} p_{ml}\Big)\widehat{u}(\mathbf{p}).$$

  Note that these are actually unbounded, closed, skew self-adjoint operators. They are densely defined on $\mathcal{H}_n$, $\mathcal{K}_n$, respectively, $\widehat{\mathcal{K}}_n$.

- The operator $\Delta$:

$$\Delta : \mathcal{H}_n \to \mathcal{H}_n, \qquad \Delta :\omega(x_1)\ldots\omega(x_n):$$
$$= \sum_{l=1}^{d} \sum_{m,m'=1}^{n} :\omega(x_1)\ldots\partial_l\omega(x_m)\ldots\partial_l\omega(x_{m'})\ldots\omega(x_n): ,$$

$$\Delta : \mathcal{K}_n \to \mathcal{K}_n, \qquad \Delta u(\mathbf{x}) = \sum_{l=1}^{d} \sum_{m,m'=1}^{n} \frac{\partial^2 u}{\partial x_{ml} \partial x_{m'l}}(\mathbf{x}),$$

$$\Delta : \widehat{\mathcal{K}}_n \to \widehat{\mathcal{K}}_n, \qquad \Delta \widehat{u}(\mathbf{p}) = -\left|\sum_{m=1}^{n} p_m\right|^2 \widehat{u}(\mathbf{p}).$$

  The operator $\Delta$ is unbounded, densely defined, self-adjoint and positive. Note that $\Delta$ is *not* the second quantized Laplacian.

- The operator $|\Delta|^{-1/2} = (-\Delta)^{-1/2}$:

$$|\Delta|^{-1/2} : \mathcal{H}_n \to \mathcal{H}_n, \qquad \text{no explicit formula,}$$
$$|\Delta|^{-1/2} : \mathcal{K}_n \to \mathcal{K}_n, \qquad \text{no explicit formula,}$$
$$|\Delta|^{-1/2} : \widehat{\mathcal{K}}_n \to \widehat{\mathcal{K}}_n, \qquad |\Delta|^{-1/2} \widehat{u}(\mathbf{p}) = \left|\sum_{m=1}^{n} p_m\right|^{-1} \widehat{u}(\mathbf{p}).$$

The operator $|\Delta|^{-1/2}$ is unbounded, densely defined, self-adjoint and positive.

- The operators $|\Delta|^{-1/2}\nabla_l$, $l = 1, \ldots, d$:

$$|\Delta|^{-1/2}\nabla_l : \mathcal{H}_n \to \mathcal{H}_n, \qquad \text{no explicit formula,} \tag{5.35}$$
$$|\Delta|^{-1/2}\nabla_l : \mathcal{K}_n \to \mathcal{K}_n, \qquad \text{no explicit formula,} \tag{5.36}$$
$$|\Delta|^{-1/2}\nabla_l : \widehat{\mathcal{K}}_n \to \widehat{\mathcal{K}}_n, \qquad |\Delta|^{-1/2}\nabla_l \widehat{u}(\mathbf{p}) = \frac{i\sum_{m=1}^{n} p_{ml}}{|\sum_{m=1}^{n} p_m|}\widehat{u}(\mathbf{p}). \tag{5.37}$$

These are *bounded* skew self-adjoint operators with operator norm

$$\left\| \, |\Delta|^{-1/2}\nabla_l \right\| = 1. \tag{5.38}$$

- The creation operators $a_l^*$, $l = 1, \ldots, d$:

$$a_l^* : \mathcal{H}_n \to \mathcal{H}_{n+1}, \qquad a_l^* :\omega(x_1)\ldots\omega(x_n): = :\partial_l\omega(0)\omega(x_1)\ldots\omega(x_n): ,$$
$$a_l^* : \mathcal{K}_n \to \mathcal{K}_{n+1}, \qquad a_l^* u(x_1, \ldots, x_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{m=1}^{n+1} \partial_l\delta(x_m) u(x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_{n+1}),$$
$$a_l^* : \widehat{\mathcal{K}}_n \to \widehat{\mathcal{K}}_{n+1}, \qquad a_l^* \widehat{u}(p_1, \ldots, p_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{m=1}^{n+1} i p_{ml} \widehat{u}(p_1, \ldots, p_{m-1}, p_{m+1}, \ldots, p_{n+1}).$$

The creation operators $a_l^*$, restricted to the subspaces $\mathcal{H}_n$, $\mathcal{K}_n$, respectively, $\widehat{\mathcal{K}}_n$ are bounded with operator norm

$$\left\| a_l^* \restriction_{\mathcal{H}_n} \right\| = \left\| a_l^* \restriction_{\mathcal{K}_n} \right\| = \left\| a_l^* \restriction_{\widehat{\mathcal{K}}_n} \right\| = \sqrt{C(0)}\sqrt{n+1}. \tag{5.39}$$

- The annihilation operators $a_l$, $l = 1, \ldots, d$:

$$a_l : \mathcal{H}_n \to \mathcal{H}_{n-1}, \qquad a_l :\omega(x_1)\ldots\omega(x_n): = \sum_{m=1}^{n} \partial_l C(x_m) :\omega(x_1)\ldots\omega(x_{m-1})\omega(x_{m+1})\ldots\omega(x_n): ,$$
$$a_l : \mathcal{K}_n \to \mathcal{K}_{n-1}, \qquad a_l u(x_1, \ldots, x_{n-1}) = \sqrt{n} \int_{\mathbb{R}^d} u(x_1, \ldots, x_{n-1}, y)\partial_l C(y)\, \mathrm{d}y,$$
$$a_l : \widehat{\mathcal{K}}_n \to \widehat{\mathcal{K}}_{n-1}, \qquad a_l \widehat{u}(p_1, \ldots, p_{n-1}) = \sqrt{n} \int_{\mathbb{R}^d} \widehat{u}(p_1, \ldots, p_{n-1}, q) i q_l \widehat{C}(q)\, \mathrm{d}q.$$

The annihilation operators $a_l$ restricted to the subspaces $\mathcal{H}_n$, $\mathcal{K}_n$, respectively, $\widehat{\mathcal{K}}_n$ are bounded with operator norm

$$\left\| a_l \restriction_{\mathcal{H}_n} \right\| = \left\| a_l \restriction_{\mathcal{K}_n} \right\| = \left\| a_l \restriction_{\widehat{\mathcal{K}}_n} \right\| = \sqrt{C(0)}\sqrt{n}. \tag{5.40}$$

Furthermore, as the notation $a_l^*$ and $a_l$ suggests, these operators are adjoint of each other.

Since all computations will be performed in the representation $\widehat{\mathcal{K}}$, we give a common core for all the unbounded operators defined above – and some others to appear in future sections:

$$\widehat{\mathcal{C}} := \oplus_{n=0}^{\infty} \widehat{\mathcal{C}}_n, \qquad \widehat{\mathcal{C}}_n := \{\widehat{u} \in \widehat{\mathcal{K}}_n : \sup_{\mathbf{p} \in \mathbb{R}^{dn}} |\widehat{u}(\mathbf{p})| < \infty\}. \tag{5.41}$$

Note that the operator $|\Delta|^{-1/2}$ is defined on the dense subspace $\widehat{\mathcal{C}}$ *only for* $d \geq 3$. Furthermore, in dimensions $d \geq 3$, the operators $|\Delta|^{-1/2} \restriction_{\widehat{\mathcal{K}}_n}$ defined on the dense subspaces $\widehat{\mathcal{C}}_n$, are *essentially self-adjoint*. This follows, e.g., from Propositions VIII.1, VIII.2 of [RS72].

Notice also that $\nabla$ is the infinitesimal generator of the *unitary group of spatial translations* while $\Delta$ is the infinitesimal generator of the Markovian semigroup of *diffusion in random scenery*

$$\exp\{z\nabla\} = T_z, \qquad T_z f(\omega) := f(\tau_z \omega),$$
$$\exp\{t\Delta\} = Q_t, \qquad Q_t f(\omega) := \int \frac{\exp\{-z^2/(2t)\}}{\sqrt{2\pi t}} f(\tau_z \omega)\, \mathrm{d}z.$$

### 5.4.4 The infinitesimal generator, stationarity, Yaglom reversibility, ergodicity

We denote by $P_t$ the semigroup of the process $\eta(t)$:

$$P_t : \mathcal{H} \to \mathcal{H}, \qquad P_t f(\omega) := \mathbf{E}\left( f(\eta(t)) \,\middle|\, \eta(0) = \omega \right).$$

Then $[0, \infty) \ni t \mapsto P_t \in \mathcal{B}(\mathcal{H})$ is a Markovian contraction semigroup on $\mathcal{H}$. In order to identify its infinitesimal generator, note that the infinitesimal change in the state of the Markov process $\eta(t)$ is due to the following three terms:

(1) infinitesimal spatial shift due to $\mathrm{d}B(t)$;

(2) infinitesimal spatial shift due to $-\mathrm{grad}\, \eta(t, 0)\, \mathrm{d}t$;

(3) infinitesimal local change in $\eta$ due to increase of local time.

Altogether

$$\eta(t + \mathrm{d}t, x) = \eta(t, x + \mathrm{d}B(t) - \mathrm{grad}\, \eta(t, 0)\, \mathrm{d}t) + V(x)\, \mathrm{d}t.$$

Hence, given a sufficiently regular function on the state space $f : \Omega \to \mathbb{R}$, we compute

$$\lim_{t\to 0} \frac{\mathbf{E}\left(f(\eta(t) - f(\eta(0))) \,\middle|\, \eta(0) = \omega\right)}{t} = \left(\frac{1}{2}\Delta - \sum_{l=1}^{d} M(\partial_l \delta_0)\nabla_l + D_V\right) f(\omega). \tag{5.42}$$

Recall (5.6) and note that hence

$$V = -C * \sum_{l=1}^{d} \partial_{ll}^2 \delta_0$$

with

$$-\sum_{l=1}^{d} \partial_{ll}^2 \delta_0 \in \mathcal{K}_1.$$

Using (5.33), (5.31) and (5.34) (in this order), we readily obtain the following expression for the infinitesimal generator of the semigroup $P_t$:

$$G := \frac{1}{2}\Delta + \sum_{l=1}^{d} \left(a_l^* \nabla_l + \nabla_l a_l\right). \tag{5.43}$$

This operator is well defined on Wick polynomials of the field $\omega(x)$ and is extended by linearity and graph closure. It is not difficult to see that it satisfies the criteria of the Hille–Yoshida theorem (see [RS72]) and thus it is indeed the infinitesimal generator of a Markovian semigroup. We omit these technical details.

The adjoint generator is

$$G^* := \frac{1}{2}\Delta - \sum_{l=1}^{d} \left(a_l^* \nabla_l + \nabla_l a_l\right). \tag{5.44}$$

Note that due to the inner coherence of the model the last two terms on the right hand side of (5.42) combine to give the tidy skew self-adjoint part of the infinitesimal generators in (5.43) and (5.44).

For later use, we introduce notation for the symmetric (self-adjoint) and anti-symmetric (skew self-adjoint) parts of the generator

$$S := -\frac{1}{2}(G + G^*) = -\frac{1}{2}\Delta, \tag{5.45}$$

$$A := \frac{1}{2}(G - G^*) = \sum_{l=1}^{d} \left(a_l^* \nabla_l + \nabla_l a_l\right) =: A_+ + A_-. \tag{5.46}$$

It is a standard – though not completely trivial – exercise to check that the operators $S$ and $A$, a priori defined on the dense subspace $\widehat{\mathcal{C}}$ are indeed essentially self-adjoint, respectively, essentially skew self-adjoint.

Note that

$$S : \mathcal{H}_n \to \mathcal{H}_n, \quad A_+ : \mathcal{H}_n \to \mathcal{H}_{n+1}, \quad A_- : \mathcal{H}_n \to \mathcal{H}_{n-1}, \quad A_\mp = -A_\pm^* \tag{5.47}$$

and

$$S \restriction_{\mathcal{H}_0} = 0, \qquad A_+ \restriction_{\mathcal{H}_0} = 0, \qquad A_- \restriction_{\mathcal{H}_0 \oplus \mathcal{H}_1} = 0. \tag{5.48}$$

It is clear that

$$G^* 1\!\!1 = 0$$

and hence, it follows that $\pi$ is indeed stationary distribution of the process $t \mapsto \eta(t)$ and $G^*$ is itself the infinitesimal generator of the stochastic semigroup $P_t^*$ of the time-reversed process.

Actually, so-called *Yaglom reversibility* holds. From (5.32), it follows that

$$G^* = JGJ. \tag{5.49}$$

This identity means that the stationary forward process $(-\infty, \infty) \ni t \mapsto \eta(t)$ and the *flipped backward process*

$$(-\infty, \infty) \ni t \mapsto \widetilde{\eta}(t) := -\eta(-t) \tag{5.50}$$

obey the same law. This is a special kind of time-reversal symmetry called Yaglom reversibility, see [Y47], [Y49] and [DFS88].

Proving ergodicity is easy: the Dirichlet form of the process $t \mapsto \eta(t)$ is

$$\mathcal{D}(f) := -(f, Gf) = -\frac{1}{2}(f, \Delta f) = \frac{1}{2} \sum_{l=1}^{d} \| \nabla_l f \|^2 . \tag{5.51}$$

So

$$\{\mathcal{D}(f) = 0\} \iff \{\nabla_l f = 0, l = 1, \dots, l\} \iff \{f = \text{const.}\pi\text{-a.s.}\},$$

since $z \mapsto \tau_z$ acts ergodically on $(\Omega, \pi)$.

This proves Theorem 5.2.1. Corollary 5.2.2 follows directly from (5.8) by the ergodic theorem.

## 5.5 Central limit theorem for additive functionals of ergodic Markov processes, graded sector condition

In the present short section, we recall the non-reversible version of the Kipnis–Varadhan central limit thoerem for additive functionals of ergodic Markov processes and the *graded sector condition* of Sethuraman, Varadhan and Yau, [SVY00].

Let $(\Omega, \mathcal{F}, \pi)$ be a probability space: the state space of a *stationary and ergodic* Markov process $t \mapsto \eta(t)$. We put ourselves in the Hilbert space $\mathcal{H} := \mathcal{L}^2(\Omega, \pi)$. Denote the *infinitesimal generator* of the semigroup of the process by $G$, which is a well-defined (possibly unbounded) closed linear operator on $\mathcal{H}$. The adjoint generator $G^*$ is the infinitesimal generator of the semigroup of the reversed (also stationary and ergodic) process

$\eta^*(t) = \eta(-t)$. It is assumed that $G$ and $G^*$ have a *common core of definition* $\mathcal{C} \subseteq \mathcal{H}$. Let $f \in \mathcal{H}$ such that $(f, 1\!\!1) = \int_\Omega f \, \mathrm{d}\pi = 0$. We ask about central limit theorem and invariance principle for

$$N^{-1/2} \int_0^{Nt} f(\eta(s)) \, \mathrm{d}s \tag{5.52}$$

as $N \to \infty$.

We denote the *symmetric* and *anti-symmetric* parts of the generators $G$, $G^*$, by

$$S := -\frac{1}{2}(G + G^*), \qquad A := \frac{1}{2}(G - G^*).$$

These operators are also extended from $\mathcal{C}$ by graph closure and it is assumed that they are well-defined self-adjoint, respectively, skew self-adjoint operators

$$S^* = S \geq 0, \qquad A^* = -A.$$

Note that $-S$ is itself the infinitesimal generator of a Markovian semigroup on $\mathcal{L}^2(\Omega, \pi)$, for which the probability measure $\pi$ is reversible (not just stationary). We assume that $-S$ is itself ergodic:

$$\mathrm{Ker}(S) = \{c1\!\!1 : c \in \mathbb{C}\}.$$

We denote by $R_\lambda \in \mathcal{B}(\mathcal{H})$ the resolvent of the semigroup $s \mapsto e^{sG}$:

$$R_\lambda := \int_0^\infty e^{-\lambda s} e^{sG} \mathrm{d}s = \big(\lambda I - G\big)^{-1}, \qquad \lambda > 0 \tag{5.53}$$

and given $f \in \mathcal{H}$ as above, we will use the notation

$$u_\lambda := R_\lambda f. \tag{5.54}$$

The following theorem yields the efficient martingale approximation of the additive functional (5.52):

**Theorem 5.5.1 (KV).** *With the notation and assumptions as before, if the following two limits hold in $\mathcal{H}$:*

$$\lim_{\lambda \to 0} \lambda^{1/2} u_\lambda = 0, \tag{5.55}$$

$$\lim_{\lambda \to 0} S^{1/2} u_\lambda =: v \in \mathcal{H}, \tag{5.56}$$

*then*

$$\sigma^2 := 2 \lim_{\lambda \to 0} (u_\lambda, f) \in [0, \infty)$$

*and there exists a zero mean, $\mathcal{L}^2$-martingale $M(t)$, adapted to the filtration of the Markov process $\eta(t)$ with stationary and ergodic increments and variance*

$$\mathbf{E}\left(M(t)^2\right) = \sigma^2 t$$

*such that*

$$\lim_{N\to\infty} N^{-1}\mathbf{E}\left(\left(\int_0^N f(\eta(s))\,\mathrm{d}s - M(N)\right)^2\right) = 0.$$

*In particular, if $\sigma > 0$, then the finite dimensional marginal distributions of the rescaled process $t \mapsto \sigma^{-1}N^{-1/2}\int_0^{Nt} f(\eta(s))\,\mathrm{d}s$ converge to those of a standard* 1d *Brownian motion.*

**Remarks.** 1. Conditions (5.55) and (5.56) of the theorem are jointly equivalent to the following

$$\lim_{\lambda,\lambda'\to 0}(\lambda+\lambda')(u_\lambda, u_{\lambda'}) = 0. \tag{5.57}$$

Indeed, straightforward computations yield:

$$(\lambda+\lambda')(u_\lambda, u_{\lambda'}) = \left\| S^{1/2}(u_\lambda - u_{\lambda'}) \right\|^2 + \lambda \left\| u_\lambda \right\|^2 + \lambda' \left\| u_{\lambda'} \right\|^2 .$$

2. The theorem is a generalization to non-reversible setup of the celebrated Kipnis–Varadhan theorem, [KV86]. To the best of our knowledge, the non-reversible formulation, proved with resolvent rather than spectral calculus, appears first – in discrete time Markov chain, rather than continuous time Markov process setup and with condition (5.57) – in [T86] where it was applied, with bare hand computations, to obtain central limit theorem for a particular random walk in random environment. Its proof follows the original proof of the Kipnis–Varadhan theorem with the difference that spectral calculus is to be replaced by resolvent calculus.

3. In continuous time Markov process setup, it was formulated in [V96] and applied to tagged particle motion in non-reversible zero mean exclusion processes. In this paper, the *(strong) sector condition* was formulated, which, together with an $H_{-1}$-bound on the function $f \in \mathcal{H}$, provide sufficient condition for (5.55) and (5.56) of Theorem 5.5.1 to hold.

4. In [SVY00], the so-called *graded sector condition* is formulated and Theorem 5.5.1 is applied to tagged particle diffusion in general (non-zero mean) non-reversible exclusion processes in $d \geq 3$.

5. For a more complete list of applications of Theorem 5.5.1 together with the strong and graded sector conditions, see the surveys [O01] and [KLO09].

Checking conditions (5.55) and (5.56) (or, equivalently, condition (5.57)) in particular applications is typically not easy. In the applications to RWRE in [T86], the conditions were checked by some tricky bare hand computations. In [V96], respectively, [SVY00], the so-called *sector condition*, respectively, the *graded sector condition* were introduced and checked for the respective models.

We recall from [SVY00] the graded sector condition. Assume that the Hilbert space $\mathcal{H} = \mathcal{L}^2(\Omega, \pi)$ is graded

$$\mathcal{H} = \overline{\oplus_{n=0}^{\infty} \mathcal{H}_n} \tag{5.58}$$

and the infinitesimal generator is consistent with this grading in the sense of (5.47).

**Theorem 5.5.2 (SVY).** *Assume that the Hilbert space and the infinitesimal generator $G = -S + A$ are graded in the sense specified above and, in addition, there exist $\gamma \in [0, 1)$ and $C < \infty$ such that for any $n \in \mathbb{N}$ and any $g \in \mathcal{H}_n$, $h \in \mathcal{H}_{n+1}$*

$$|(h, A_+ g)| \leq C n^{\gamma} \sqrt{(h, Sh)} \sqrt{(g, Sg)}. \tag{5.59}$$

*If $f \in \mathcal{H}$ with $(f, \mathbb{1}) = 0$ is such that*

$$\| S^{-1/2} f \| := \lim_{\lambda \to 0} (f, u_\lambda) < \infty, \tag{5.60}$$

*then* (5.55) *and* (5.56) *hold and, as consequence, the conclusions of Theorem 5.5.1 are valid.*

## 5.6 Proof of the central limit theorem

The proof of Theorem 5.2.3 consists of three parts. First, in Subsection 5.6.1, we prove diffusive lower bound on the variance of the displacement $X(t)$. We need this in order to exclude the possibility that the a priori martingale part of the displacement and the martingale approximation of the compensator in the limit just cancel out. (As it is well known, this happens for example in tagged particle diffusion in 1d simple symmetric exclusion process with nearest neighbour jumps, see [A83].) Then, in Subsection 5.6.2, we prove the $H_{-1}$-bound (5.60) for our particular case. Finally, in Subsection 5.6.3, we check conditions (5.59) of Theorem 5.5.2 for our particular model.

### 5.6.1 Diffusive lower bound

Similarly to the decomposition (5.8), for $s, t \in \mathbb{R}$ with $s < t$, let

$$M(s, t) := X(t) - X(s) - \int_s^t \varphi(\eta(u))\, \mathrm{d}u = B(t) - B(s). \tag{5.61}$$

**Lemma 5.6.1.**

1. *Fix $s \in \mathbb{R}$. The process $[s, \infty) \ni t \mapsto M(s, t)$ is a forward martingale with respect to the forward filtration $\{\mathcal{F}_{(-\infty,t]} : t \geq s\}$ of the process $t \mapsto \eta(t)$.*

2. *Fix $t \in \mathbb{R}$. The process $(-\infty, t] \ni s \mapsto M(s, t)$ is a backward martingale with respect to the backward filtration $\{\mathcal{F}_{[s,\infty)} : s \leq t\}$ of the process $t \mapsto \eta(t)$.*

*Proof.* There is nothing to prove about the first statement: the integral on the right-hand side of (5.61) was chosen exactly so that it compensates the conditional expectation of the infinitesimal increments of $X(t)$.

We turn to the second statement, which does need a proof. This consists of the following ingredients:

(1) The displacements are preserved on the flipped backward trajectories $t \mapsto \widetilde{\eta}(t)$ defined in (5.50):

$$\widetilde{X}(t) - \widetilde{X}(s) = X(-t) - X(-s). \tag{5.62}$$

(2) The forward process $t \mapsto \eta(t)$ and flipped backward process $t \mapsto \widetilde{\eta}(t)$ are identical in law (Yaglom reversibility).

(3) The function $\omega \mapsto \varphi(\omega)$ is odd with respect to the flip-map $\omega \mapsto -\omega$.

Putting these facts together (in this order), we obtain

$$\begin{aligned}\lim_{h\to 0}(-h)^{-1}\mathbf{E}\left(X(s-h) - X(s) \mid \mathcal{F}_{[s,\infty)}\right) &= \lim_{h\to 0} h^{-1}\mathbf{E}\left(-\widetilde{X}(-s+h) + \widetilde{X}(-s) \mid \widetilde{\mathcal{F}}_{(-\infty,-s]}\right) \\ &= -\varphi(\widetilde{\eta}(-s)) = \varphi(\eta(s)).\end{aligned}$$

□

From Lemma 5.6.1, it follows directly that for any $s < t$, the random variables $M(s,t)$ and $\int_s^t \varphi(\eta(u))\,\mathrm{d}u$ are *uncorrelated*, because

$$\mathbf{E}\left(M(s,t)\int_s^t \varphi(\eta(u))\,\mathrm{d}u\right) = \int_s^t \left(\mathbf{E}\left(M(s,u)\varphi(\eta(u))\right) + \mathbf{E}\left(M(u,t)\varphi(\eta(u))\right)\right)\mathrm{d}u. \tag{5.63}$$

The first expectation on the right-hand side of (5.63) can be written as

$$\mathbf{E}\left(\mathbf{E}\left(M(s,u)\varphi(\eta(u)) \mid \mathcal{F}_{[s,u]}\right)\right) = \mathbf{E}\left(\varphi(\eta(u))\mathbf{E}\left(M(s,u) \mid \mathcal{F}_{[s,u]}\right)\right),$$

which is 0 by the martingale property. Similarly, the second term in (5.63) vanishes by the backward martingale property. Therefore,

$$\begin{aligned}\mathbf{E}\left((X(t) - X(s))^2\right) &= \mathbf{E}\left((M(s,t))^2\right) + \mathbf{E}\left(\left(\int_s^t \varphi(\eta(u))\,\mathrm{d}u\right)^2\right) \\ &= (t-s) + \mathbf{E}\left(\left(\int_s^t \varphi(\eta(u))\,\mathrm{d}u\right)^2\right).\end{aligned} \tag{5.64}$$

Hence, the lower bound in (5.24).

### 5.6.2 Diffusive upper bound: $H_{-1}$-bound

We recall a general result proved in [SVY00]. See also the surveys [O01], [KLO09] and further references cited therein.

Let $t \mapsto \xi(t)$ be the *reversible* Markov process on the same state space $(\Omega, \pi)$ as the original $\eta(t)$ which has the infinitesimal generator $-S$.

**Lemma 5.6.2 (SVY).** *Let $\varphi \in \mathcal{L}^2(\Omega, \pi)$ with $\int \varphi \, \mathrm{d}\pi = 0$. Then*

$$\limsup_{t\to\infty} t^{-1}\mathbf{E}\left(\left(\int_0^t \varphi(\eta(s))\,\mathrm{d}s\right)^2\right) \le \lim_{t\to\infty} t^{-1}\mathbf{E}\left(\left(\int_0^t \varphi(\xi(s))\,\mathrm{d}s\right)^2\right) = 2\|S^{-1/2}\varphi\|^2. \tag{5.65}$$

In our case,

$$S = -\frac{1}{2}\Delta$$

and the reversible process $t \mapsto \xi(t)$ is the so-called *diffusion in random scenery* process. That means:

$$\xi(t) := \tau_{Z_t}\omega$$

where $t \mapsto Z_t$ is a Brownian motion in $\mathbb{R}^d$ of covariance $\delta_{ij}$, independent of the field $\omega$. The function $\varphi : \Omega \to \mathbb{R}$ is $\varphi(\omega) = \omega(0)$. Thus, the upper bound in (5.65) will be

$$\lim_{t\to\infty} t^{-1}\mathbf{E}\left(\left(\int_0^t \varphi(\xi(s))\,\mathrm{d}s\right)^2\right) = \lim_{t\to\infty} t^{-1}\mathbf{E}\left(\left(\int_0^t \omega(Z_s)\,\mathrm{d}s\right)^2\right) = \int_{\mathbb{R}^d} |p|^{-2}\,\widehat{V}(p)\,\mathrm{d}p. \tag{5.66}$$

Here, the last step is straightforward computation with expectation taken over the Brownian motion $Z(t)$ *and* over the random scenery $\omega$. The integral on the right-hand side is the same as in (5.25), and thus, (5.66) yields the upper bound in (5.24). □

### 5.6.3 Graded sector condition

As a first remark, note that condition (5.59) is equivalent to

$$\left\| S^{-1/2} A_+ S^{-1/2} \restriction_{\mathcal{H}_n} \right\| \le C n^{\gamma} \tag{5.67}$$

where the operator $S^{-1/2} A_+ S^{-1/2} \restriction_{\mathcal{H}_n}$ is meant as first defined on a dense subspace of $\mathcal{H}_n$ and extended by continuity. In our case, the dense subspace will be $\widehat{\mathcal{C}}_n$ specified in (5.41) and

$$S^{-1/2} A_+ S^{-1/2} = \sum_{l=1}^{d} |\Delta|^{-1/2} a_l^* \nabla_l |\Delta|^{-1/2}. \tag{5.68}$$

The operators $\nabla_l |\Delta|^{-1/2}$ map the subspaces $\widehat{\mathcal{C}}_n$ to themselves and are bounded, see (5.37) and (5.38). In order to bound the norm of the operator $|\Delta|^{-1/2} a_l^* : \mathcal{H}_n \to \mathcal{H}_{n+1}$, let

$\widehat{u} \in \widehat{\mathcal{C}}_n$, then

$$|\Delta|^{-1/2} a_l^* \widehat{u}(p_1, \ldots, p_{n+1}) = \frac{i}{\sqrt{n+1}} \frac{1}{\left|\sum_{m=1}^{n+1} p_m\right|} \sum_{m=1}^{n+1} p_{ml} \widehat{u}(p_1, \ldots, p_{m-1}, p_{m+1}, \ldots, p_n). \tag{5.69}$$

Hence

$$\begin{aligned}
&(n+1) \left\| \, |\Delta|^{-1/2} a_l^* \widehat{u} \, \right\|^2 \\
&= \int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} \frac{1}{\left|\sum_{m=1}^{n+1} p_m\right|^2} \left| \sum_{m=1}^{n+1} p_{ml} \widehat{u}(p_1, \ldots, p_{m-1}, p_{m+1}, \ldots, p_n) \right|^2 \prod_{m=1}^{n+1} \frac{\widehat{V}(p_m)}{|p_m|^2} \mathrm{d}p_1 \ldots \mathrm{d}p_{n+1} \\
&\leq (n+1)^2 \int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} \frac{1}{\left|\sum_{m=1}^{n+1} p_m\right|^2} |p_{n+1,l}|^2 \, |\widehat{u}(p_1, \ldots, p_n)|^2 \prod_{m=1}^{n+1} \frac{\widehat{V}(p_m)}{|p_m|^2} \mathrm{d}p_1 \ldots \mathrm{d}p_{n+1} \\
&= (n+1)^2 \int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} |\widehat{u}(p_1, \ldots, p_n)|^2 \prod_{m=1}^{n} \frac{\widehat{V}(p_m)}{|p_m|^2} \left( \int_{\mathbb{R}^d} \frac{p_{n+1,l}^2}{|p_{n+1}|^2} \frac{\widehat{V}(p_{n+1})}{\left|\sum_{m=1}^{n+1} p_m\right|^2} \mathrm{d}p_{n+1} \right) \mathrm{d}p_1 \ldots \mathrm{d}p_n.
\end{aligned} \tag{5.70}$$

In the second line, Schwarz's inequality and the symmetry of the function $\widehat{u}(p_1, \ldots, p_n)$ is used.

The innermost integral of the last expression in (5.70) is bounded above by

$$C^2 := \sup_{p \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\widehat{V}(p+q)}{|q|^2} \mathrm{d}q < \infty.$$

Thus, for $\widehat{u} \in \widehat{\mathcal{C}}_n$

$$\left\| \, |\Delta|^{-1/2} a_l^* \widehat{u} \, \right\|^2 \leq C^2 (n+1) \, \| \, \widehat{u} \, \|^2 .$$

Hence, by continuous extension,

$$\left\| \, |\Delta|^{-1/2} a_l^* \restriction_{\mathcal{H}_n} \right\| \leq C \sqrt{n+1} \tag{5.71}$$

and (5.67) with $\gamma = 1/2$ follows. □

# Chapter 6

# Further developments and conclusion

In this final chapter, we report some recent results concerning the MSAW and the SRBP. The theorems about the MSAW are from the paper [HTV11], but they are *not part of the present thesis*, and we quote them without proof.

## 6.1 Central limit theorem for the MSAW in three and higher dimensions

We recall the definition of local times on the lattice $\mathbb{Z}^d$ in continuous time (4.1), and we modify it by adding some initial values $\ell(0,x) \in \mathbb{R}$ for each $x \in \mathbb{Z}^d$:

$$\ell(t,x) := \ell(0,x) + |\{s \in [0,t) : X(s) = x\}|.$$

The definition of jump rates (4.2) remains valid.

Similarly to (5.5) for the SRBP model, we consider the local time profile as seen from the position of the random walker:

$$\eta(t) = \big(\eta(t,x)\big)_{x\in\mathbb{Z}^d}, \qquad \eta(t,x) := \ell(t, X(t)+x).$$

It is obvious that $t \mapsto \eta(t)$ is a càdlàg Markov process on the state space

$$\Omega := \{\omega = \big(\omega(x)\big)_{x\in\mathbb{Z}^d} : \omega(x) \in \mathbb{R}, (\forall\varepsilon > 0) \lim_{|x|\to\infty} |x|^{-\varepsilon} |\omega(x)| = 0\}.$$

Note that we allow initial values $\ell(0,x) \in \mathbb{R}$ for the occupation time measure and thus $\ell(t,x)$ need not be non-negative.

Next, we define a probability measure on $\Omega$ which will turn out to be *stationary* and ergodic for the Markov process $t \mapsto \eta(t)$. Let

$$R : \mathbb{R} \to [0,\infty), \qquad R(u) := \int_0^u \frac{w(v) - w(-v)}{2}\,\mathrm{d}v.$$

$R$ is strictly convex and even. We denote by $\mathrm{d}\pi(\omega)$ *the unique centered Gibbs measure* (Markov field) on $\Omega$ defined by the conditional specifications for $\Lambda \subset \mathbb{Z}^d$ finite:

$$\mathrm{d}\pi\left(\omega_\Lambda \;\middle|\; \omega_{\mathbb{Z}^d\setminus\Lambda}\right) = Z_\Lambda^{-1}\exp\left\{-\frac{1}{2}\sum_{\substack{x,y\in\Lambda\\|x-y|=1}} R(\omega(x)-\omega(y)) - \sum_{\substack{x\in\Lambda,y\in\Lambda^c\\|x-y|=1}} R(\omega(x)-\omega(y))\right\}\mathrm{d}\omega_\Lambda. \tag{6.1}$$

Note that the (translation invariant) Gibbs measure given by the specifications (6.1) exists only in three and more dimensions. The measure $\mathrm{d}\pi$ is invariant under the spatial shifts and the dynamical system which consists of the probability space $(\Omega, \pi)$ with the translations is *ergodic*.

In the particular case when $r(u) = u$, $R(u) = u^2/2$, the measure $\mathrm{d}\pi(\omega)$ is the distribution of the massless free Gaussian field on $\mathbb{Z}^d$, $d \geq 3$ with expectations and covariances

$$\int_\Omega \omega(x)\,\mathrm{d}\pi(\omega) = 0, \qquad \int_\Omega \omega(x)\omega(y)\,\mathrm{d}\pi(\omega) = (-\Delta)^{-1}_{x,y}$$

where $\Delta$ is the lattice Laplacian.

**Theorem 6.1.1.** *The probability measure* $\mathrm{d}\pi(\omega)$ *is stationary and ergodic for the Markov process* $t \mapsto \eta(t) \in \Omega$.

The law of large numbers for the displacement of the random walker drops out for free:

**Corollary 6.1.2.** *For $\pi$-almost all initial profiles $\ell(0,\cdot)$, almost surely,*

$$\lim_{t\to\infty}\frac{X(t)}{t} = 0.$$

The main results refer to the diffusive scaling limit of the displacement.

**Theorem 6.1.3.** *1. Under some technical conditions for the rate function,*

$$0 < \gamma \leq \inf_{|e|=1}\varliminf_{t\to\infty} t^{-1}\mathbf{E}\left((e\cdot X(t))^2\right) \leq \sup_{|e|=1}\varlimsup_{t\to\infty} t^{-1}\mathbf{E}\left((e\cdot X(t))^2\right) < \infty.$$

*2. Assume that*

$$r(u) = u, \qquad s(u) = s_4u^4 + s_2u^2 + s_0$$

*and we also make the technical assumption that $s_4/\gamma$ is sufficiently small. Then, the matrix of asymptotic covariances*

$$\sigma^2_{kl} := \lim_{t\to\infty} t^{-1}\mathbf{E}\left(X_k(t)X_l(t)\right)$$

*exists and it is non-degenerate. The finite dimensional distributions of the rescaled displacement process*

$$X_N(t) := N^{-1/2}X(Nt)$$

*converge to those of a $d$-dimensional Brownian motion with covariance matrix $\sigma^2$.*

## 6.2 Conclusion

In order to provide a complete picture for the reader, we remark that the SRBP model in one and two dimensions has also been investigated recently.

In one dimension, Tarrès, Tóth and Valkó proved essentially the bounds

$$C_1 t^{5/4} \leq \mathbf{E}\left(|X(t)|^2\right) \leq C_2 t^{3/2} \tag{6.2}$$

for the displacement of the SRBP defined in (1.3)–(1.4) with some $0 < C_1 < C_2 < \infty$ under some conditions on the function $V$, see [TTV11].

On the other hand, Tóth and Valkó proved in [TV10] among others that, for the two-dimensional SRBP, the super-diffusive bounds

$$C_3 t \log\log t \leq \mathbf{E}\left(|X(t)|^2\right) \leq C_4 t \log t \tag{6.3}$$

hold with some $0 < C_3 < C_4 < \infty$.

These bounds (6.2) and (6.3) are still not of the order conjectured in the 1980's for the MSAW and formulated in Section 1.1, but they give robust estimates which do not depend on the particularities of the model.

The existence of these recent results also shows that the self-repelling random walks and processes are at the cutting edge of modern probability theory.

For possible directions of future research, there is a wide variety of open questions. The conjectures of Section 1.1 are still unknown in general. The results of the present thesis and those of the quoted references are valid under some restrictive assumptions. For example, the powerful tool of Ray−Knight approach cannot be extended beyond the models treated here for some combinatorial reasons. One may ask for some robust argument for proving super-diffusive behaviour of the MSAW in one (and two) dimension.

Concerning the case above the critical dimension, the functional analytic tools used in Chapter 5 can be applied for *stationary* initialization of local times. It is a natural question to consider different (most interestingly empty) initial condition. The recent paper [CP10] treats similar issues and proves central limit theorem for additive functionals of reversible Markov processes for almost all initial values, but those methods do not seem to be directly applicable here and they are not valid for the empty initial condition. Another direction of future research might be the extension of the functional analytic methods for other types of models, e.g. random walks in random environment which is already the starting point of a new research project.

# Acknowledgements

I am especially grateful to my supervisor, prof. Bálint Tóth for introducing me to probability theory from the beginning and to self-interacting random walks later, for giving me interesting problems and useful suggestions during my PhD years and for his continuous and generous support.

I thank Illés Horváth, my fellow PhD student and co-author for the fruitful discussions and collaboration.

# Bibliography


[APP83] D. Amit, G. Parisi, L. Peliti: Asymptotic behavior of the 'true' self-avoiding walk. *Phys. Rev. B*, **27**: 1635–1645 (1983)

[A83] R. Arratia: The motion of a tagged particle in the simple symmetric exclusion system on $\mathbb{Z}$. *Ann. Probab.*, **11**: 362–373 (1983)

[BN02] K. Burdzy, D. Nualart: Brownian motion reflected on Brownian motion. *Probab. Theory Related Fields*, **122**: 471–493 (2002)

[BT91] M. van den Berg, B. Tóth. Exponential estimates for the Wiener sausage. *Probab. Theory Related Fields*, **88**: 249–259 (1991)

[CL95] M. Cranston, Y. Le Jan: Self-attracting diffusions: two case studies. *Math. Ann.*, **303**: 87–93 (1995)

[CM96] M. Cranston, T. S. Mountford: The strong law of large numbers for a Brownian polymer. *Ann. Probab.*, **24**: 1300–1323 (1996)

[CP10] C. Cuny, M. Peligrad: Central limit theorem started at a point for stationary processes and additive functionals of reversible Markov chains. *J. Theoret. Probab.*, **23**: 1–18 (2010)

[DFS88] R. L. Dobrushin, J. Fritz, Yu. M. Suhov: A. N. Kolmogorov: The founder of the theory of reversible Markov processes. *Uspekhi Mat. Nauk*, **43:6**: 167–188 (1988) [English translation: *Russian Math. Surveys*, **43:6**: 157–182]

[D95] R. Durrett: *Probability: Theory and Examples.* Second Edition, Duxbury Press, 1995

[DR92] R. T. Durrett, L. C. G. Rogers: Asymptotic behavior of Brownian polymers. *Probab. Theory Related Fields*, **92**: 337–349 (1992)

[FINR04] L. R. G. Fontes, M. Isopi, C.M. Newman, K. Ravishankar: The Brownian Web: characterization and convergence. *Ann. Probab.*, **32**: 2857–2883 (2004)

[HTV11] I. A. Horváth, B. Tóth, and B. Vető: Diffusive limits for "true" (or myopic) self-avoiding random walks and self-repellent Brownian polymers in $d \geq 3$. To appear in *Probab. Theory Related Fields* (2011)

[J97] S. Janson: *Gaussian Hilbert Spaces.* Cambridge University Press, 1997

[KLO09] T. Komorowski, C. Landim, S. Olla: *Book in preparation*, Springer

[KO03] T. Komorowski, S. Olla: On the sector condition and homogenization of diffusions with a Gaussian drift. *J. Funct. Anal.*, **197**: 179–211 (2003)

[K63] F. B. Knight: Random walks and a sojourn density process of Brownian motion. *Transactions of the AMS*, **109**: 56–86 (1963)

[KV86] C. Kipnis, S. R. S. Varadhan: Central limit theorem for additive functionals of reversible Markov processes with applications to simple exclusion. *Comm. Math. Phys.*, **106**: 1–19 (1986)

[MS93] N. Madras, G. Slade: *The Self-Avoiding Walk*, Birkhuser, Boston, 1993

[MT08] T. S. Mountford, P. Tarrès: An asymptotic result for Brownian polymers. *Ann. Inst. H. Poincaré Probab. Statist.*, **44**: 29–46 (2008)

[NR06] C. M. Newman and K. Ravishankar: Convergence of the Tóth lattice filling curve to the Tóth – Werner plane filling curve. *ALEA – Lat. Am. J. Probab. Math. Stat.*, **1**: 333–346 (2006)

[NRW87] J. R. Norris, L. C. G. Rogers, D. Williams: Self-avoiding walk: a Brownian motion model with local time drift. *Probab. Theory Related Fields*, **74**: 271–287 (1987)

[N84] E. Nummelin: *Irreducible Markov Chains and Non-Negative Operators.* Cambridge University Press, 1984.

[OP83] S. P. Obukhov, L. Peliti: Renormalisation of the "true" self-avoiding walk. *J. Phys. A*, **16**: L147–L151 (1983)

[O01] S. Olla: Central limit theorems for tagged particles and for diffusions in random environment. In: F. Comets, É. Pardoux (eds): *Milieux aléatoires Panor. Synthèses*, **12**, Soc. Math. France, Paris, 2001.

[PP87] L. Peliti, L. Pietronero: Random walks with memory. *Riv. Nuovo Cimento*, **10**: 1–33 (1987)

[P07] R. Pemantle: A survey of random processes with reinforcement. *Probab. Surv.*, **4**: 1–79 (2007)

[R63] D. Ray: Sojourn times of a diffusion process. *Illinois J. Math.*, **7**: 615–630 (1963)

[RS72] M. Reed, B. Simon: *Methods of Modern Mathematical Physics Vol 1, 2.* Academic Press New York, 1972–75.

[RY99] D. Revuz, M. Yor: *Continuous Martingales and Brownian Motion.* Third Edition, Springer, 1999

[SVY00] S. Sethuraman, S. R. S. Varadhan, H-T. Yau: Diffusive limit of a tagged particle in asymmetric simple exclusion processes. *Comm. Pure Appl. Math.*, **53**: 972–1006 (2000)

[S74] B. Simon: *The* $P(\varphi)_2$ *Euclidean (Quantum) Field Theory.* Princeton University Press, 1974.

[STW00] F. Soucaliuc, B. Tóth, W. Werner: Reflection and coalescence between independent one-dimensional Brownian paths. *Ann. Inst. H. Poincaré Probab. Statist.*, **36**: 509–545 (2000)

[SW02] F. Soucaliuc, W. Werner: A note on reflecting Brownian motion. *Electron. Commun. Probab.*, **7**: 117–122 (2002)

[T04] P. Tarrès: Vertex-reinforced random walk on $\mathbb{Z}$ eventually gets stuck on five points. *Ann. Probab.*, **32**: 2650–2701 (2004)

[TTV11] P. Tarrès, B. Tóth, B. Valkó: Diffusivity bounds for 1d Brownian polymers. To appear in *Ann. Probab.* (2011)

[T97] B. Tóth: Limit theorems for weakly reinforced random walks. *Studia Sci. Math. Hungar.*, **33**: 321–337 (1997)

[T86] B. Tóth: Persistent random walk in random environment. *Probab. Theory Related Fields* **71**: 615–625 (1986)

[T99] B. Tóth: Self-interacting random motions – a survey. In P. Révész and B. Tóth, editors, *Random Walks*, vol. 9 of *Bolyai Society Mathematical Studies*, pages 349–384. János Bolyai Mathematical Society, Budapest, 1999.

[T01] B. Tóth: Self-interacting random motions. In *Proceedings of the 3rd European Congress of Mathematics, Barcelona*, vol. 1, pages 555–565. Birkhäuser, Boston-Basel-Berlin, 2001.

[T95] B. Tóth: The 'true' self-avoiding walk with bond repulsion on $\mathbb{Z}$: limit theorems. *Ann. Probab.*, **23**: 1523–1556 (1995)

[TV10] B. Tóth, B. Valkó: Superdiffusive bounds on self-repellent Brownian polymers and diffusion in the curl of the Gaussian free field in $d = 2$. `arXiv:1012.5698` (2010)

[TV11] B. Tóth, B. Vető: Continuous time 'true' self-avoiding random walk on $Z$. *ALEA Lat. Am. J. Probab. Math. Stat.*, **8**: 59–75 (2011)

[TV08] B. Tóth, B. Vető: Self-repelling random walk with directed edges on $\mathbb{Z}$, *Electron. J. Probab.*, **13**: 1909–1926 (2008)

[TV07] B. Tóth, B. Vető: Skorohod-reflection of Brownian Paths and $\text{BES}^3$, *Acta Sci. Math. (Szeged)* **73**: 781–788 (2007)

[TW98] B. Tóth, W. Werner: The true self-repelling motion. *Probab. Theory Related Fields*, **111**: 375–452 (1998)

[V96] S. R. S. Varadhan: Self-diffusion of a tagged particle in equilibrium of asymmetric mean zero random walks with simple exclusion. *Ann. Inst. H. Poincaré Probab. Statist.*, **31**: 273–285 (1996)

[V09] B. Vető: The "True" Self-Avoiding Random Walk in $\mathbb{Z}$, *Wolfram Demonstration Project*, `http://demonstrations.wolfram.com/TheTrueSelfAvoidingRandomWalkInZ/` (2009)

[W07] J. Warren: Dyson's Brownian motions, interwining and interlacing, *Electron. J. Probab.*, **12** 573–590 (2007)

[Y47] A. M. Yaglom: On the statistical treatment of Brownian motion. (Russian) *Doklady Akad. Nauk SSSR*, **56**: 691–694 (1947)

[Y49] A. M. Yaglom: On the statistical reversibility of Brownian motion. (Russian) *Mat. Sbornik*, **24**: 457–492 (1949)